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A COUNTEREXAMPLE TO A COMPLEMENTATION PROBLEM

J. Bourgain*

Abstract

The existence is shown of subspaces of L^1 which are isomorphic to an $L^1(\mu)$-space and are not complemented. A more precise local statement is also given.

1. Introduction

The question we are dealing with is the following:

PROBLEM 1: Let μ and ν be measures and $T : L^1(\mu) \to L^1(\nu)$ an isomorphic embedding. Does there always exist a projection of $L^1(\nu)$ onto the range of T?

and was raised in [1], [4], [5] and [21].

This problem has the following finite dimensional reformulation (cfr. [4]).

PROBLEM 2: Does there exist for each $\lambda < \infty$ some $C < \infty$ such that given a finite dimensional subspace E of $L^1(\nu)$ satisfying $d(E, \ell^1(\dim E)) \leq \lambda$ (d = Banach-Mazur distance), one can find a projection $P : L^1(\nu) \to E$ with $\|P\| \leq C$?

In [4], L. Dor obtained a positive solution to problem 1 provided $\|T\| \|T^{-1}\| < \sqrt{2}$. It was shown by L. Dor and T. Starbird (cfr. [5]) that

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any l^1-subspace of $L^1(\nu)$ which is generated by a sequence of probabilistically independent random variables is complemented. A slight improvement of this result will be given in the remarks below, where we show that problem 2 is affirmative under the additional hypothesis that E is spanned by independent variables. Our main purpose is to show that the general solution to the above questions is negative. Examples of uncomplemented l^p-subspaces of L^p $(1 < p < \infty)$ were already discovered (see [24] for the cases $2 < p < \infty$ and $1 < p < 4/3$ and [1] for $1 < p < 2$).

2. The Example

We first introduce some notation. For each positive integer N, denote G_N the group $\{1, -1\}^N$ equipped with its Haar measure m_N.

For $1 \leq n \leq N$, the n^{th} Rademacker function r_n on G_N is defined by $r_n(x) = x_n$ for all $x \in G_N$. To each subset S of $\{1, 2, \ldots, N\}$ corresponds a Walsh function $w_S = \Pi_{n \in S}\, r_n$ and $L^1(G_N)$ is generated by this system of Walsh functions.

For fixed $0 \leq \epsilon \leq 1$, let $\mu = \bigotimes_n \mu_n$ be the product measure on G_N, where $\mu_n(1) = \dfrac{1+\epsilon}{2}$ and $\mu_n(-1) = \dfrac{1-\epsilon}{2}$ for all $n = 1, \ldots, N$. This measure μ is called sometimes the ϵ-biased coin-tossing measure (cfr. [30]).

Let now $T_\epsilon : L^1(G_N) \to L^1(G_N)$ be the convolution operator corresponding to μ. Thus $(T_\epsilon f)(x) = (f * \mu(x) = \int_{G_N} f(x, y)\mu(\mathrm{d}y)$ for all $f \in L^1(G_N)$.

It is clear that T_ϵ is a positive operator of norm 1 and easily verified that $T_\epsilon(w_S) = \epsilon^{|S|} w_s$, where $|S|$ denotes the cordinality of the set S. Another way of introducing T_ϵ is by using Riesz-products.

Before describing the example, we give some lemma's.

LEMMA 1: *If* $f \in L^1(G_N)$, *then* $\|T_\epsilon f\|_2 \leq |\int f \,\mathrm{d}m_N| + \epsilon \|f\|_2$.

PROOF: Take $f = a_\phi + \Sigma_{S \neq \phi}\, a_S w_S$ the Walsh expansion of f. Then

$$T_\epsilon f = a_\phi + \sum_{S \neq \phi} a_S \epsilon^{|S|} w_S$$

and hence $\|T_\epsilon f\|_2^2 = |a_\phi|^2 + \sum_{s \neq \phi} |a_S|^2 \epsilon^{2|S|} \leq |a_\phi|^2 + \epsilon^2 \|f\|_2^2.$

The required inequality follows.

LEMMA 2: *Let $f_1, \ldots, f_d$ be functions in $L^1(G_N)$ such that for each $i = 1, \ldots, d$*

1. $\int f_i \, \mathrm{d}m_N = 0$.
2. $\int_{A_i} |f_i| \, \mathrm{d}m_N \geq \delta \|f_i\|_1$ *where* $A_i = [|f_i| \geq d\|f_i\|_1]$.

Then

$$\int_{G_N \times \cdots \times G_N} |f_1(x_1) + \cdots + f_d(x_d)| \, \mathrm{d}m_N(x_1) \ldots \mathrm{d}m_N(x_d) \geq \frac{\delta}{6} \sum_{i=1}^{d} \|f_i\|_1.$$

PROOF: For $i = 1, \ldots, d$, take $D_i = G_N \backslash A_i$ and let C_i be the subset of $G_N \times \cdots \times G_N$ defined by $C_i = B_1 \times \cdots \times B_{i-1} \times A_i \times B_{i+1} \times \cdots \times B_d$. Remark that $m_N(A_i) \leq 1/d$ and hence $m_N(B_i) \geq 1 - 1/d$. Let $r_1, \ldots, r_d$ be Rademacker functions on $[0, 1]$. By unconditionality, we get

$$\begin{aligned}
&\int_{G_N \times \cdots \times G_N} \left| \sum_{i=1}^{d} f_i(x_i) \right| \mathrm{d}m_N(x_1) \ldots \mathrm{d}m_N(x_d) \\
&\quad \geq \frac{1}{2} \int_0^1 \int_{G_N \times \cdots \times G_N} \left| \sum_{i=1}^{d} r_i(t) f_i(x_i) \right| \mathrm{d}m_N(x_1) \ldots \mathrm{d}m_N(x_d) \, \mathrm{d}t \\
&\quad \geq \frac{1}{2} \sum_i \int_{C_i} |f_i(x_i)| \, \mathrm{d}m_N(x_1) \ldots \mathrm{d}m_N(x_d) \\
&\quad \geq \frac{1}{2} \left(1 - \frac{1}{d}\right)^{d-1} \sum_i \int_{A_i} |f_i(x)| \, \mathrm{d}m_N(x) \geq \frac{\delta}{6} \sum_i \|f_i\|_1,
\end{aligned}$$

as required.

For each $\nu \in G_N$, define the function $e_\nu = \Pi_{n=1}^{N} (1 + \nu_n r_n)$ on G_N. Thus $(e_\nu)_{\nu \in G_N}$ generates $L^1(G_N)$ and is isometrically equivalent to the $\ell^1(2^N)$-basis.

LEMMA 3: *For fixed $0 \leq \epsilon \leq 1$ and $\kappa > 0$, the following holds*

$$m_N[T_\epsilon(e_\nu) > \kappa] < \kappa^{-1/2} \left(1 - \frac{\epsilon^2}{4}\right)^{N/2}.$$

PROOF: It is easily verified that $T_\epsilon(e_\nu) = \Pi_{n=1}^{N} (1 + \epsilon \nu_n r_n)$. If we let $\Gamma = \Pi_{n=1}^{N} (1 + \epsilon r_n)$, then by independency

$$\int \sqrt{\Gamma} \, \mathrm{d}m_N = 2^{-N} (\sqrt{1 + \epsilon} + \sqrt{1 - \epsilon})^N < \left(1 - \frac{\epsilon^2}{4}\right)^{N/2}$$

and thus

$$m_N[T_\epsilon(e_\nu) > \kappa] = m_N[\sqrt{\Gamma} > \sqrt{\kappa}] \leqslant \kappa^{-1/2} \left(1 - \frac{\epsilon^2}{4}\right)^{N/2}.$$

We use the symbol $\bigoplus$ to denote the direct sum in ℓ^1-sense. For fixed N and d, take

$$X = \underbrace{L^1(G_N)\bigoplus\cdots\bigoplus L^1(G_N)}_{d \text{ copies}} \quad\text{and}\quad Y = \underbrace{L^1(G_N\times\cdots\times G_N)}_{d \text{ factors}}.$$

Consider the maps

$$\alpha : X \to \ell^1(d)$$

$$\beta : X \to Y$$

and for $0 \leq \epsilon \leq 1$

$$\gamma_\epsilon : X \to X$$

defined by

$$\alpha(f_1\bigoplus\cdots\bigoplus f_d) = \left(\int f_1\, dm_N, \ldots, \int f_d\, dm_N\right)$$

$$\beta(f_1\bigoplus\cdots\bigoplus f_d) = \sum_{i=1}^{d}\left(f_i(x_i) - \int f_i\, dm_N\right)$$

where $(x_1,\ldots,x_d) \in G_N\times\cdots\times G_N$ is the product variable

$$\gamma_E(f_1\bigoplus\cdots\bigoplus f_d) = (f_1 - T_\epsilon f_1)\bigoplus\cdots\bigoplus(f_d - T_\epsilon f_d).$$

Obviously $\|\alpha\| \leq 1$, $\|\beta\| \leq 2$ and $\|\gamma_\epsilon\| \leq 2$.

Let $\Lambda_\epsilon : x \to \ell^1(d)\bigoplus Y\bigoplus X$ be the map $\alpha\bigoplus\beta\bigoplus\gamma_\epsilon$, clearly satisfying $\|\Lambda_\epsilon\| \leq 5$.

LEMMA 4: *Under the above notations,* $\|\Lambda_\epsilon(\varphi)\| \geq \frac{1}{24}|\varphi\|_1$ *for each* $\varphi \in X$, *whenever* $0 < \epsilon \leq 1/4d$.

PROOF: Assume $\varphi = f_1\bigoplus\cdots\bigoplus f_d$ and take for each $i = 1,\ldots,d$

$$g_i = f_i - \int f_i\, dm_N$$

$A_i = [|g_i| \geq d\|g_i\|_1]$, $B_i = G_N\backslash A_i$, $g_i' = g_i\chi_{A_i}$ and $g_i'' = g_i\chi_{B_i}$.

Let further $I = \{i = 1,\ldots,d;\ \|g_i'\|_1 > \frac{1}{4}\|g_i\|_1\}$ and $J = \{1,\ldots,d\}\backslash I$.

Using Lemma 2, we find that

$$\|\beta(f_1 \oplus \cdots \oplus f_d)\|_1 \geq \int_{G_N \times \cdots \times G_N} \left|\sum_{i \in I} g_i(x_i)\right| \mathrm{d}m_N(x_1) \ldots \mathrm{d}m_N(x_d)$$
$$\geq \frac{1}{24} \sum_{i \in I} \|g_i\|_1.$$

On the other hand, by Lemma 1

$$\|T_\epsilon g_i\|_1 \leq \|T_\epsilon g_i'\|_1 + \left|\int g_i'' \,\mathrm{d}m_N\right| + \epsilon\|g_i''\|_2 \leq 2\|g_i'\|_1 + \epsilon d\|g_i\|_1$$

and hence for $i \in J$

$$\|f_i - T_\epsilon f_i\|_1 = \|g_i - T_\epsilon g_i\|_1 \geq \|g_i\|_1 - \|T_\epsilon g_i\|_1 \geq \tfrac{1}{4}\|g_i\|_1.$$

Consequently

$$\|\gamma_\epsilon(f_1 \oplus \cdots \oplus f_d)\|_1 \geq \sum_{i \in J} \|f_i - T_\epsilon f_i\|_1 \geq \frac{1}{4} \sum_{i \in J} \|g_i\|_1.$$

Combination of these inequalities leads to

$$\|\Lambda_\epsilon(\varphi)\|_1 \geq \sum_{i=1}^{d} \left|\int f_i \,\mathrm{d}m_N\right| + \frac{1}{24} \sum_{i=1}^{d} \|g_i\|_1 \geq \frac{1}{24} \sum_{i=1}^{d} \|f_i\|_1 = \frac{1}{24}\|\varphi\|_1$$

proving the lemma.

COROLLARY 5: *Again under the above notations, denote R_ϵ the range of Λ_ϵ.* Then $d(R_\epsilon, \ell^1(d.2^N)) \leq \frac{1}{120}$ *provided* $0 < \epsilon \leq 1/4d$.

Our next aim is to show that R_ϵ is a badly complemented subspace of $\ell^1(d) \oplus Y \oplus X$ for a suitable choice of N, d and ϵ.

LEMMA 6: *Fix any positive integer $d \geq 4$, take $N = d^{6d}$ and let $\epsilon = 1/4d$. Then $\|P| \geq d/384$ for any projection P from $\ell^1(d) \oplus Y \oplus X$ onto R.*

PROOF: Define for each $\nu \in G_N$

$$\xi_\nu = \frac{1}{d} \sum_{j=0}^{d-1} T_{\epsilon^j}(e_\nu) \quad \text{and} \quad A_\nu = [\xi_\nu > \tfrac{1}{4}].$$

Since $A_\nu \subset \cup_{j=0}^{d-1} [T_{\epsilon^j}(a_\nu) > \frac{1}{4}]$, application of Lemma 3 gives that

$$m_N(A_\nu) \le \sum_{j=0}^{d-1} m_N[T_{\epsilon^j}(e_\nu) > \tfrac{1}{4}] \le 2d\left(1-\frac{\epsilon^{2d}}{4}\right)^{N/2}$$

and hence, by the choice of N and ϵ

$$m_N(A_\nu) < \tfrac{1}{2},$$

as an easy computation shows.

It follows that if $\psi_\nu = \xi_\nu - 1$, then

$$\|\psi_\nu\|_1 \ge \int_{A_\nu} \xi_\nu \,\mathrm{d}m_N - m_N(A_\nu) \ge \int \xi_\nu \,\mathrm{d}m_N - \tfrac{1}{4} - m_N(A_\nu) > \tfrac{1}{4}.$$

Assuming P a projection from $\ell^1(d) \oplus Y \oplus X$ onto R_ϵ, one may consider the operator $Q = \Lambda_\epsilon^{-1}$ from $\ell^1(d) \oplus Y \oplus X$ into X.

For each $i = 1, \ldots, d$ and $\nu \in G_N$, let φ_ν^i be ψ_ν seen as element of the i^{th} component $L^1(G_N)$ in the direct sum X. Thus $\alpha(\varphi_\nu^i) = 0$, $\beta(\varphi_\nu^i) = \psi_\nu(x_i)$ and $\gamma(\varphi_\nu^i) = \varphi_\nu^i - T_\epsilon(\varphi_\nu^i)$.

By well-known results concerning operators on L^1-spaces, we get

$$\begin{aligned}
&d \int \sum_\nu |\psi_\nu| \,\mathrm{d}m_N \\
&= \int \max_i \Big(\sum_\nu |Q\Lambda_\epsilon(\varphi_\nu^i)|\Big) \,\mathrm{d}m_N \oplus \cdots \mathrm{d}m_N \\
&\le \int \max_i |Q| \Big(\sum_\nu |\Lambda_\epsilon(\varphi_\nu^i)| \,\mathrm{d}m_N \oplus \cdots \oplus \mathrm{d}m_N \\
&\le \|Q\| \left\{ \int \max_i \Big(\sum_\nu |\psi_\nu(x_i)| \Big) \,\mathrm{d}m_N(x_1) \ldots \mathrm{d}m_N(x_d) \right. \\
&\qquad \left. + \sum_i \sum_\nu \int |\varphi_\nu^i - T_\epsilon(\varphi)| \,\mathrm{d}m_N \right\}.
\end{aligned}$$

Remark that, by symmetry, $\Sigma_\nu |\psi_\nu|$ is a constant function. Because $\frac{1}{4} < \|\psi_\nu\|_1 \le 2$ and

$$\|\psi_\nu - T_\epsilon(\psi_\nu)\|_1 = \|\xi_\nu - T_\epsilon(\xi_\nu)\|_1 = \frac{1}{d}\|e_\nu - T_{\epsilon^d}(e_\nu)\|_1 \le \frac{2}{d},$$

we find using Lemma 4

$$d \sum_\nu \|\psi_\nu\|_1 \le 24\|P\| \Big(\sum_\nu \|\psi_\nu\|_1 + 2^{N+1} \Big)$$

and hence

$$\|P\| \geq d \frac{\frac{1}{4}2^N}{24(2^{N+1}+2^{N+1})} = \frac{d}{384}$$

completing the proof.

From Corollary 5 and Lemma 6, it follows that

THEOREM 7: *There exists a constant $0 < C < \infty$ such that whenever $\tau > 0$ and D is a positive integer which is large enough, one can find a D-dimensional subspace E of L^1 satisfying $d(E, \ell^1(D)) \leq C$ and $\|P\| \geq C^{-1}(\log\log D)^{1-\tau}$ whenever P is a projection from L^1 onto E.*

This provides in particular a negative solution to Problem 1 and Problem 2 stated in the Introduction.

3. Remarks and Questions

1. Following L. Dor, one may define local and uniform moduli for functions and subspaces of an $L^1(\mu)$-space.

For a function f in $L^1(\mu)$ and $\rho > 0$, take

$$\alpha(f, \rho) = \inf\left\{\mu(A); \int_A |f|\, d\mu \geq \rho\|f\|_1\right\}.$$

If now E is a subspace of $L^1(\mu)$ and $\rho > 0$, let

$$\alpha(E, \rho) = \sup\{\alpha(f, \rho); f \in E\}$$

and

$$\beta(E, \rho) = \inf\left\{\mu(A); \int_A |f|\, d\mu \geq \rho\|f\|_1 \text{ for each } f \in E\right\}.$$

Call $\alpha(E, \rho)$ a local modulus and $\beta(E, \rho)$ a uniform modulus of the space E.

Based on the ideas presented in the preceding section, the following can be proved

LEMMA 8: *There exist a sequence (E_n) of finite dimensional subspaces of L^1 and constants $C < \infty$ and $c > c$, such that*

1. $d(E_n, \ell^1(\dim E_n)) \leq C$.

2. $\lim_{n\to\infty} \alpha(E_n, c) = 0$.
3. *For each* $\rho > 0$, $\inf_n \beta(E_n, \rho) > 0$.

As was pointed out by Dor [6], this leads to the existence of a non-complemented ℓ^1-subspace of L^1.

2. In fact, one may choose the spaces E_n of Lemma 8 in such a way that they are well-complemented and probabilistically independent. This allows us to construct a non-complemented ℓ^1-direct sum of uniformly complemented, independent, uniform ℓ^1-isomorphs. Thus the next result concerning independent functions can not be extended to independent ℓ^1-copies.

THEOREM 9: *If E is an ℓ^1-subspace of $L^1(\mu)$ spanned by independent variables, then E is complemented in $L^1(\mu)$ by a projection P whose norm $\|P\|$ can be bounded in function of $d(E, \ell^1(\dim E))$ (cfr. [5]).*

There is an easy reduction to the case where E is generated by a sequence (f_k) of normalized, independent and mean zero variables. Using then the uniqueness up to equivalence of unconditional bases in ℓ^1-spaces (see [14]), it turns out that this sequence (f_k) is a "good" ℓ^1-bases for E, or more precisely there is some constant $M < \infty$, M only depending on $d(E, \ell^1(\dim E))$, so that

$$M^{-1} \sum_k |a_k| \leq \left\| \sum_k a_k f_k \right\| \leq \sum_k |a_k|$$

whenever (a_k) is a finite sequence of scalars.

Assume $\mathscr{E}_k$ $(k = 1, 2, \ldots)$ independent σ-algebra's such that f_k is $\mathscr{E}_k$-measurable. The main ingredient of the next lemma is the result [4].

LEMMA 10: *There exists a sequence (A_k) of μ-measurable sets, satisfying*

1. $A_k \in \mathscr{E}_k$ *for each* k,
2. $\int_{A_k} f_k \, d\mu \geq \rho$ *for each* k,
3. $\Sigma_k\, \mu(A_k) \leq K$,

where $\rho > 0$ and $K < \infty$ only depend on M and hence only on $d(E, \ell^1(\dim E))$.

The proof of this lemma is contained in [5], Section 3. So we will not give it here. Let us now pass to the

Proof of Theorem 9: We may clearly make the additional assumption that $\mu(A_k) < \frac{1}{3}$.

For each k, let $\mathcal{F}_k = \mathcal{G}(\mathcal{E}_1, \ldots, \mathcal{E}_k)$ the σ-algebra generated by $\mathcal{E}_1, \ldots, \mathcal{E}_k$.

Take

$$B_1 = A_1 \quad \text{and} \qquad B_k = A_k \backslash \cup_{\ell<k} A_\ell \quad \text{for } k > 1.$$

Clearly $B_k \in \mathcal{F}_k$ for each k. Remark also that

$$\int_{B_k} f_k \, \mathrm{d}\mu = \int f_k \chi_{A_k} \prod_{\ell<k} (1 - \chi_{A_\ell}) = \prod_{\ell<k} (1 - \mu(A_\ell)) \int_{A_k} f_k$$

and hence

$$\int_{B_k} f_k \, \mathrm{d}\mu = \sigma_k \geq \exp(-3K)\rho.$$

Define

$$\Delta_1[f] = E[f \mid \mathcal{F}_1] \quad \text{and} \quad \Delta_k[f] = E[f \mid \mathcal{F}_k] - E[f \mid \mathcal{F}_{k-1}] \text{ for } k > 1.$$

Thus

$$\Delta_k[f_\ell] = \delta_{k,\ell} f_\ell.$$

Next, take $P : L^1(\mu) \to E$ given by $P(f) = \Sigma_k \sigma_k^{-1} < \Delta_k[f], B_k > f_k$. It is clear that P is a projection. We estimate its norm

$$\begin{aligned}
\|P\| &\leq \left\| \sum_k \sigma_k^{-1} |\Delta_k[\chi_{B_k}]| \right\|_\infty \\
&\leq \frac{\exp 3K}{\rho} \left\| \sum_k \chi_{B_k} + \sum_k \mu(A_k) \right\|_\infty \\
&\leq (1 + K) \frac{\exp 3K}{\rho}.
\end{aligned}$$

3. Our example leaves the following questions unanswered

Problem 3: What is the biggest λ such that problem 1 has a positive solution provided $\|T\| \|T^{-1}\| > \lambda$?

For E subspace of L^1, define

$$\pi(E) = \inf\{\|P\|;\ P : L^1 \to E \text{ is a projection}\}.$$

Take further for fixed $n = 1, 2, \ldots$ and $\lambda < \infty$

$$\gamma(n, \lambda) = \sup\{\pi(E);\ \dim E = n \text{ and } d(E, \ell^1(n)) \leq \lambda\}.$$

PROBLEM 4: Find estimations on the numbers $\gamma(n, \lambda)$. At this point, it does not seem even clear that for fixed $\lambda < \infty$ the following holds

$$\lim_{\nu \to \infty} \frac{\gamma(n, \lambda)}{\sqrt{n}} = 0.$$

Let us mention the following fact, which may be of some interest for further investigations

PROPOSITION 10: *Given $\lambda < \infty$, one can find constants $c > 0$ and $C < \infty$ such that if E is a finite dimensional subspace of L^1 satisfying* $d(E, \ell^1(\dim E)) \leq \lambda$, then E has a subspace F for which the following holds:

1. $d(F, \ell^1(\dim F)) \leq \lambda$
2. $\dim F \geq c \dim E$
3. *There exists a projection $P : L^1 \to F$ with $\|P\| \leq C$.*

4

PROBLEM 5: Let G be an uncountable compact abelian group and E a translation invariant subspace of $L^1(G)$, such that E is isomorphic to $L^1(G)$. Must E be complemented?

Related to this question is the following one, due to G. Pisier [19].

PROBLEM 6: Let G be the Cantor group and define E as the subspace of $L^1(G)$ generated by the Walsh-functions w_S where $|S| \geq 2$.

Obviously, E is uncomplemented. What about the following

a. Is E an $\mathscr{L}^1$-space?

b. Is E isomorphic to $L^1(G)$?

It can be shown that E satisfies the Dunford–Pettis property (see [13] for definition and related facts).

5

Easy modifications of the construction given in the second section also allow us to obtain badly complemented $\ell^p(n)$-subspaces of L^p for $1 < p < 2$.

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