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A COUNTEREXAMPLE TO A COMPLEMENTATION PROBLEM

J. Bourgain*

Abstract

The existence is shown of subspaces of $L^1$ which are isomorphic to an $L^1(\mu)$-space and are not complemented. A more precise local statement is also given.

1. Introduction

The question we are dealing with is the following:

**Problem 1:** Let $\mu$ and $\nu$ be measures and $T : L^1(\mu) \to L^1(\nu)$ an isomorphic embedding. Does there always exist a projection of $L^1(\nu)$ onto the range of $T$?

and was raised in [1], [4], [5] and [21].

This problem has the following finite dimensional reformulation (cfr. [4]).

**Problem 2:** Does there exist for each $\lambda < \infty$ some $C < \infty$ such that given a finite dimensional subspace $E$ of $L^1(\nu)$ satisfying $d(E, \ell^1(\dim E)) \leq \lambda$ ($d =$ Banach-Mazur distance), one can find a projection $P : L^1(\nu) \to E$ with $\|P\| \leq C$?

In [4], L. Dor obtained a positive solution to problem 1 provided $\|T\|\|T^{-1}\| < \sqrt{2}$. It was shown by L. Dor and T. Starbird (cfr. [5]) that

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any $l^1$-subspace of $L^1(\nu)$ which is generated by a sequence of probabilistically independent random variables is complemented. A slight improvement of this result will be given in the remarks below, where we show that problem 2 is affirmative under the additional hypothesis that $E$ is spanned by independent variables. Our main purpose is to show that the general solution to the above questions is negative. Examples of uncomplemented $l^p$-subspaces of $L^p$ ($1 < p < \infty$) were already discovered (see [24] for the cases $2 < p < \infty$ and $1 < p < 4/3$ and [1] for $1 < p < 2$).

2. The Example

We first introduce some notation. For each positive integer $N$, denote $G_N$ the group $\{1, -1\}^N$ equipped with its Haar measure $m_N$. For $1 \leq n \leq N$, the $n^{th}$ Rademacher function $r_n$ on $G_N$ is defined by $r_n(x) = x_n$ for all $x \in G_N$. To each subset $S$ of $\{1, 2, \ldots, N\}$ corresponds a Walsh function $w_S = \prod_{n \in S} r_n$ and $L^1(G_N)$ is generated by this system of Walsh functions.

For fixed $0 \leq \epsilon \leq 1$, let $\mu = \otimes_n \mu_n$ be the product measure on $G_N$, where $\mu_n(1) = \frac{1 + \epsilon}{2}$ and $\mu_n(-1) = \frac{1 - \epsilon}{2}$ for all $n = 1, \ldots, N$. This measure $\mu$ is called sometimes the $\epsilon$-biased coin-tossing measure (cfr. [30]).

Let now $T_\epsilon : L^1(G_N) \to L^1(G_N)$ be the convolution operator corresponding to $\mu$. Thus $(T_\epsilon f)(x) = (f * \mu)(x) = \int_{G_N} f(x, y) \mu(dy)$ for all $f \in L^1(G_N)$.

It is clear that $T_\epsilon$ is a positive operator of norm 1 and easily verified that $T_\epsilon w_S = e^{i|S|} w_S$, where $|S|$ denotes the cardinality of the set $S$. Another way of introducing $T_\epsilon$ is by using Riesz-products.

Before describing the example, we give some lemma's.

Lemmas 1: If $f \in L^1(G_N)$, then $\|T_\epsilon f\|_2 \leq |\int f \ d\mu_N| + \epsilon \|f\|_2$.

Proof: Take $f = a_\phi + \sum_{S \neq \phi} a_\phi w_S$ the Walsh expansion of $f$. Then

$$T_\epsilon f = a_\phi + \sum_{S \neq \phi} a_\phi e^{i|S|} w_S$$

and hence $\|T_\epsilon f\|_2^2 = |a_\phi|^2 + \sum_{S \neq \phi} |a_\phi|^2 e^{2|S|} \leq |a_\phi|^2 + \epsilon^2 \|f\|_2^2$.

The required inequality follows.
LEMMA 2: Let $f_1, \ldots, f_d$ be functions in $L^1(G_N)$ such that for each $i = 1, \ldots, d$

1. $\int f_i \, dm_N = 0$.
2. $\int_{A_i} |f_i| \, dm_N \geq \delta \|f_i\|$ where $A_i = \{|f_i| \geq d \|f_i\|\}$.

Then

$$\int_{G_N \times \cdots \times G_N} |f_1(x_1) + \cdots + f_d(x_d)| \, dm_N(x_1) \cdots dm_N(x_d) \geq \frac{\delta}{6} \sum_{i=1}^{d} \|f_i\|.$$ 

PROOF: For $i = 1, \ldots, d$, take $D_i = G_N \setminus A_i$ and let $C_i$ be the subset of $G_N \times \cdots \times G_N$ defined by $C_i = B_1 \times \cdots \times B_i \times A_i \times B_{i+1} \times \cdots \times B_d$. Remark that $m_N(A_i) \leq \frac{1}{d}$ and hence $m_N(B_i) \geq 1 - \frac{1}{d}$. Let $r_1, \ldots, r_d$ be Rademacher functions on $[0, 1]$. By unconditionality, we get

$$\int_{G_N \times \cdots \times G_N} \left| \sum_{i=1}^{d} f_i(x_i) \right| \, dm_N(x_1) \cdots dm_N(x_d)$$

$$\leq \frac{1}{2} \int_0^1 \int_{G_N \times \cdots \times G_N} \left| \sum_{i=1}^{d} r_i(t)f_i(x_i) \right| \, dm_N(x_1) \cdots dm_N(x_d) \, dt$$

$$\leq \frac{1}{2} \sum_{i=1}^{d} \int_{C_i} |f_i(x_i)| \, dm_N(x_1) \cdots dm_N(x_d)$$

$$\leq \frac{1}{2} \left(1 - \frac{1}{d}\right)^{d-1} \sum_{i=1}^{d} \int_{A_i} |f_i(x)| \, dm_N(x) \geq \frac{\delta}{6} \sum_{i=1}^{d} \|f_i\|,$$

as required.

For each $\nu \in G_N$, define the function $e_\nu = \prod_{n=1}^{N} (1 + \nu r_n)$ on $G_N$. Thus $(e_\nu)_{\nu \in G_N}$ generates $L^1(G_N)$ and is isometrically equivalent to the $\ell^1(2^N)$-basis.

LEMMA 3: For fixed $0 \leq \epsilon \leq 1$ and $\kappa > 0$, the following holds

$$m_N[T_\epsilon(e_\nu) > \kappa] < \kappa^{-1/2} \left(1 - \frac{\epsilon^2}{4}\right)^{N/2}.$$ 

PROOF: It is easily verified that $T_\epsilon(e_\nu) = \prod_{n=1}^{N} (1 + \epsilon r_n)$. If we let $\Gamma = \prod_{n=1}^{N} (1 + \epsilon r_n)$, then by independency

$$\int \sqrt{\Gamma} \, dm_N = 2^{-N} (\sqrt{1+\epsilon} + \sqrt{1-\epsilon})^N < \left(1 - \frac{\epsilon^2}{4}\right)^{N/2}$$

and thus

$$m_N[T_\epsilon(e_\nu) > \kappa] = m_N[\sqrt{\Gamma} > \sqrt{\kappa}] \leq \kappa^{-1/2} \left(1 - \frac{\epsilon^2}{4}\right)^{N/2}.$$
We use the symbol $\oplus$ to denote the direct sum in $\ell^1$-sense. For fixed $N$ and $d$, take

$$X = L^1(G_N) \oplus \cdots \oplus L^1(G_N) \quad \text{and} \quad Y = L^1(G_N \times \cdots \times G_N).$$

Consider the maps

$$\alpha : X \to \ell^1(d)$$
$$\beta : X \to Y$$

and for $0 \leq \epsilon \leq 1$

$$\gamma_\epsilon : X \to X$$

defined by

$$\alpha(f_1 \oplus \cdots \oplus f_d) = \left( \int f_1 \, dm_N, \ldots, \int f_d \, dm_N \right)$$
$$\beta(f_1 \oplus \cdots \oplus f_d) = \sum_{i=1}^d \left( f_i(x_i) - \int f_i \, dm_N \right)$$

where $(x_1, \ldots, x_d) \in G_N \times \cdots \times G_N$ is the product variable

$$\gamma_\epsilon(f_1 \oplus \cdots \oplus f_d) = (f_1 - T_1 f_1) \oplus \cdots \oplus (f_d - T_d f_d).$$

Obviously $\|\alpha\| \leq 1$, $\|\beta\| \leq 2$ and $\|\gamma_\epsilon\| \leq 2$.

Let $\Lambda_\epsilon : x \to \ell^1(d) \oplus Y \oplus X$ be the map $\alpha \oplus \beta \oplus \gamma_\epsilon$, clearly satisfying $\|\Lambda_\epsilon\| \leq 5$.

**Lemma 4:** Under the above notations, $\|\Lambda_\epsilon(\varphi)\| \geq \frac{1}{2\epsilon} \|\varphi\|_1$ for each $\varphi \in X$, whenever $0 < \epsilon \leq 1/4d$.

**Proof:** Assume $\varphi = f_1 \oplus \cdots \oplus f_d$ and take for each $i = 1, \ldots, d$

$$g_i = f_i - \int f_i \, dm_N$$

$A_i = \{|g_i| \geq d \|g_i\|_1\}$, $B_i = G_N \setminus A_i$, $g'_i = g_i \chi_{A_i}$ and $g''_i = g_i \chi_{B_i}$.

Let further $I = \{i = 1, \ldots, d; \|g_i\|_1 > \frac{1}{2} \|g_i\|_1\}$ and $J = \{1, \ldots, d\} \setminus I$.

Using Lemma 2, we find that
\begin{align*}
\|\beta(f_1 \oplus \cdots \oplus f_d)\| & \geq \int_{G_N \times \cdots \times G_N} \left| \sum_{i=1}^{k} g_i(x_i) \right| d\mu_N(x_1) \cdots d\mu_N(x_d) \\
n & \geq \frac{1}{24} \sum_{i=1}^{k} \|g_i\|.
\end{align*}

On the other hand, by Lemma 1
\begin{align*}
\|T_\epsilon g_i\| & \leq \|T_\epsilon g\| + \int g_i'' d\mu_N + \|g_i''\| \leq 2\|g_i\| + \epsilon d\|g_i\|
\end{align*}
and hence for \(i \in J\)
\begin{align*}
\|f_i - T_\epsilon f_i\| = \|g_i - T_\epsilon g_i\| & \geq \|g_i\| - \|T_\epsilon g_i\| \geq \frac{1}{4}\|g_i\|.
\end{align*}
Consequently
\begin{align*}
\|\gamma(f_1 \oplus \cdots \oplus f_d)\| & \geq \sum_{i \in J} \|f_i - T_\epsilon f_i\| \geq \frac{1}{4} \sum_{i \in J} \|g_i\|.
\end{align*}

Combination of these inequalities leads to
\begin{align*}
\|A_\epsilon(\varphi)\| & \geq \sum_{i=1}^{d} \int f_i d\mu_N + \frac{1}{24} \sum_{i=1}^{d} \|g_i\| \geq \frac{1}{24} \sum_{i=1}^{d} \|f_i\| = \frac{1}{24} \|\varphi\|
\end{align*}
proving the lemma.

**Corollary 5:** Again under the above notations, denote \(R_\epsilon\) the range of \(A_\epsilon\). Then \(d(R_\epsilon, \ell^1(d, 2^N)) \leq \frac{1}{20}\) provided \(0 < \epsilon \leq 1/4d\).

Our next aim is to show that \(R_\epsilon\) is a badly complemented subspace of \(\ell^1(d) \oplus Y \oplus X\) for a suitable choice of \(N\), \(d\) and \(\epsilon\).

**Lemma 6:** Fix any positive integer \(d \geq 4\), take \(N = d^{6d}\) and let \(\epsilon = 1/4d\). Then \(\|P\| \geq d/384\) for any projection \(P\) from \(\ell^1(d) \oplus Y \oplus X\) onto \(R\).

**Proof:** Define for each \(\nu \in G_N\)
\begin{align*}
\xi_\nu &= \frac{1}{d} \sum_{i=0}^{d-1} T_i(e_\nu) \quad \text{and} \quad A_\nu = [\xi_\nu > \frac{1}{2}].
\end{align*}
Since \(A_\nu \subset \bigcup_{i=0}^{d-1} [T_i(a_\nu) > \frac{1}{2}]\), application of Lemma 3 gives that
and hence, by the choice of \( N \) and \( \epsilon \)

\[ m_N(A_v) < \frac{1}{2}, \]

as an easy computation shows.

It follows that if \( \psi_v = \xi_v - 1 \), then

\[ \| \psi_v \|_1 \geq \int_{A_v} \xi_v \, dm_N - m_N(A_v) \geq \int \xi_v \, dm_N - \frac{1}{4} - m_N(A_v) > \frac{1}{4}. \]

Assuming \( P \) a projection from \( \ell^1(d) \oplus Y \oplus X \) onto \( R^* \), one may consider the operator \( Q = A^{-1} \) from \( \ell^1(d) \oplus Y \oplus X \) into \( X \).

For each \( i = 1, \ldots, d \) and \( v \in G_N \), let \( \varphi_i \) be \( \psi_v \) seen as element of the \( i \)-th component \( L^1(G_N) \) in the direct sum \( X \). Thus \( \alpha(\varphi_i) = 0 \), \( \beta(\varphi_i) = \psi_v(x_i) \) and \( \gamma(\varphi_i) = \varphi_i - T_\epsilon(\varphi_i) \).

By well-known results concerning operators on \( L^1 \)-spaces, we get

\[
d \int \sum_v |\psi_v| \, dm_N = \int \max_i \left( \sum_v |QA_i(\varphi_i)| \right) \, dm_N + \cdots + \sum_v \int |\varphi_v - T_\epsilon(\varphi)| \, dm_N \]

Remark that, by symmetry, \( \sum_v |\psi_v| \) is a constant function. Because \( \frac{1}{4} < \| \psi_v \|_1 \leq 2 \) and

\[
\| \psi_v - T_\epsilon(\psi_v) \|_1 = \| \xi_v - T_\epsilon(\xi_v) \|_1 = \frac{1}{d} \| e_v - T_\epsilon(e_v) \|_1 \leq \frac{2}{d},
\]

we find using Lemma 4

\[
d \sum_v \| \psi_v \|_1 \leq 2d \| P \| \left( \sum_v \| \psi_v \|_1 + 2^{N+1} \right)
\]
and hence
\[ \|P\| \geq d \frac{\frac{1}{2} 2^N}{24(2^{N+1} + 2^{N+1})} = \frac{d}{384} \]
completing the proof.

From Corollary 5 and Lemma 6, it follows that

**Theorem 7**: There exists a constant $0 < C < \infty$ such that whenever $\tau > 0$ and $D$ is a positive integer which is large enough, one can find a $D$-dimensional subspace $E$ of $L^1$ satisfying $d(E, \ell^1(D)) \leq C$ and $\|P\| \geq C^{-1}(\log \log D)^{1-\tau}$ whenever $P$ is a projection from $L^1$ onto $E$.

This provides in particular a negative solution to Problem 1 and Problem 2 stated in the Introduction.

3. Remarks and Questions

1. Following L. Dor, one may define local and uniform moduli for functions and subspaces of an $L^1(\mu)$-space. For a function $f$ in $L^1(\mu)$ and $\rho > 0$, take

   \[ \alpha(f, \rho) = \inf \left\{ \mu(A) ; \int_A |f| \, d\mu \geq \rho \|f\|_1 \right\}. \]

   If now $E$ is a subspace of $L^1(\mu)$ and $\rho > 0$, let

   \[ \alpha(E, \rho) = \sup \{ \alpha(f, \rho) ; f \in E \} \]

   and

   \[ \beta(E, \rho) = \inf \left\{ \mu(A) ; \int_A |f| \, d\mu \geq \rho \|f\|, \text{ for each } f \in E \right\}. \]

   Call $\alpha(E, \rho)$ a local modulus and $\beta(E, \rho)$ a uniform modulus of the space $E$.

   Based on the ideas presented in the preceding section, the following can be proved

   **Lemma 8**: There exist a sequence $(E_n)$ of finite dimensional subspaces of $L^1$ and constants $C < \infty$ and $c > c$, such that

   1. $d(E_n, \ell^1(\dim E_n)) \leq C$. 

2. \( \lim_{n \to \infty} \alpha(E_n, c) = 0. \)

3. For each \( \rho > 0, \inf_n \beta(E_n, \rho) > 0. \)

As was pointed out by Dor [6], this leads to the existence of a non-complemented \( \ell^1 \)-subspace of \( L^1 \).

2. In fact, one may choose the spaces \( E_n \) of Lemma 8 in such a way that they are well-complemented and probabilistically independent. This allows us to construct a non-complemented \( \ell^1 \)-direct sum of uniformly complemented, independent, uniform \( \ell^1 \)-isomorphs. Thus the next result concerning independent functions can not be extended to independent \( \ell^1 \)-copies.

**Theorem 9:** If \( E \) is an \( \ell^1 \)-subspace of \( L^1(\mu) \) spanned by independent variables, then \( E \) is complemented in \( L^1(\mu) \) by a projection \( P \) whose norm \( \|P\| \) can be bounded in function of \( d(E, \ell^1(\dim E)) \) (cfr. [5]).

There is an easy reduction to the case where \( E \) is generated by a sequence \( (f_k) \) of normalized, independent and mean zero variables. Using then the uniqueness up to equivalence of unconditional bases in \( \ell^1 \)-spaces (see [14]), it turns out that this sequence \( (f_k) \) is a “good” \( \ell^1 \)-bases for \( E \), or more precisely there is some constant \( M < \infty, M \) only depending on \( d(E, \ell^1(\dim E)) \), so that

\[
M^{-1} \sum_k |a_k| \leq \left\| \sum_k a_k f_k \right\| \leq \sum_k |a_k|
\]

whenever \( (a_k) \) is a finite sequence of scalars.

Assume \( \mathcal{E}_k \) \((k = 1, 2, \ldots)\) independent \( \sigma \)-algebra’s such that \( f_k \) is \( \mathcal{E}_k \)-measurable. The main ingredient of the next lemma is the result [4].

**Lemma 10:** There exists a sequence \( (A_k) \) of \( \mu \)-measurable sets, satisfying

1. \( A_k \in \mathcal{E}_k \) for each \( k \),
2. \( J_{A_k} f_k \, d\mu \geq \rho \) for each \( k \),
3. \( \Sigma_k \mu(A_k) \leq K \),

where \( \rho > 0 \) and \( K < \infty \) only depend on \( M \) and hence only on \( d(E, \ell^1(\dim E)) \).

The proof of this lemma is contained in [5], Section 3. So we will not give it here. Let us now pass to the
PROOF OF THEOREM 9: We may clearly make the additional assumption that \( \mu(A_k) < \frac{1}{3} \).

For each \( k \), let \( \mathcal{F}_k = \mathcal{G}(\mathcal{E}_1, \ldots, \mathcal{E}_k) \) the \( \sigma \)-algebra generated by \( \mathcal{E}_1, \ldots, \mathcal{E}_k \).

Take

\[
B_1 = A_1 \quad \text{and} \quad B_k = A_k \setminus \bigcup_{\ell < k} A_{\ell} \quad \text{for } k > 1.
\]

Clearly \( B_k \in \mathcal{F}_k \) for each \( k \). Remark also that

\[
\int_{B_k} f_k \, d\mu = \int f_k \chi_{A_k} \prod_{\ell < k} (1 - \chi_{A_\ell}) = \prod_{\ell < k} (1 - \mu(A_\ell)) \int_{A_k} f_k
\]

and hence

\[
\int_{B_k} f_k \, d\mu = \sigma_k \geq \exp(-3K)\rho.
\]

Define

\[
\Delta_k[f] = E[f \mid \mathcal{F}_1] \quad \text{and} \quad \Delta_k[f] = E[f \mid \mathcal{F}_k] - E[f \mid \mathcal{F}_{k-1}] \quad \text{for } k > 1.
\]

Thus

\[
\Delta_k[f_{\ell}] = \delta_{k,\ell} f_{\ell}.
\]

Next, take \( P : L^1(\mu) \to E \) given by \( P(f) = \sum_k \sigma_k^{-1} \Delta_k[f], \ B_k > f_k \). It is clear that \( P \) is a projection. We estimate its norm

\[
\|P\| \leq \left\| \sum_k \sigma_k^{-1} \Delta_k[\chi_{B_k}] \right\|_{\infty}
\]

\[
\leq \frac{\exp 3K}{\rho} \left\| \sum_k \chi_{B_k} + \sum_k \mu(A_k) \right\|_{\infty}
\]

\[
\leq (1 + K) \frac{\exp 3K}{\rho}.
\]

3. Our example leaves the following questions unanswered.

PROBLEM 3: What is the biggest \( \lambda \) such that problem 1 has a positive solution provided \( \|T\| \|T^{-1}\| > \lambda \)?

For \( E \) subspace of \( L^1 \), define
\( \pi(E) = \inf \{ \| P \| ; \ P : L^1 \to E \text{ is a projection} \} \).

Take further for fixed \( n = 1, 2, \ldots \) and \( \lambda < \infty \)

\[ \gamma(n, \lambda) = \sup \{ \pi(E); \ \dim E = n \ \text{and} \ d(E, \ell^1(n)) \leq \lambda \} \].

**PROBLEM 4:** Find estimations on the numbers \( \gamma(n, \lambda) \). At this point, it does not seem even clear that for fixed \( \lambda < \infty \) the following holds

\[ \lim_{n \to \infty} \frac{\gamma(n, \lambda)}{\sqrt{n}} = 0. \]

Let us mention the following fact, which may be of some interest for further investigations

**PROPOSITION 10:** Given \( \lambda < \infty \), one can find constants \( c > 0 \) and \( C < \infty \) such that if \( E \) is a finite dimensional subspace of \( L^1 \) satisfying \( d(E, \ell^1(\dim E)) \leq \lambda \), then \( E \) has a subspace \( F \) for which the following holds:

1. \( d(F, \ell^1(\dim F)) \leq \lambda \)
2. \( \dim F \geq c \dim E \)
3. There exists a projection \( P : L^1 \to F \) with \( \| P \| \leq C \).

**PROBLEM 5:** Let \( G \) be an uncountable compact abelian group and \( E \) a translation invariant subspace of \( L^1(G) \), such that \( E \) is isomorphic to \( L^1(G) \). Must \( E \) be complemented?

Related to this question is the following one, due to G. Pisier [19].

**PROBLEM 6:** Let \( G \) be the Cantor group and define \( E \) as the subspace of \( L^1(G) \) generated by the Walsh-functions \( w_S \) where \( |S| \geq 2 \).

Obviously, \( E \) is uncomplemented. What about the following

a. Is \( E \) an \( L^1 \)-space?

b. Is \( E \) isomorphic to \( L^1(G) \)?

It can be shown that \( E \) satisfies the Dunford-Pettis property (see [13] for definition and related facts).
Easy modifications of the construction given in the second section also allow us to obtain badly complemented \( \ell^p(n) \)-subspaces of \( L^p \) for \( 1 < p < 2 \).

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