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O. GABBER
A. JOSEPH

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ON THE BERNSTEIN-GELFAND-GELFAND RESOLUTION AND THE DUFLO SUM FORMULA

O. Gabber and A. Joseph

Abstract

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. In ([8], Prop. 12) Duflo gave a remarkable sum formula interrelating induced ideals. The main result of this paper provides a natural generalization of this formula and more precisely gives a resolution for certain primitive quotients of the enveloping algebra $U(\mathfrak{g})$. The proof has three distinct steps. One, the extension of the Bernstein-Gelfand-Gelfand (in short, B.G.G.) resolution of a simple finite dimensional $U(\mathfrak{g})$ module to certain simple highest weight modules. Two, the description of the so-called $\mathfrak{k}$-finite part of the space of homomorphisms of any one Verma module to any other. Three, the proof of exactness of a certain functor. The last can be viewed as a non-trivial generalization of the fact that a Verma module with dominant highest weight is projective in the so-called $\mathcal{O}$ category. A by-product gives some results on a problem of Kostant relating $U(\mathfrak{g})$ to the $\mathfrak{k}$-finite part of the space of endomorphisms of a simple highest weight module.

1. Preliminaries

1.1: Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra for $\mathfrak{g}$, R the set of non-zero roots, $R^+ \subset R$ a system of positive roots, $B \subset R^+$ the set of simple roots, ρ the half sum of the positive roots, $s_\alpha \in \operatorname{Aut}(\mathfrak{h}^*)$ the reflection corresponding to the root $\alpha \in R$, and W the group generated by the $s_\alpha : \alpha \in B$. Let X_α be the element of a

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Chevalley basis for $\mathfrak{g}$ corresponding to the root α and set

$$\mathfrak{n}^+ = \sum_{\alpha\in R^+} \mathbb{C}X_\alpha,\ \mathfrak{n}^- = \sum_{\alpha\in R^+} \mathbb{C}X_{-\alpha},\ \mathfrak{b} = \mathfrak{h}\oplus\mathfrak{n}^+.$$

1.2: For each $\lambda \in \mathfrak{h}^*$, set $R_\lambda = \{\alpha \in R : 2(\lambda,\alpha)/(\alpha,\alpha) \in \mathbb{Z}\}$ (which is itself a root system) and $R_\lambda^+ = R_\lambda \cap R^+$, with $B_\lambda \subset R_\lambda^+$ the corresponding set of simple roots. Call λ regular (resp. dominant) if $(\lambda,\alpha) \neq 0$ (resp. $(\lambda,\alpha) \geq 0$) for all $\alpha \in R^+$. For each $B' \subset B_\lambda$, let $W_{B'}$ be the subgroup of W generated by the $s_\alpha : \alpha \in B'$ and $w_{B'}$ the largest element of $W_{B'}$ with respect to its Bruhat order $\leq$ (as defined in [7], 7.7.3). If $B' = B_\lambda$ we write $W_{B'} = W_\lambda$, $w_{B'} = w_\lambda$. Let $M(\lambda)$ denote the Verma module with highest weight λ-ρ associated to the quadruplet $\mathfrak{g}$, $\mathfrak{h}$, B, λ (see [7], 7.1.4), $\overline{M(\lambda)}$ the unique maximal submodule of $M(\lambda)$, and set $L(\lambda) = M(\lambda)/\overline{M(\lambda)}$, $J(\lambda) = \operatorname{Ann} L(\lambda)$. For each $\mathfrak{h}$ module V we set $V_\lambda = \{v \in V : Hv = (\lambda, H)v$, for all $H \in \mathfrak{h}\}$. Let e_λ denote the canonical generator of $M(\lambda)$ (which has weight $\lambda - \rho$). Set $R_\lambda^0 = \{\alpha \in R : (\alpha,\lambda) = 0\}$.

1.3: Let $u \mapsto \check{u}$ (resp. $u \mapsto {}^t u$) denote the involutory antiautomorphism of $U(\mathfrak{g})$ defined by $\check{X} = -X : X \in \mathfrak{g}$ (resp. ${}^tX_\alpha = X_{-\alpha} : \alpha \in R$, ${}^tH = H : H \in \mathfrak{h}$). Identify $U := U(\mathfrak{g}) \otimes U(\mathfrak{g})$ canonically with $U(\mathfrak{g}\times\mathfrak{g})$. Define $j : \mathfrak{g} \to \mathfrak{g}\times\mathfrak{g}$ through $j(X) = (X, -{}^tX)$, set $\mathfrak{k} = j(\mathfrak{g})$, so $U(\mathfrak{k})$ may be regarded as a subalgebra of U. Let $\mathfrak{k}^\wedge$ denote the set of equivalence classes of finite dimensional irreducible representations of $\mathfrak{k}$. For each locally finite $\mathfrak{k}$ module L and each $\sigma \in \mathfrak{k}^\wedge$, we let L_σ denote the isotypical component of type σ of L. Let $\iota : U(\mathfrak{k}) \to U(\mathfrak{g})$ be the $\mathbb{C}$ algebra isomorphism sending $(-{}^tX, X)$ to X for every $X \in \mathfrak{g}$. If $R \xrightarrow[\varphi]{} S$ is a ring homomorphism and M is a left S module, we let M^φ denote the left R module which consists of the underlying abelian group $|M|$ of M together with the operation $(r, m) \mapsto \varphi(r)\cdot m$ of R on $|M|$.

1.4: Let $\mathcal{O}$ denote the category of finitely generated $U(\mathfrak{g})$ modules which are $\mathfrak{h}$ semisimple and $\mathfrak{b}$ locally finite (see [1-3, 6]). Each $M \in 0b\mathcal{O}$ has finite length [2]. This category has enough projectives and so the extension groups $\operatorname{Ext}^k(\cdot\,,\cdot)$ relative to $\mathcal{O}$ are thereby defined. Let $Z(\mathfrak{g})$ denote the centre of $U(\mathfrak{g})$. Then $\operatorname{Max} Z(\mathfrak{g})$ is isomorphic to $\mathfrak{h}^*/W$ such that for each $\lambda \in \mathfrak{h}^*$, $\hat{\lambda} := W\lambda$ corresponds to the element $Z(\mathfrak{g}) \cap J(\lambda)$ of $\operatorname{Max} Z(\mathfrak{g})$. Let $\mathcal{O}_{\hat{\lambda}}$ denote the subcategory of $\mathcal{O}$ of all modules annihilated by a power of this maximal ideal. Each $M \in 0b\mathcal{O}$ admits a primary decomposition and we denote by $p_{\hat{\lambda}} : 0b\,\mathcal{O} \to 0b\,\mathcal{O}_{\hat{\lambda}}$ the projection onto the primary component defined by $\hat{\lambda}$. It is an exact functor on $\mathcal{O}$.

1.5: Given $M, N \in Ob\,\mathcal{O}$, consider $\operatorname{Hom}_{\mathbb{C}}(M, N)$ (resp. $(M \otimes N)^*$) as a U module through $((a\otimes b)\cdot x)m = ({}^t\check{a}x\check{b})m$ (resp. $((a\otimes b)\cdot y,$

$m \otimes n) = (y, \check{a}m \otimes \check{b}n))$ where $a, b \in U(g)$, $m \in M$, $n \in N$, $x \in \mathrm{Hom}(M, N)$, $y \in (M \otimes N)^*$. We remark that $(M(-\lambda) \otimes M(-\mu))^*$ is isomorphic to the $\mathfrak{g} \times \mathfrak{g}$ module co-induced from the $\mathfrak{b} \times \mathfrak{b}$ module $\mathbb{C}_{\lambda+\rho,\mu+\rho}$. Let $L(M, N)$ (resp. $L(M \otimes N)^*$) denote the set of all $\mathfrak{k}$-finite elements of $\mathrm{Hom}(M, N)$ (resp. $(M \otimes N)^*$) which we remark is again a U module. For $\lambda, \mu \in \mathfrak{h}^*$, we set $L(\lambda, \mu) = L(M(-\lambda) \otimes M(-\mu))^*$.

1.6: Let E be a finite dimensional $U(\mathfrak{g})$ module and given $M \in 0b\,\mathcal{O}$, consider $E \otimes M$ as a $U(\mathfrak{g})$ module through the diagonal action. One has $E \otimes M \in 0b\,\mathcal{O}$ and the functor $M \mapsto E \otimes M$ is exact. Again one has the natural isomorphisms

$$\mathrm{Hom}_{\mathfrak{g}}(E, \mathrm{Hom}_{\mathbb{C}}(M, N)) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(E \otimes M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(M, E^* \otimes N).$$

The latter gives on taking projective resolutions natural isomorphisms

$$\mathrm{Ext}^k(E \otimes M, N) \xrightarrow{\sim} \mathrm{Ext}^k(M, E^* \otimes N) : k \in \mathbb{N}.$$

1.7: Let $\mathcal{H}$ denote the category of all U modules which satisfy the following properties. One, each $L \in 0b\,\mathcal{H}$ is locally finite as a $\mathfrak{k}$ module. Two, $\dim L_\sigma < \infty$ for each $\sigma \in \mathfrak{k}^\wedge$. Three, each $L \in 0b\,\mathcal{H}$ admits a finite filtration such that the centre of U acts by scalars on each subquotient. Clearly $\mathcal{H}$ is stable under tensoring with finite dimensional U modules. It follows from the classification ([21], I, Sect. 4) of the simple modules in $\mathcal{H}$ that each $L \in 0b\,\mathcal{H}$ has finite length (for example, as shown in ([2], 4.2)). For each $M, N \in 0b\,\mathcal{O}$, one has $L(M, N) \in \mathcal{H}$. Indeed, the first property holds by construction. The second obtains from the isomorphism valid for any simple finite dimensional $U(\mathfrak{g})$ module E, namely $\mathrm{Hom}_{\mathfrak{k}}(E', L(M, N)) \xleftarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(E \otimes M, N)$, the last space being finite dimensional (since $E \otimes M, N$ have finite length). The third obtains by taking composition series for M, N. We have shown that

LEMMA: *For each $M, N \in 0b\,\mathcal{O}$, the U module $L(M, N)$ has finite length.*

1.8: Observe that $\tau : a \mapsto {}^t\check{a}$ is an involutory automorphism of $U(\mathfrak{g})$. Given $M \in 0b\,\mathcal{O}$, we let $\delta(M)$ denote the submodule of $(M^*)^\tau$ of all $\mathfrak{h}$ finite elements. Through the existence of a non-degenerate contravariant form on $L(\lambda)$ (see [11], 1.6), one has $L(\lambda) \cong \delta(L(\lambda))$. In particular $E^* \cong E^\tau$ for any finite dimensional module E. Again each $M \in 0b\,\mathcal{O}$ has finite length, so $\delta(M) \in 0b\,\mathcal{O}$ and $\delta(M)$ has the same composition factors as M (with the same multiplicities).

1.9: For each $M, N \in \mathrm{Ob}\,\mathcal{O}$, define $\sigma : \mathrm{Hom}_{\mathbb{C}}(M, (N^*)^\tau) \to (N \otimes M)^*$ through $(\sigma(x), m \otimes n) = (xm, n)$. From $(\sigma((a \otimes b) \cdot x), m \otimes n) = (((a \otimes b) \cdot x)m, n) = ({}^t\check{a}x\check{b}m, n) = (x\check{b}m, \check{a}n) = (\sigma(x), \check{a}n \otimes \check{b}m) = ((a \otimes b) \cdot \sigma(x), n \otimes m)$, it follows that σ is a U module homomorphism. Again σ is obviously injective. Given $y \in (N \otimes M)^*$, then for each $m \in M$ the map $g(y, m) : n \mapsto (y, n \otimes m)$ of N to $\mathbb{C}$ is $\mathbb{C}$-linear. It follows that the map $\eta(y) : m \mapsto g(y, m)$ of M to $(N^*)^\tau$ is $\mathbb{C}$-linear and the map $\eta : y \mapsto \eta(y)$ is inverse to σ.

LEMMA: *The map σ restricts to a U module isomorphism of $L(M, \delta(N))$ onto $L(N \otimes M)^*$. In particular $L(N \otimes M)^*$ has finite length as a U module.*

If $x \in L(M, \delta(N))$, then $\sigma(x)$ is obviously $\mathfrak{k}$-finite. Conversely for each $y \in L(N \otimes M)^*$, $m \in M$, $X \in \mathfrak{g}$, we have $X(\eta(y)m) = \eta(j({}^t\check{X})y)m + \eta(y)Xm$, and so the local finiteness of $\mathfrak{h}$ on M implies that $\eta(y)m \in \delta(N)$. Hence the surjectivity of the restriction of σ. The last part follows from 1.7.

1.10: Define an ordering on $\mathbb{Z}B$ through $\mu \geq \nu$ if $\mu - \nu \in \mathbb{N}B$. Given $M \in \mathrm{Ob}\,\mathcal{O}$, set $\Omega(M) = \{\lambda \in \mathfrak{h}^* : M_\lambda \neq 0\}$. If $M \neq 0$, then $\Omega(M)$ admits at least one maximal element. Note that $H_0(\mathfrak{n}^-, M) = M/\mathfrak{n}^- M$ is a locally finite semisimple $\mathfrak{h}$ module.

LEMMA: *Suppose $M, N \in \mathrm{Ob}\,\mathcal{O}$ with N a submodule of M. If $H_0(\mathfrak{n}^-, M)$, $H_0(\mathfrak{n}^-, N)$ are isomorphic as $\mathfrak{h}$ modules, then $M = N$.*

Assume $Q := M/N \neq 0$. Let $\mu \in \Omega(Q)$ be maximal. Through the maximality of μ one has $(\mathfrak{n}^- M)_\mu = \Sigma X_{-\alpha} M_{\mu+\alpha} = \Sigma X_{-\alpha} N_{\mu+\alpha} = (\mathfrak{n}^- N)_\mu$. Yet $\dim N_\mu/(\mathfrak{n}^- N)_\mu = \dim M_\mu/(\mathfrak{n}^- M)_\mu$, by hypothesis. This gives $M_\mu = N_\mu$, which is a contradiction.

1.11: For each $M \in \mathrm{Ob}\,\mathcal{O}$, let $[M]$ denote the corresponding element in the Grothendieck group $\mathcal{G}$ of $\mathcal{O}$. For each $\hat{\lambda} \in \mathfrak{h}^*/W$, let $\mathcal{G}_{\hat{\lambda}}$ denote the subgroup of $\mathcal{G}$ corresponding to $\mathcal{O}_{\hat{\lambda}}$. It is well-known that $\{[L(\mu)] : \mu \in \hat{\lambda}\}$ is a basis for $\mathcal{G}_{\hat{\lambda}}$. Again each $M(\lambda) : \lambda \in \mathfrak{h}^*$ has finite length with simple factors amongst the $L(\mu) : \mu \in \hat{\lambda}$ and we denote by $[M(\lambda) : L(\mu)]$ the number of times $L(\mu)$ occurs in $M(\lambda)$. The resulting matrix is invertible (by [7], 7.6.23) and (by [7], 7.6.14) one has

$$[E \otimes M(\lambda)] = \sum_{\nu \in \Omega(E)} [M(\lambda + \nu)] \dim E_\nu$$

for any finite dimensional $U(\mathfrak{g})$ module E.

1.12: Let $P(R)$ denote the lattice of integral weights. Let $P(R)^+$ (resp. $P(R)^{++}$) denote the dominant (resp. dominant and regular) elements of $P(R)$. For each $\nu \in P(R)$, let $E(\nu)$ denote a (unique up to isomorphism) simple finite dimensional $U(\mathfrak{g})$ module with extreme weight ν. The map $\nu \mapsto E(\nu)^\iota$ identifies the $\mathfrak{k}^\wedge$ of classes of finite dimensional simple $U(\mathfrak{k})$ modules with $P(R)/W$ and hence with $P(R)^+$. Frobenius reciprocity gives $\dim \mathrm{Hom}_{\mathfrak{k}}(E(\nu)^\iota, L(\lambda, \mu)) = \dim E(\nu)_{\mu-\lambda}$ for all $\lambda, \mu \in \mathfrak{h}^*$, $\nu \in P(R)$. In particular, $L(\lambda, \mu) \neq 0$ if and only if $\lambda - \mu \in P(R)$. Now assume $\lambda - \mu \in P(R)$. Then by 1.9, $L(\lambda, \mu)$ has finite length. Since $\dim E(\lambda - \mu)_{\lambda-\mu} = 1$, it follows that $L(\lambda, \mu)$ admits a unique simple subquotient, which we denote by $V(\lambda, \mu)$, satisfying $\dim \mathrm{Hom}_{\mathfrak{k}}(E(\lambda - \mu), V(\lambda, \mu)) = 1$. We shall need the following

THEOREM:

(i) *Every simple module in $\mathscr{H}$ is isomorphic to some $V(\lambda, \mu)$.*

(ii) *$V(\lambda, \mu)$ is isomorphic to $V(\lambda', \mu')$ if and only if there exists $w \in W$ such that $\lambda' = w\lambda$, $\mu' = w\mu$.*

(iii) *Suppose $\lambda \in \mathfrak{h}^*$ is dominant. Then if $L(M(\lambda), L(\mu)) \neq 0$ (which holds in particular if λ is regular), it is isomorphic to $V(-\mu, -\lambda)$. Furthermore every simple $V \in 0b\mathscr{H}$ is so obtained.*

(i), (ii) are just ([9], I, 4.1, 4.5) and (iii) follows from ([14], 4.7) and (i), (ii).

1.13: Given $-\lambda \in \mathfrak{h}^*$ dominant, then for each $-\mu \in -\lambda + P(R)$ dominant we define following Jantzen ([11], Sect. 2) a translation operator T_λ^μ: $\mathcal{O} \to \mathcal{O}$ through $T_\lambda^\mu M = p_{\hat\mu}(E(\mu - \lambda) \otimes p_{\hat\lambda}(M))$. If $R_\lambda^0 \subset R_\mu^0$, then for all $w \in W_\lambda$ we have $T_\lambda^\mu M(w\lambda) \cong M(w\mu)$ (see [10], 2.10). Let E be a finite dimensional $U(\mathfrak{g})$ module. Through the natural U module isomorphisms $L(M, N) \otimes (\mathbb{C} \otimes E) \xrightarrow{\sim} L(M \otimes E^*, N)$, $L(M, N) \otimes (E \otimes \mathbb{C}) \xrightarrow{\sim} L(M, N \otimes E^\tau)$, it is obvious how to define exact functors on $\mathscr{H}$ satisfying $R_\lambda^\mu L(M, N) \cong L(T_\lambda^\mu M, N)$, $S_\lambda^\theta L(M, N) \cong L(M, T_\lambda^\mu N)$ for M, $N \in 0b\mathcal{O}$. Again by 1.6, T_λ^μ is both left and right adjoint to T_μ^λ.

1.14: For each $j \in \mathbb{N}$, $\mu \in \mathfrak{h}^*$, $N \in 0b\mathcal{O}$, one has $\mathrm{Ext}^j(M(\mu), N) \cong H^j(\mathfrak{n}^+, N)_{\mu-\rho} \cong (H_j(\mathfrak{n}^-, \delta(N))_{\mu-\rho})^*$, the first isomorphism being due to Delorme ([6], Thm. 2), the second a formal consequence of the appropriate standard complexes.

1.15: Take $\lambda, \mu \in \mathfrak{h}^*$ and let us note the almost obvious fact that $L(M(\lambda), M(\mu)) = 0$ unless $\lambda - \mu \in P(R)$. This latter condition further implies that $W_\lambda = W_\mu$.

LEMMA: *Fix* $-\lambda$, $-\mu \in \mathfrak{h}^*$ *dominant with* $\lambda - \mu \in P(R)$. *Then for each* $w \in W_\lambda$ *and each finite dimensional* $U(\mathfrak{g})$ *module E one has*

$$\dim \operatorname{Hom}_{\mathfrak{k}}(E^{\iota}, L(M(w_\lambda\lambda), M(w\mu))) = \dim \operatorname{Hom}_{\mathfrak{k}}(E^{\iota}, L(M(w^{-1}w_\lambda\lambda), M(\mu))).$$

We show that both sides equal $\dim E_{w\mu - w_\lambda\lambda}$. For the right hand side this follows from the fact that $M(\mu)$ is simple (and so isomorphic to $\delta M(\mu)$), 1.9 and 1.12, noting that $\Omega(E)$ is W stable. The left hand side equals (by 1.7) $\dim \operatorname{Hom}_{\mathfrak{g}}(M(w_\lambda\lambda), E^* \otimes M(w\mu))$; since $M(w_\lambda\lambda)$ is projective in $\mathcal{O}$, we have by 1.11 that the latter equals $\dim(E^*)_{w_\lambda\lambda - w\mu} = \dim E_{w\mu - w_\lambda\lambda}$.

Remarks: Although this also follows from ([14], 4.10) the above proof is much simpler. It is not difficult to extend the above to a further proof of ([14], 4.3) and hence of Duflo's theorem ([8], Thm. 1); but then this becomes essentially the proof given in ([3], 4.4).

1.16: Take $\lambda \in \mathfrak{h}^*$ dominant. $Z(\mathfrak{g})$ acts on $M(\lambda)$ by a homomorphism $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$. Let $C = \lambda + P(R)$, and let $\mathcal{O}_C$ be the full subcategory of $\mathcal{O}$ consisting of those modules M that satisfy $\Omega(M) \subset C$. Define a functor $T : \mathcal{O}_C \to \mathcal{H}$ by $T(N) = L(M(\lambda), N)$ (cf. 1.7). T is exact since any $M(\lambda) \otimes E$ (E being a finite dimensional $U(\mathfrak{g})$ module) is projective in $\mathcal{O}$. Let $\mathcal{H}$ consisting of those $M \in 0b(\mathcal{H})$ on which $1 \otimes Z(\mathfrak{g})$ acts through $1 \otimes z \mapsto \chi_\lambda(\check{z})$. The image of T lies in $\mathcal{H}_{\hat{\lambda}}$, and in the following theorem we view $\mathcal{H}_{\hat{\lambda}}$ as the target category of T.

THEOREM:

(i) *T has a left adjoint* T'.

(ii) *The unit map* $\eta : Id_{\mathcal{H}_{\hat{\lambda}}} \to TT'$ *is an isomorphism of functors.*

(iii) *If* λ *is regular, then T is an equivalence of categories.*

We indicate a proof for the theorem, which has also been proved by Bernstein and Gelfand ([3], 6.3, 6.1 (ii), 5.9 (i)).

(i). If $M \in 0b(\mathcal{H}_{\hat{\lambda}})$, we make M into a two-sided $U(\mathfrak{g})$ module by $amb = ({}^t\check{a} \otimes \check{b}) \cdot m$ for all $m \in M$, $a, b \in U(g)$. Define $T'(M) = M \otimes_A M(\lambda)$, where $A = U(\mathfrak{g})/U(\mathfrak{g})\ker(\chi_\lambda)$. Now $T'(M) \in 0b(\mathcal{O}_C)$ because if $E \subset M$ is a finite dimensional $\mathfrak{k}$ stable generating subspace (so $M = EU(\mathfrak{g})$), then we get a surjective $\mathfrak{g}$ linear map $E^{\iota^{-1}} \otimes M(\lambda) \to T'(M)$. If $M \in 0b(\mathcal{H}_{\hat{\lambda}})$ and $N \in 0b(\mathcal{O}_C)$, one defines an isomorphism $\zeta(M, N) : (\operatorname{Hom}_{\mathfrak{g}}(M \otimes_A M(\lambda), N) \to \operatorname{Hom}_U(M, L(M(\lambda), N))$ by $\zeta(\varphi) = (m \mapsto \varphi(m \otimes u)))$. This makes T' a left adjoint to T.

(ii). We have to show that for any $M \in 0b(\mathcal{H}_{\hat{\lambda}})$ the map $\eta(M) : M \to$

$L(M(\lambda), M \otimes_A M(\lambda))$ (given by $m \mapsto (n \mapsto m \otimes n)$)) is bijective. We make A into a U module by $(a \otimes b) \cdot x = {}^t\check{a}x\check{b}$, for all $x \in A$, a, $b \in U(g)$. Then $\eta(A)$ is an isomorphism by ([13], 6.4).

If E is a finite dimensional $U(\mathfrak{g})$ module we have natural isomorphisms

$$T(E \otimes N) \tilde{\leftarrow} (E^\tau \otimes \mathbb{C}) \otimes T(N), N \in 0b(\mathcal{O}_C)$$

$$T'((E^\tau \otimes \mathbb{C}) \otimes M) \tilde{\leftarrow} E \otimes T'(M), M \in 0b(\mathcal{H}_{\hat{\lambda}}).$$

Using these isomorphisms, one shows that if $\eta(M)$ is an isomorphism then so is $\eta((E^\tau \otimes \mathbb{C}) \otimes M)$. In particular, $\eta((E^\tau \otimes \mathbb{C}) \otimes A)$ is an isomorphism. This implies that $\eta(M)$ is an isomorphism for any $M \in 0b(\mathcal{H}_{\hat{\lambda}})$, by observing that TT' is right exact and that for suitable finite dimensional $U(\mathfrak{g})$ modules E_1, E_2 there exists an exact sequence $(E_1 \otimes \mathbb{C}) \otimes A \to (E_2 \otimes \mathbb{C}) \otimes A \to M \to 0$ in $\mathcal{H}_{\hat{\lambda}}$.

(iii). We have to show that the counit map $\epsilon : T'T \to Id_{\mathcal{O}_C}$ is also an isomorphism of functors. The composition $T \xrightarrow{\eta T} TT'T \xrightarrow{T\epsilon} T$ is Id_T, so by (ii) $T\epsilon$ is an isomorphism. Thus, as T is exact, $0 = T(\ker(\epsilon(N))) = T(\operatorname{coker}(\epsilon(N)))$ for any $N \in 0b(\mathcal{O}_C)$. So it remains to show that if $N \in 0b(\mathcal{O}_C)$ and $TN = 0$ then $N = 0$. Indeed, if $N \neq 0$, then N contains a simple submodule $L(\mu) : \mu \in C$, so $TN \supset TL(\mu)$; but by ([14], 4.7) $TL(\mu) \neq 0$, and we get a contradiction.

2. The generalized B.G.G. resolution

Throughout this section we fix $-\lambda \in \mathfrak{h}^*$ dominant and regular.

2.1: Given $\alpha \in B_\lambda$, one can choose $\nu_\alpha \in P(R)$ such that $-\lambda_\alpha := -\lambda + \nu_\alpha$ is dominant and $(\beta, \lambda_\alpha) = 0 : \beta \in R^+$ is equivalent to $\beta = \alpha$. Following Vogan ([22]) we set $\theta_\alpha = T^\lambda_{\lambda_\alpha} \circ T^{\lambda_\alpha}_\lambda : \mathcal{O} \to \mathcal{O}$. Using 1.13, θ_α is left adjoint to θ_α. So we obtain natural isomorphisms

$$\operatorname{Ext}^j(\theta_\alpha M, N) \tilde{\rightarrow} \operatorname{Ext}^j(M, \theta_\alpha N) : j \in \mathbb{N}, M, N \in 0b\,\mathcal{O}.$$

2.2: For each $w \in W_\lambda$, let $l(w)$ denote the reduced length of w with respect to B_λ. For each $w, w' \in W_\lambda$, we define an expression $P_{w,w'}$ in the indeterminate q through

$$P_{w,w'}(q) = \sum_{k=0}^{\infty} q^{(l(w')-l(w)-k)/2} \dim \operatorname{Ext}^k(M(w\lambda), L(w'\lambda)).$$

A result of Casselman and Schmid (proved also in [6], Thm. 4) implies that $P_{w,w'}(q)$ is polynomial in $q^{1/2}$. Kazhdan and Lusztig ([19], Conj. 1.5) have further conjectured that $P_{w,w'}(q)$ is polynomial in q and that this polynomial is determined by a particular purely combinatorial procedure which uses only the description of W_λ as a Coxeter group. This has been shown to follow from certain other conjectures ([10], [23]); but for the moment remains an open problem. Here we just establish one of the identities which would follow from the Kazhdan-Lusztig conjecture.

LEMMA: *For each $w, w' \in W_\lambda$, $\alpha \in B_\lambda$, such that $w's_\alpha < w'$, one has that $P_{ws_\alpha,w'}(q) = P_{w,w'}(q)$. In particular, for each $B' \subset B_\lambda$, $w \in W_{B'}$, one has that $P_{w,w_{B'}}(q) = 1$.*

We can assume $ws_\alpha < w$, without loss of generality. Then the conclusion of the lemma is equivalent to the identity

$$(*) \qquad \dim \operatorname{Ext}^{j+1}(M(ws_\alpha\lambda), L(w'\lambda)) = \dim \operatorname{Ext}^j(M(w\lambda)), L(w'\lambda)), j \in \mathbb{N}.$$

Under the hypothesis $w's_\alpha < w'$, it follows that $M(w's_\alpha\lambda)$ is a submodule of $M(w'\lambda)$ and by ([10], 2.10a) that $\theta_\alpha M(w'\lambda) \twoheadrightarrow \theta_\alpha M(w's_\alpha\lambda)$. Hence $\theta_\alpha(M(w'\lambda)/M(w's_\alpha\lambda)) = 0$. Since $L(w'\lambda)$ is a quotient of $M(w'\lambda)/M(w's_\alpha\lambda)$, it follows that $\theta_\alpha L(w'\lambda) = 0$, and so $\operatorname{Ext}^j(\theta_\alpha M(w\lambda), L(w'\lambda)) = 0$ by 2.1. In particular $L(w\lambda)$ is not a quotient of $\theta_\alpha M(w\lambda)$. Then from ([11], 2.17) we obtain an exact sequence

$$0 \to M(w\lambda) \to \theta_\alpha M(w\lambda) \to M(ws_\alpha\lambda) \to 0$$

from which the corresponding long exact sequence for $\operatorname{Ext}^*(\cdot, L(w'\lambda))$ gives $(*)$.

2.3: From 1.8, 1.14 and 2.2 we obtain

COROLLARY: *For each $B' \subset B_\lambda$, $w \in W_\lambda$, one has*

$$\dim H_j(\mathfrak{n}^-, L(w_{B'}\lambda))_{w\lambda-\rho} = \begin{cases} 1: w \in W_{B'}, j = l(w_{B'}) - l(w), \\ 0: \textit{otherwise.} \end{cases}$$

Remarks. As is well-known the remaining weight spaces of $H_j(\mathfrak{n}^-, L(w_{B'}\lambda))$ are null. This follows from the action of $Z(\mathfrak{g})$ and the fact that $w\lambda - \lambda \in \mathbb{Z}B$ implies $w \in W_\lambda$. This result then generalizes the Bott-Kostant formula established for finite dimensional simple modules (i.e. when $-\lambda \in P(R)^{++}$ and $B' = B$).

2.4: Fix $B' \subset B_\lambda$ and set $s = l(w_{B'})$. Then for each $j \in \mathbb{N}$, set $W^j_{B'} =$

$\{w \in W_{B'} : l(w) = j\}$, and

$$C_j = \bigoplus_{w \in W^j_{B'}} M(w\lambda).$$

As $M(w\lambda)$ is $U(\mathfrak{n}^-)$ free, we have for each $y \in W_{B'}$ that

$$\dim H_t(\mathfrak{n}^-, C_j)_{y\lambda-\rho} = \begin{cases} 1 : t = 0, j = l(y), \\ 0 : \text{otherwise.} \end{cases}$$

2.5: For each $w \in W_{B'}$, fix a $U(\mathfrak{g})$ module embedding $i_w : M(w\lambda) \hookrightarrow M(w_{B'}\lambda)$. For $w, w' \in W_{B'}$ such that $w \leq w'$, let $i_{w,w'} : M(w\lambda) \to M(w'\lambda)$ be the embedding such that $i_{w'} \circ i_{w,w'} = i_w$.

Fix $j \in \{1, 2, \ldots, s\}$ and consider a $U(\mathfrak{g})$ module map $\partial_j : C_{j-1} \to C_j$ defined by $(x_w)_{w \in W^{j-1}_B} \overset{\partial_j}{\mapsto} (y_{w'})_{w' \in W^j_{B'}}$ when

$$y_{w'} = \sum_{w \leq w'} c^j_{w,w'} i_{w,w'}(x_w), \quad x_w \in M(w\lambda),$$

where $c^j_{w,w'} \in \mathbb{Z}$ is non-zero and defined whenever $w \leq w'$, $w \in W^{j-1}_{B'}$, $w' \in W^j_{B'}$.

LEMMA: *The natural surjection $H_0(\mathfrak{n}^-, C_{j-1}) \to H_0(\mathfrak{n}^-, \operatorname{Im} \partial_j)$ is bijective.*

Set $K = \ker \partial_j$, $V = W^{j-1}_{B'}$. We have an exact sequence $0 \to K/K \cap \mathfrak{n}^- C_{j-1} \to C_{j-1}/\mathfrak{n}^- C_{j-1} \to \partial_j C_{j-1}/\mathfrak{n}^-(\partial_j C_{j-1}) \to 0$, so the lemma is equivalent to $K \subset \bigoplus_{w \in V} \mathfrak{n}^- M(w\lambda)$, or to $K \subset \bigoplus_{w \in V} \overline{M(w\lambda)}$, that is to

$$\overline{K} := \operatorname{Im}(K \to \bigoplus_{w \in V} L(w\lambda)) = 0.$$

If $\overline{K} \neq 0$, there exists $w \in V$ such that $[\overline{K} : L(w\lambda)] > 0$, and so $[K : L(w\lambda)] > 0$. Yet equality of lengths in V implies through ([7], 7.6.23) that $[C_j : L(w\lambda)] = 1$, so $0 = [C_j/K : L(w\lambda)] = [\partial_j C_{j-1} : L(w\lambda)]$. On the other hand since there exists $w' \in W^j_{B'}$ such that $w \leq w'$ and by hypothesis we then have $c^j_{w,w'} \neq 0$, it follows that ∂_j is injective on the summand $M(w\lambda)$ of C_{j-1}. Thus $\partial_j C_{j-1}$ contains a copy of $M(w\lambda)$, which implies $[\partial_j C_{j-1} : L(w\lambda)] \geqslant 1$. This contradiction proves the lemma.

2.6: An appropriate combinatorial property of the Bruhat ordering enables one to choose the $c^j_{w,w'}$ of 2.5 such that $\partial_j \partial_{j-1} = 0$, for all $j = 2, \ldots, s$. (See [2], Sect. 11 or [7], 7.8.14). Furthermore

PROPOSITION: *The sequence*

$$0 \to C_0 \to C_1 \to \cdots C_{j-1} \xrightarrow{\partial_j} C_j \to \cdots \to C_s \to L(w_{B'}\lambda) \to 0$$

is exact.

Set $X_{s+1} = Y_{s+1} = L(w_{B'}\lambda)$, $Y_s = \ker(C_s \to X_{s+1})$, and for each $j \in \{1, 2, \ldots, s\}$, set $X_j = \operatorname{Im} \partial_j$, $Y_{j-1} = \ker \partial_j$. For each $j \in \{1, 2, \ldots, s+1\}$, X_j is a submodule of Y_j and we show that $X_j = Y_j$. Fix $r \geq 1$ and assume that this has been established for all $j > r$. This means that we have the short exact sequences

$$0 \to X_j \to C_j \to X_{j+1} \to 0 : r < j \leqslant s.$$

By 2.4, the associated long exact sequence for homology implies for all $\mu \in \mathfrak{h}^*$ and $r < j \leq s$ that

$$\dim H_t(\mathfrak{n}^-, X_j)_\mu = \begin{cases} \dim H_{t+1}(\mathfrak{n}^-, X_{j+1})_\mu & : t > 0 \\[2ex] \dim H_1(\mathfrak{n}^-, X_{j+1})_\mu - \dim H_0(\mathfrak{n}^-, X_{j+1})_\mu \\ \quad + \dim H_0(\mathfrak{n}^-, C_j)_\mu & : t = 0. \end{cases}$$

Then from 2.3 and 2.4 we obtain

$$\dim H_t(\mathfrak{n}^-, X_j)_\mu = \begin{cases} 1 : \mu = w\lambda,\ w \in W_{B'},\ l(w) = j - t - 1, \\ 0 : \text{otherwise.} \end{cases}$$

for all $j > r$ and in particular for $j = r + 1$.

Finally from the long exact sequence associated to $0 \to Y_r \to C_r \to X_{r+1} \to 0$, 2.4 and the above we eventually obtain

$$\dim H_0(\mathfrak{n}^-, Y_r)_\mu = \dim H_0(\mathfrak{n}^-, C_{r-1})_\mu$$

for all $\mu \in \mathfrak{h}^*$. Then by 2.5, $H_0(\mathfrak{n}^-, Y_r)$ and $H_0(\mathfrak{n}^-, X_r)$ are isomorphic as $\mathfrak{h}$-modules and so $X_r = Y_r$ by 1.10. Noting that ∂_1 is injective completes the proof of the proposition.

Remark. This generalizes the B.G.G. resolution originally established [2] for the case $-\lambda \in P(R)^{++}$, $B' = B$. The original proof is different to ours and can only be generalized to the case when $B' \subset B$ (see [20] for this). The present proof was found following conversations with M. Duflo and P. Delorme.

3. Mappings of Verma modules

3.1: Take $-\lambda \in \mathfrak{h}^*$ dominant. Then $M(\lambda)$ is a simple module and so isomorphic to $\delta(M(\lambda))$. Then by 1.9 one has for all $\mu \in \mathfrak{h}^*$ that

$$L(M(\mu), M(\lambda)) = L(M(\mu), \delta M(\lambda)) = L(-\lambda, -\mu),$$

up to isomorphisms. This relationship of mappings of Verma modules to the principal series has been known for some time. Here we consider the most general form this takes when $-\lambda$ is not necessarily dominant. Some results in this direction were already obtained in ([5], 5.5) and in ([14], 4.10).

3.2: Fix $-\lambda$, $-\mu \in \mathfrak{h}^*$ dominant with $\lambda - \mu \in P(R)$ (recall 1.15). Choose w_1, $w_2 \in W_\lambda$ and $\alpha \in B_\lambda$ such that $s_\alpha w_1 > w_1$, $s_\alpha w_2 < w_2$. The second relation implies that $M(s_\alpha w_2\lambda)$ is a submodule of $M(w_2\lambda)$.

LEMMA: *Under the above hypotheses, one has* $L(M(w_1\mu), M(w_2\lambda)/M(s_\alpha w_2\lambda)) = 0$.

Equivalently for any finite dimensional $U(\mathfrak{g})$ module E one has $\mathrm{Hom}_\mathfrak{g}(M(w_1\mu), (E \otimes (M(w_2\lambda)/M(s_\alpha w_2\lambda)))) = 0$. To establish this it is enough to show that $L(w_1\mu)$ is not a subquotient of $p_{\hat\mu}(E \otimes M(w_2\lambda)/M(s_\alpha w_2\lambda))$. Now by 1.11 and the invariance of $\Omega(E)$ under W one has

$$[E \otimes (M(w_2\lambda)/M(s_\alpha w_2\lambda))] = \sum_{\nu \in \Omega(E)} ([M(w_2(\lambda+\nu))] - [M(s_\alpha w_2(\lambda+\nu))])(\dim E_\nu),$$

and so

$$(*) \qquad [p_{\hat\mu}(E \otimes M(w_2\lambda)/M(s_\alpha w_2\lambda))] = \sum_{\substack{w \in W_\lambda: \\ w\mu \in \lambda + \Omega(E)}} (\dim E_{w\mu-\lambda})([M(w_2w\mu)] - [M(s_\alpha w_2 w\mu)]).$$

Through the hypothesis $s_\alpha w_1 > w_1$, one has by ([10], 5.19) that

$$(**) \qquad [M(w_2w\mu):L(w_1\mu)] = [M(s_\alpha w_2w\mu):L(w_1\mu)].$$

Combined with (*) this establishes the assertion of the lemma.

Remarks. A technically easier proof of (**) follows from ([11], 2.16) and ([3], 4.5 (6)). Again the analysis of ([14], 5.4) can be combined with the operators of coherent continuation to give an alternative proof of the fact that $L(w_1\mu)$ is not a subquotient of $E \otimes (M(w_2\lambda)/M(s_\alpha w_2\lambda))$.

3.3: Let W be a Coxeter group with S the corresponding set of simple reflections and length function $l(\cdot)$. It is well-known that there exists an associative product $*$ on W uniquely defined through

$$w_* w' = ww' \quad \text{if} \quad l(w) + l(w') = l(ww'),$$

$$s_* s = s \quad \text{if} \quad s \in S.$$

(Up to a sign, these are the defining relations for the generators of the "singular Hecke algebra" obtained say from ([19], Sect. 1) by putting $q = 0$.)

LEMMA: *For all $w, y \in W$, $s \in S$, one has*

(i) $$s_* w = \begin{cases} sw : sw > w, \\ w \ \ : sw < w. \end{cases}$$

(ii) $$w_* s = \begin{cases} ws : ws > w, \\ w \ \ : ws < w. \end{cases}$$

(iii) $$(w_* y)^{-1} = y^{-1} * w^{-1}.$$

(iv) $$w_* w' \geq w', w.$$

The top lines of (i), (ii) are immediate from the definition of $*$. For the bottom line in say (ii), set $w' = sw$. Then $sw' > w'$ and so $s_* w = s_*(s_* w') = (s_* s)_* w' = s_* w' = w$.

We prove (iii) by induction on $l(w)$. For $l(w) = 0, 1$, it follows from (i), (ii). Otherwise write $w = s_* z : l(z) < l(w)$. Then $(w_* y)^{-1} = ((s_* z)_* y)^{-1} = (s_*(z_* y))^{-1} = ((z_* y)^{-1}{}_* s) = (y^{-1}{}_* z^{-1})_* s = y^{-1}{}_*(z^{-1}{}_* s) = y^{-1}{}_* w^{-1}$. (iv) follows from (i), (ii).

3.4: Fix $-\lambda \in \mathfrak{h}^*$ dominant. For all $w_1, w_2 \in W_\lambda$, one has from 3.3 (iv) that $w_2^{-1}{}_* w_1 w_\lambda \geq w_1 w_\lambda$ and so $w_3 := (w_2^{-1}{}_* w_1 w_\lambda) w_\lambda \leq w_1$.

PROPOSITION: *Assume $-\lambda, -\mu \in \mathfrak{h}^*$ dominant with $\lambda - \mu \in P(R)$. Given $w_1, w_2 \in W_\lambda$, define $w_3 \in W_\lambda$ as above. Then the U-module* *homomorphism of $L(M(w_1\lambda), M(w_2\mu))$ into $L(M(w_3\lambda), M(w_2\mu))$ defined by restriction is injective with image $L(M(w_3\lambda), M(\mu))$.*

The assertion is clear for $w_2 = 1$. If $w_2 \neq 1$, choose $\alpha \in B_\lambda$ such that $s_\alpha w_2 < w_2$. If $s_\alpha w_1 > w_1$, then by 3.2 the natural embedding $L(M(w_1\lambda), M(s_\alpha w_2\mu)) \hookrightarrow L(M(w_1\lambda), M(w_2\mu))$ is surjective. If $s_\alpha w_1 < w_1$, then by ([13], 6.1) the map of $L(M(w_1\lambda), M(w_2\mu))$ into $L(M(s_\alpha w_1\lambda), M(w_2\mu))$ defined by restriction is injective and so, by 3.2 again, we obtain an embedding of $L(M(w_1\lambda), M(w_2\mu))$ into $L(M(s_\alpha w_1\lambda), M(s_\alpha w_2\mu))$. In either case we obtain an embedding of $L(M(w_1\lambda), M(w_2\mu))$ into $L(M((s_\alpha {}_* w_1 w_\lambda)w_\lambda\lambda), M(s_\alpha w_2\mu))$, and so by induction an embedding into $L(M(w_3\lambda), M(\mu))$. On the other hand we can take $\alpha \in B_\lambda$ such that $s_\alpha w_1 < w_1$. Then a similar argument gives an embedding of $L(M(s_\alpha w_1\lambda), M((s_\alpha {}_* w_2)\mu))$ into $L(M(w_1\lambda), M(w_2\mu))$. By induction this gives an embedding of $L(M(w_\lambda\lambda), M((w_\lambda w_1^{-1} {}_* w_2)\mu))$ into $L(M(w_1\lambda), M(w_2\mu))$ which we saw above further embeds in $L(M(w_3\lambda), M(\mu))$, both maps having been defined by restriction. Now by 3.3, we have $(w_2^{-1} * w_1 w_\lambda)^{-1} = w_\lambda w_1^{-1} {}_* w_2$ and so by 1.15 the combined map is surjective. Consequently the second map must also be surjective, proving the assertion.

3.5: Assume $-\lambda$, $-\mu \in \mathfrak{h}^*$ dominant with $\lambda - \mu \in P(R)$ and fix $B' \subset B_\lambda$.

Corollary: *For each $w \in W_{B'}$ and each finite dimensional $U(\mathfrak{g})$ module E, one has*

$$\dim \operatorname{Hom}_{\mathfrak{g}}(Q, M(w\mu)) = \dim \operatorname{Hom}_{\mathfrak{g}}(Q, \delta M(w\mu)),$$

where $Q = E \otimes M(w_{B'}\lambda)$.

From $l(ww_\lambda) = l(w_\lambda) - l(w)$ for all $w \in W_\lambda$ and an analogous assertion for $W_{B'}$, we obtain $l(w^{-1}w_{B'}w_\lambda) = l(w^{-1}) + l(w_{B'}w_\lambda)$. Since $w_\lambda^2 = 1$, it follows from the definition of $*$ that $(w^{-1} {}_* w_{B'}w_\lambda)w_\lambda = w^{-1}w_{B'}$, so by 3.4, 3.1 one has the isomorphisms $L(M(w_{B'}\lambda), M(w\mu)) \rightrightarrows L(M(w^{-1}w_{B'}\lambda), M(\mu)) \rightrightarrows L(-\mu, -w^{-1}w_{B'}\lambda)$. On the other hand, by 1.9 we have $L(M(w_{B'}\lambda), \delta(M(w\mu))) \cong L(-w\mu, -w_{B'}\lambda)$. Combined with 1.7 and 1.12, these isomorphisms imply the assertion of the corollary.

3.6: Take λ, μ, w_1, w_2, α as in 3.2.

Lemma:

(i) $L(M(w_2\lambda)/M(s_\alpha w_2\lambda), \delta M(w_1\mu)) = 0.$

(ii) $L(L(w_2\lambda), L(w_1\mu)) = 0$.

(iii) *The map of* $L(-w_1\mu, -w_2\lambda)$ *into* $L(-w_1\mu, -s_\alpha w_2\lambda)$ *defined by restriction is injective.*

For (i), observe that $L(w_1\mu)$ is the unique simple submodule of $\delta M(w_1\mu)$, so it suffices to show for any finite dimensional module E that

$$(*) \qquad [E \otimes (M(w_2\lambda)/M(s_\alpha w_2\lambda)) : L(w_1\mu)] = 0.$$

This obtains by an argument parallel to 3.2. Hence (i). Through the embedding $\mathrm{Hom}_{\mathfrak{g}}(E \otimes L(w_2\lambda), L(w_1\mu)) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(E \otimes (M(w_2\lambda)/M(s_\alpha w_2\lambda)), L(w_1\mu))$ and $(*)$ we obtain (ii). Recalling 1.9, (i) gives (iii).

Remark. When $\alpha \in B$, the result in (iii) is due to Zelobenko (see [8], Lemmes 4, 5).

3.7: We conclude this section with a result of obvious importance which by virtue of ([4], 2.14) is a far reaching generalization of 3.6 (ii). We start with the following

LEMMA: *For all* $\lambda, \mu, \nu \in \mathfrak{h}^*$ *one has*

(i) $L(L(\mu), L(\lambda)) \neq 0 \Leftrightarrow L(L(\lambda), L(\mu)) \neq 0$.

(ii) $L(L(\mu), L(\lambda))\, L(L(\nu), L(\mu)) = 0$ *implies that one of these modules must vanish.*

(i) follows from the isomorphism $\delta(L\mu) \xleftarrow{\sim} L(\mu)$. (ii) follows from the simplicity of $L(\mu)$.

3.8: PROPOSITION: *Let* $\lambda \in \mathfrak{h}^*$ *be dominant and regular. Then for each* $w, y \in W_\lambda$, *one has*

$$L(L(w\lambda), L(y\lambda)) \neq 0 \Leftrightarrow J(w^{-1}\lambda) = J(y^{-1}\lambda).$$

Suppose $L(L(w\lambda), L(y\lambda)) \neq 0$. Then there exists a finite dimensional $U(\mathfrak{g})$ module E such that $\mathrm{Hom}_{\mathfrak{g}}(L(w\lambda), L(y\lambda) \otimes E) \neq 0$ and so $L(w\lambda)$ is a submodule of $L(y\lambda) \otimes E$. It follows that $L(M(\lambda), L(w\lambda))$ is a submodule of $L(M(\lambda), L(y\lambda) \otimes E)$. Hence the right annihilator of $L(M(\lambda), L(w\lambda))$ contains the right annihilator J of $L := L(M(\lambda), L(y\lambda) \otimes E)$. Since L is isomorphic to $L(M(\lambda), L(y\lambda)) \otimes (E^\tau \otimes \mathbb{C})$, it follows that J coincides with the right annihilator of $L(M(\lambda), L(y\lambda))$. By ([14], 4.7, 4.12) this gives $J(w^{-1}\lambda) \supset J(y^{-1}\lambda)$. By 3.7 (i), interchange of w, y gives the reverse inclusion.

Suppose $J(w^{-1}\lambda) = J(y^{-1}\lambda)$. By ([8], Prop. 8) $U(\mathfrak{g})/J(w^{-1}\lambda)$ has a unique U submodule which is furthermore isomorphic to some $V(-\sigma\lambda, -\lambda)$ with σ an involution of W_λ. By ([14], 4.12) it is clear that $J(\sigma\lambda) = J(w^{-1}\lambda)$. After Vogan ([24], 3.5) there exists a finite dimensional $U(\mathfrak{g})$ module E such that $U(\mathfrak{g})/J(w^{-1}\lambda)$ (and hence $V(-\sigma\lambda, -\lambda)$) is a submodule of $V(-w^{-1}\lambda, -\lambda) \otimes (\mathbb{C} \otimes E)$. From 1.12(i), we have $V(-w^{-1}\lambda, -\lambda) \cong V(-\lambda, -w\lambda)$, and so $V(-\sigma\lambda, -\lambda)$ is a submodule of $V(-w\lambda, -\lambda) \otimes (E \otimes \mathbb{C})$. Then by 1.12 (iii), $L(M(\lambda), L(\sigma\lambda))$ is a submodule of $L(M(\lambda), L(w\lambda)) \otimes (E \otimes \mathbb{C}))$ which is isomorphic to $L(M(\lambda), L(w\lambda) \otimes E^\tau)$. The resulting injection $i: L(M(\lambda), L(\sigma\lambda)) \to L(M(\lambda), L(w\lambda) \otimes E^\tau)$ must come by 1.16(iii) by applying T to an injection $L(\sigma\lambda) \to L(w\lambda) \otimes E^\tau$. Hence $L(L(\sigma\lambda), L(w\lambda)) \neq 0$. Interchanging w, y and using 3.7 gives $L(L(w\lambda), L(y\lambda)) \neq 0$, as required.

4. Exactness of the functor $L(M(w_{B'}\lambda), \cdot)$.

In this section we fix $-\lambda \in \mathfrak{h}^*$ dominant and $B' \subset B_\lambda$. Set $\Lambda = \{\mu \in \lambda + P(R): -\mu \text{ is dominant}\}$.

4.1: Let $\mathcal{O}_\Lambda^{B'}$ denote the subcategory of $\mathcal{O}$ consisting of all those modules (necessarily of finite length) whose simple factors are amongst the $L(w\mu): \mu \in \Lambda,\ w \in W_{B'}$. By ([6], Thm. 4(iv)) it follows that the $M(w_{B'}\mu): \mu \in \Lambda$ are projective in $\mathcal{O}_\Lambda^{B'}$. On the other hand $\mathcal{O}_\Lambda^{B'}$ is not closed under tensoring with finite dimensional $U(\mathfrak{g})$ modules. Nevertheless we have the

PROPOSITION: *Suppose $M_1, M_2, M_3 \in \mathrm{Ob}\, \mathcal{O}_\Lambda^{B'}$, with*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

exact. Then

$$0 \to L(M(w_{B'}\lambda), M_1) \to L(M(w_{B'}\lambda), M_2) \to L(M(w_{B'}\lambda), M_3) \to 0$$

is exact.

This is proved in sections 4.2, 4.3.

4.2: A module $M \in \mathrm{Ob}\,\mathcal{O}$ is said to admit a p-filtration if it has a finite filtration with factors isomorphic to Verma modules. For example, by ([7], 7.6.14) $E \otimes M(\mu)$ (E finite dimensional, $\mu \in \mathfrak{h}^*$) has a p-filtration.

LEMMA: *Suppose $Q \in \mathrm{Ob}\,\mathcal{O}$ admits a p-filtration. Then for all $\mu \in \mathfrak{h}^*$, $k > 0$, one has*

$$\mathrm{Ext}^k(Q, \delta M(\mu)) = 0.$$

It is enough to prove the assertion for Q a Verma module, say $M(\nu): \nu \in \mathfrak{h}^*$. By 1.14, $\mathrm{Ext}^k(M(\nu), \delta M(\mu)) = (H_k(\mathfrak{n}^-, M(\mu))_{\nu-\rho})^*$, up to isomorphism, so the assertion follows from the fact that $M(\mu)$ is $U(\mathfrak{n}^-)$ free.

4.3: Let E be a finite dimensional $U(\mathfrak{g})$ module and set $Q = E \otimes M(w_{B'}\lambda)$ and fix $\mu \in \Lambda$. We show that $\mathrm{Ext}^1(Q, L(y\mu)) = 0: y \in W_{B'}$ by induction on $l(y)$. This will establish 4.1. When $l(y) = 0$, that is $y = 1$, we have $L(\mu) \cong M(\mu) \cong \delta M(\mu)$ and so the assertion follows from 4.2. Now fix $w \in W_{B'}$ and suppose the assertion proved for all $y \in W_{B'}$ such that $l(y) < l(w)$. In particular this gives

$$\mathrm{Ext}^1(Q, \overline{M(w\mu)}) = 0. \tag{1}$$

From the exact sequence

$$0 \to L(w\mu) \to \delta M(w\mu) \to \delta\overline{M(w\mu)} \to 0$$

and 4.2 we obtain an exact sequence

$$0 \to \mathrm{Hom}(Q, L(w\mu)) \to \mathrm{Hom}(Q, \delta M(w\mu)) \to \mathrm{Hom}(Q, \delta\overline{M(w\mu)}) \to \tag{2}$$

$$\to \mathrm{Ext}^1(Q, L(w\mu)) \to 0.$$

From the exact sequence

$$0 \to \overline{M(w\mu)} \to M(w\mu) \to L(w\mu) \to 0,$$

and (1) we obtain an exact sequence

$$0 \to \mathrm{Hom}(Q, \overline{M(w\mu)}) \to \mathrm{Hom}(Q, M(w\mu)) \to \mathrm{Hom}(Q, L(w\mu)) \to 0. \tag{3}$$

Combining (2) and (3) gives

$$\begin{aligned}\dim \operatorname{Ext}^1(Q, L(w\mu)) = \{&\dim \operatorname{Hom}(Q, \overline{\delta M(w\mu)}) - \\ &- \dim \operatorname{Hom}(Q, \overline{M(w\mu)})\} \\ &- \{\dim \operatorname{Hom}(Q, \delta M(w\mu)) \\ &- \dim \operatorname{Hom}(Q, M(w\mu))\}.\end{aligned}$$

The first term in curly brackets vanishes by the induction hypothesis and the fact that $\overline{\delta M(w\mu)}$ and $\overline{M(w\mu)}$ have the same composition factors which are amongst the $L(y\mu): y < w$. The second term vanishes by 3.5.

4.4: Let M be a simple $U(\mathfrak{g})$ module. The natural action of $U(\mathfrak{g})$ in M defines an embedding of $U(\mathfrak{g})/\operatorname{Ann} M$ into $\operatorname{Hom}(M, M)$ and in fact the image lies in the $\mathfrak{k}$-finite part $L(M, M)$. Kostant has asked if the image is exactly $L(M, M)$. This is generally false ([5], 6.5; [13], 9.3, 9.4); yet it is quite important to ascertain when it does hold, especially for highest weight modules.

THEOREM: *For each* $-\lambda \in \mathfrak{h}^*$ *dominant and* $B' \subset B$, *one has*

$$U(g)/J(w_{B'}\lambda) = L(L(w_{B'}\lambda), L(w_{B'}\lambda)).$$

By 4.1, $L(M(w_{B'}\lambda), L(w_{B'}\lambda))$ is a quotient of $L(M(w_{B'}\lambda), M(w_{B'}\lambda))$ and the latter by ([14], 3.6) identifies with $U(\mathfrak{g})/\operatorname{Ann} M(w_{B'}\lambda)$. Since $L(L(w_{B'}\lambda), L(w_{B'}\lambda))$ is a submodule of $L(M(w_{B'}\lambda), L(w_{B'}\lambda))$, this proves the theorem.

Remark. In the special case when $B' \subset B$ the above result is due to Conze-Berline and Duflo ([5], 2.12, 6.3). Their proof does not admit further generalization since it uses induction from the parabolic subalgebra defined by B'. When $B' = B_\lambda$ with λ regular, the result is noted in ([12], 5.7).

4.5: For $\mu \in \mathfrak{h}^*$, we write $A_\mu := U(\mathfrak{g})/J(\mu)$, $A'_\mu := L(L(\mu), L(\mu))$. The embedding of A_μ into A'_μ extends ([13], 4.3) to an embedding of Fract A_μ into Fract A'_μ. In order to compute the scale factors in the Goldie polynomial defined by the Goldie rank of A_μ (see [15], 5.12) it is useful to know when Fract A_μ = Fract A'_μ.

Since $J(\mu)$ is a prime ideal, A_μ admits a unique simple submodule V_μ which furthermore ([8], Prop. 4) has annihilator $J_\mu := \check{J}(\mu) \otimes U(g) + U(g) \otimes \check{J}(\mu)$. We let $l_0(A'_\mu)$ denote the number of factors in a U composition series of A'_μ having annihilator J_μ.

LEMMA: $l_0(A'_\mu) = 1 \Leftrightarrow \text{Fract } A_\mu = \text{Fract } A'_\mu$.

If M is a finitely generated left $U(\mathfrak{g})$ module, let $\operatorname{Dim} M$ denote its Gelfand-Kirillov dimension over $U(\mathfrak{g})$ as defined in ([17], 2.1). Now let M be a simple U subquotient of A'_μ, which by k-finiteness is a finitely generated left $U(\mathfrak{g})$ module. By ([17], 1.4, 3.1 and 3.3 Remark) we have $\operatorname{Dim} M = \operatorname{Dim}(U(\mathfrak{g})/\operatorname{Ann}_{U(\mathfrak{g})} M)$. Since $\operatorname{Ann}_{U(\mathfrak{g})} M \supset \check{J}(\mu)$, it follows from the primeness of $\check{J}(\mu)$ that $\operatorname{Ann}_{U(\mathfrak{g})} M = \check{J}(\mu)$ if and only if $\operatorname{Dim} M = \operatorname{Dim} V_\mu$. A similar argument on the right, taking account of ([8], Prop. 4), shows that $\operatorname{Ann} M = J_\mu$ if and only if $\operatorname{Dim} M = \dim V_\mu$. Let S denote the set of regular elements of A_μ. Since A'_μ is $\mathfrak{k}$-finite and has finite length as a U module, it follows from ([18], 3.7) that S is an Ore subset of the regular elements of A'_μ and $S^{-1}A'_\mu = \operatorname{Fract} A'_\mu$. Hence it remains to show that $S^{-1}M = 0$ if and only if $\operatorname{Dim} M < \operatorname{Dim} V_\mu = \operatorname{Dim} U(\mathfrak{g})/J(\mu)$. This follows from ([16], 5.1, 5.2(i)).

4.6: Retain the above notation and take $\nu \in \mu + P(R)$ in the upper closure of the W_μ facette containing μ (for this see [11], 2.6).

LEMMA: *Set* $H^\nu_\mu = R^\nu_\mu S^\nu_\mu$ *(notation* 1.13). *Then*

(i) $H^\nu_\mu A'_\mu = A'_\mu$.

(ii) $H^\nu_\mu A_\nu = A_\nu$.

(iii) $l_0(A'_\mu) = l_0(A'_\nu)$.

(iv) $H^\nu_\mu V_\mu = V_\nu$.

(v) $\operatorname{Fract} A_\mu = \operatorname{Fract} A'_\mu \Leftrightarrow \operatorname{Fract} A_\nu = \operatorname{Fract} A'_\nu$.

By ([11], 2.10, 2.11) we have under the hypothesis of the lemma the isomorphisms $T^\nu_\mu L(\mu) \cong L(\nu)$ (resp. $T^\nu_\mu M(\mu) \cong M(\nu)$) and so by 1.13 the isomorphisms $H^\nu_\mu A'_\mu = A'_\nu$ (resp. $H^\nu_\mu L(M(\mu), M(\mu)) = L(M(\nu), M(\nu))$). Hence (i). Since $L(M(\mu), M(\mu)) \leftarrow U(\mathfrak{g})/\operatorname{Ann} M(\mu)$ by ([13], 6.4) and A_μ is the image of $U(\mathfrak{g})/\operatorname{Ann} M(\mu)$ in A'_μ, exactness of H^ν_μ gives (ii). Now let K be a simple U subquotient of A'_μ. Then by 1.12, K is isomorphic to some $L(M(\lambda_1), L(\lambda_2)) : \lambda_1, \lambda_2 \in \mathfrak{h}^*$ with λ_1 dominant. Furthermore from the action of the centre of U it easily follows that $\lambda_1, \lambda_2 \in W\mu$. Then from ([11], 2.10, 2.11) and 1.12, 1.13, it follows that either $H^\nu_\mu K = 0$ or is a simple subquotient of A_ν; then, by an argument similar to that given in ([4], 2.11), $H^\nu_\mu K$ has the same Gelfand-Kirillov dimension as K. Moreover by a trivial extension of ([4], 2.4), whether or not $H^\nu_\mu K = 0$ depends only on $\operatorname{Ann} K$. Hence (iii), (iv). Finally (v) follows from (iii).

4.7: COROLLARY: *Fix* $-\lambda \in \mathfrak{h}^*$ *dominant, regular and take* $B' \subset B_\lambda$. *Then for each* $\alpha \in B'$, *one has*

$$\operatorname{Fract} U(g)/J(w_{B'} s_\alpha \lambda) = \operatorname{Fract} L(L(w_{B'} s_\alpha \lambda), L(w_{B'} s_\alpha \lambda)).$$

With respect to λ, α define ν_α as in 2.1. Then apply 4.6(v) to 4.4 with $\mu = w_{B'}s_\alpha\lambda$, $\nu = w_{B'}s_\alpha(\lambda - \nu_\alpha) = w_{B'}(\lambda - \nu_\alpha)$.

4.8: For each $w \in W_\lambda$, set $S(w) = \{\alpha \in R_\lambda^+ : w\alpha \in R_\lambda^-\}$. Define an ordering $\subseteq$ on W_λ through $y \subseteq w$ given $S(y^{-1}) \subseteq S(w^{-1})$. One checks that $y \subseteq w$ implies $y \leq w$ and that $y \subseteq w \Leftrightarrow (y_*^{-1} w w_\lambda) w_\lambda = y^{-1}w$. Thus the obvious generalization of 4.1 shows that $L(M(w\lambda), \cdot)$ is exact when restricted to the subcategory of $\mathcal{O}_\lambda^\wedge$ of all modules with simple factors $L(y\lambda) : y \in W_\lambda$ where y satisfies $y' \leq y \Rightarrow y' \subseteq w$. Since $s_\alpha \leq y$, $\forall \alpha \in$ supp y it follows that supp $y \subset S(w^{-1})$, that is $y \in W_{B'}$ where $B' = B_\lambda \cap S(w^{-1})$. Though this rather weak generalization is probably not the best the corresponding assertion with $\subseteq$ replaced by $\leq$ is false for it implies that Kostant's problem has always a positive answer (which is false by ([5], 6.5)) for simple highest weight modules. This is in spite of the fact that $\operatorname{Ext}^1(M(w\lambda), L(y\lambda)) = 0$ if $w \geq y$.

5. Main theorem

5.1: Fix $-\lambda$, $-\mu \in \mathfrak{h}^*$ dominant, μ regular, with $\lambda - \mu \in P(R)$. Take $B' \subset B_\lambda$. Let $s = l(w_{B'})$, and for each $j \in \{0, 1, 2, \ldots, s\}$ set $D_j = \bigoplus_{w \in W_{B'}^j} L(M(w_{B'}\lambda), M(w\mu))$. Finally put

$$L = L(M(w_{B'}\lambda), L(w_{B'}\mu)).$$

THEOREM: *There is a long exact sequence*

$$0 \to D_0 \to D_1 \to \cdots \to D_s \to L \to 0.$$

Apply 4.1 to 2.6.

5.2: When $\lambda = \mu$ in 5.1, we have that $L = U(\mathfrak{g})/J(w_{B'}\lambda)$ by 4.4. Again by 3.4 one has that

$$D_s = L(M(w_{B'}\lambda), M(w_{B'}\lambda)) = U(\mathfrak{g})/J(\lambda)$$

$$D_{s-1} = \bigoplus_{\alpha \in B'} L(M(w_{B'}\lambda), M(w_{B'}s_\alpha\lambda)) = \bigoplus_{\alpha \in B'} L(M(s_\alpha\lambda), M(\lambda))$$

$$= \bigoplus_{\alpha \in B'} J(s_\alpha\lambda).$$

In view of the definition of the maps in 5.1 this gives the

COROLLARY: *For each $B' \subset B$, one has*

$$\sum_{\alpha \in B'} J(s_\alpha \lambda) = J(w_{B'}\lambda).$$

Remark. When $B' \subset B$, this result is due to Duflo ([8], Prop. 12). When $B' = B_\lambda$, it is just ([12], 4.4, 4.5). By ([12], 4.5) it implies that $J(w_{B'}\lambda)/J(\lambda)$ is an idempotent ideal and has exactly card B' distinct maximal submodules.

5.3: Again take $\lambda = \mu$ in 5.1. Then by 3.1, 3.4

$$D_{\mathfrak{j}} = \bigoplus_{w \in W_{B'}^{\mathfrak{j}}} L(M(w_{B'}\lambda), M(w\lambda)) = \bigoplus_{w \in W_{B'}^{\mathfrak{j}}} L(-\lambda, -w^{-1} w_{B'}\lambda).$$

Combined with 1.12 this gives the following multiplicity formula for simple $\mathfrak{k}$ submodules of $U(\mathfrak{g})/J(w_{B'}\lambda)$.

COROLLARY: *Fix $-\lambda \in \mathfrak{h}^*$ dominant and regular. Then for each $\nu \in P(R)$ one has*

$$\dim \mathrm{Hom}_{\mathfrak{k}}(E(\nu), U(\mathfrak{g})/J(w_{B'}\lambda)) = \sum_{w \in W_{B'}} (\det w) \dim E(\nu)_{\lambda - w\lambda}.$$

Remarks. When $B' \subset B$, Conze-Berline and Duflo ([5], 2.12, 6.3) gave a formula for the left hand side above. Their formula obtains from 4.4 and Frobenius reciprocity with respect to induction from the parabolic subalgebra defined by B'. The equivalence of these two formulae imply a combinatorial statement concerning weight subspaces of finite dimensional $U(\mathfrak{g})$ modules.

6. Duality

6.1: Some of our results can be given a dual form with the help of the following. Fix $\lambda, \mu \in \mathfrak{h}^*$ with $\lambda - \mu \in P(R)$. Then (see 6.3) $L(\lambda, \mu) \times L(-\lambda, -\mu)$ admits a bilinear form $\langle , \rangle$ satisfying $\langle (a \otimes b)x, y \rangle = \langle x, (\check{a} \otimes \check{b})y \rangle$, for all $x \in L(\lambda, \mu)$, $y \in L(-\lambda, -\mu)$, $a, b \in U(\mathfrak{g})$. For each $\sigma, \tau \in k^\wedge$, $\langle , \rangle$ restricts to a $\mathfrak{k}$-invariant bilinear form on $L(\lambda, \mu)_\sigma \times L(-\lambda, -\mu)_\tau$ which is non-degenerate if τ is contragradient to σ and zero otherwise.

6.2: To apply 6.1 to the comparison of mappings of principal series modules we start with the following observation. Suppose $\lambda, \lambda' \in \mathfrak{h}^*$ are chosen so that we have an embedding of $M(\lambda')$ into $M(\lambda)$. Then there exists $a \in U(\mathfrak{n}^-)_{\lambda'-\lambda}$ such that $ae_\lambda = e_{\lambda'}$. (Furthermore a is unique up to a non-zero scalar which can be fixed canonically as follows. First, under the above hypothesis, $\lambda - \lambda'$ is a non-negative integral linear combination of the $\alpha \in B$ (with say coefficients k_α) and second, with respect to the canonical filtration of $U(\mathfrak{n}^-)$, the leading term of a is just

$$\prod_{\alpha \in B} X_{-\alpha}^{k_\alpha}$$

up to a non-zero scalar ([21], Lemma 1). Fix this scalar to be one.)

LEMMA: *There exists an embedding of* $M(-\lambda)$ *into* $M(-\lambda')$ *and* $\check{a}e_{-\lambda'} = se_{-\lambda}$, *with* $s = \pm 1$.

Fix $\alpha \in B$. Then $[X_\alpha, a]e_\lambda = 0$ and so $[X_\alpha, a] \in \operatorname{Ann} e_\lambda = U(\mathfrak{g})\mathfrak{n}^+ + \Sigma_{\beta \in B} U(g)(H_\beta - (\lambda - \rho, H_\beta))$. Since $a \in U(\mathfrak{n}^-)_{\lambda'-\lambda}$ and α is simple, we have in fact the more precise result, namely

$$[X_\alpha, a] \in U(\mathfrak{g})_\eta(H_\alpha - (\lambda - \rho, H_\alpha)),$$

where $\eta = \lambda' - \lambda + \alpha$. Hence

$$[X_\alpha, \check{a}] \in (H_\alpha + (\lambda - \rho, H_\alpha))U(\mathfrak{g})_\eta = U(\mathfrak{g})_\eta(H_\alpha + (\lambda + \eta - \rho, H_\alpha)).$$

Yet $-(\lambda + \eta - \rho, H_\alpha) = -(\lambda' + \rho, H_\alpha)$, and so $X_\alpha \check{a} e_{-\lambda'} = [X_\alpha, \check{a}]e_{-\lambda'} = 0$. Since α was arbitrary, it follows that $\check{a}e_{-\lambda'}$ is a highest weight vector (necessarily non-zero) of weight $(\lambda' - \lambda) - (\lambda' + \rho) = -(\lambda + \rho)$ and hence proportional to the canonical generator $e_{-\lambda}$ of $M(-\lambda)$ embedded in $M(-\lambda')$ "canonically" as above. Comparison of leading terms shows that the constant of proportionality is just $(-1)^{\Sigma k_\alpha}$.

Remark. Of course the first part also obtains from ([7], 7.6.23). When $B_\lambda \subset B$, the second part can also be derived from ([7], 7.8.8).

6.3: The bilinear form referred to in 6.1 has been defined purely algebraically in ([7], 9.6.9) for the case $\lambda = \mu$. We describe the modifications needed in the general case. In this we denote by t, u, v elements of $U(\mathfrak{k})$, a, b elements of $U(\mathfrak{g})$, θ an element of $U(\mathfrak{k})^*$, f an element of $L := L(M(\lambda) \otimes M(\mu))^*$.

Define an action of $U(\mathfrak{k})\otimes U(\mathfrak{k})$ on $U(\mathfrak{k})^*$ through $((u\otimes v)\cdot\theta)(t)=\theta(\check{u}tv)$ and set

$$U(\mathfrak{k})_l=U(\mathfrak{k})\otimes\mathbb{C},\quad U(\mathfrak{k})_r=\mathbb{C}\otimes U(\mathfrak{k}).$$

By ([7], 2.7.12) the sum of the simple finite dimensional $U(\mathfrak{k})_l$ submodules of $U(\mathfrak{k})^*$ coincides with the sum of the simple finite dimensional $U(\mathfrak{k})_r$ submodules of $U(\mathfrak{k})^*$, and we denote this subspace by $L(U(\mathfrak{k})^*)$. Let $\epsilon: U(\mathfrak{k})\to\mathbb{C}$ be the augmentation. $\mathbb{C}\epsilon$ occurs as the unique one dimensional subrepresentation of $L(U(\mathfrak{k})^*)$. Let $\varphi_0: L(U(\mathfrak{k})^*)\to\mathbb{C}$ be the linear form on $L(U(\mathfrak{k})^*)$ which takes the value 1 on ϵ and zero on the $U(\mathfrak{k})\otimes U(\mathfrak{k})$ stable complement of $\mathbb{C}\epsilon$ in $L(U(\mathfrak{k})^*)$.

Now for each $\nu\in\mathfrak{h}^*$, define $T_\nu=\{\theta\in U(\mathfrak{k})^*:\theta(uj(H))=(\nu,H)\theta(u)$, for all $H\in\mathfrak{h}$, $u\in U(\mathfrak{k})\}$ which is a $U(\mathfrak{k})_l$ module. With $\nu=\lambda-\mu$, $f\in L$, we define $\theta_f\in T_\nu$ through $\theta_f(u)=f(u(e_\lambda\otimes e_u))$. Then the map $f\mapsto\theta_f$ is a $U(\mathfrak{k})$ module isomorphism of L onto the $U(\mathfrak{k})_l$ finite part $L(T_\nu)$ of T_ν. (For this see [7], 5.5.8 or [8], Sect. I,2). Now take $\lambda'\in\mathfrak{h}^*$ such that $M(\lambda')\subset M(\lambda)$ and $a\in U(\mathfrak{n}^-)_{\lambda'-\lambda}$ as in 6.2. Then for all $f\in L$, we have $((1\otimes j(a))\cdot\theta_f)(u)=\theta_f(uj(a))=f(uj(a)(e_\lambda\otimes e_\mu))=f(u(ae_\lambda\otimes e_\mu))$, since $a\in U(\mathfrak{n}^-)$ and ${}^tXe_\mu=0$ for all $X\in\mathfrak{n}^-$.

Let $\psi: L\to L':=L(M(\lambda')\otimes M(\mu))$ be defined by restriction. Set $\nu'=\lambda'-\mu$, and define for any $f'\in L'$ the element $\theta_{f'}\in T_{\nu'}$ as above. Then for all $f\in L$, we have $\theta_{\psi(f)}(u)=\psi(f)(u(e_{\lambda'}\otimes e_\mu))=f(u(e_{\lambda'}\otimes e_\mu))$. Since $ae_\lambda=e_{\lambda'}$, this gives

$$(*)\qquad (1\otimes j(a))\cdot\theta_f=\theta_{\psi(f)}.$$

Similarly let $\psi': L(M(-\lambda')\otimes M(-\mu))^*\to L(M(-\lambda)\otimes M(-\mu))^*$ be defined by restriction. Then for each $g'\in L(M(-\lambda')\otimes M(-\mu))^*$, we have $\theta_{g'}\in T_{-\nu'}$, $\theta_{\psi'(g')}\in T_{-\nu}$, and by $(*)$ and 6.2 we get

$$(1\otimes j(\check{a}))\cdot\theta_{g'}=s\theta_{\psi'(g')}.$$

Using ([7], 2.7.7) we have $T_{-\nu}T_\nu\subset T_0$ and so $L(T_{-\nu})L(T_\nu)\subset L(T_0)$. Similarly $L(T_{-\nu'})L(T_{\nu'})\subset L(T_0)$. By ([7], 2.7.7), the invariance of φ_0 under $U(\mathfrak{k})_r$ gives (noting $j(\check{a})=j(a)^\vee$) that

$$\varphi_0(((1\otimes j(\check{a}))\cdot\theta_{g'})\theta_f)=\varphi_0(\theta_{g'}((1\otimes j(a))\cdot\theta_f)).$$

Just as in ([7], 9.6.9) using ([7], 2.7.15, 9.6.8) and the reductivity of $\mathfrak{k}$, one checks that the form $\langle g, f\rangle \mapsto \varphi_0(\theta_g\theta_f)$ on $L(\lambda, \mu)\times L(-\lambda, -\mu)$ has the properties claimed in 6.1. Furthermore with respect to the above maps we have the

LEMMA: *The diagram*

$$\begin{tikzcd} L(\lambda', \mu)\times L(-\lambda, -\mu) \arrow[r, "1\times\psi"] \arrow[d, "s\psi'\times 1"] & L(\lambda', \mu)\times L(-\lambda', -\mu) \arrow[d] \\ L(\lambda, \mu)\times L(-\lambda, -\mu) \arrow[r] & \mathbb{C} \end{tikzcd}$$

commutes. That is $s\langle\psi'(g'), f\rangle = \langle g', \psi(f)\rangle$.

Indeed

$$s\langle\psi'(g'), f\rangle = \varphi_0(s\theta_{\psi'(g')}\theta_f) =$$
$$\varphi_0(((1\otimes j(\check{a}))\cdot\theta_{g'})\theta_f) = \varphi_0(\theta_{g'}(1\otimes j(a))\cdot\theta_f)) =$$
$$= \varphi_0(\theta_{g'}\theta_{\psi(f)}) = \langle g', \psi(f)\rangle.$$

Remark. A similar result holds for the second variable.

6.4: Take λ, μ, w_1, w_2, α as in 3.2. Under the hypothesis of 3.2, it follows that $M(-w_2\lambda)$ is a submodule of $M(-s_\alpha w_2\lambda)$. Applying the analogue of 6.3 with respect to second variable to 3.6 we obtain

COROLLARY: *The map* $L(w_1\mu, s_\alpha w_2\lambda)\to L(w_1\mu, w_2\lambda)$ *defined by restriction is surjective.*

Remark. When $\alpha\in B$, this was given in ([4], V, 1.11).

6.5: Both 3.6 and 6.4 admit analogous assertions for the first variable. This gives the commutative diagram of restriction maps

$$\begin{tikzcd} & & 0 & \\ 0 \arrow[r] & L(-w_1\mu, -w_2\lambda) \arrow[r] & L(-w_1\mu, -s_\alpha w_2\lambda) \arrow[u] & \\ & L(-s_\alpha w_1\mu, -w_2\lambda) \arrow[r] \arrow[u] & L(-s_\alpha w_1\mu, -s_\alpha w_2\lambda) \arrow[r] \arrow[u] & 0 \\ & & 0 \arrow[u] & \end{tikzcd}$$

which implies an isomorphism of $L(-w_1\mu, -w_2\lambda)$ with $L(-s_\alpha w_1\mu, -s_\alpha w_2\lambda)$. The intertwining operators of ([8], Sect. I, 2) also give an isomorphism between those modules.

Index of notation

Symbols frequently used are given below in order of appearance.

1.1 $\mathfrak{g}$, $\mathfrak{h}$, R, R^+, B, ρ, s_α, W, X_α, $\mathfrak{n}^+$, $\mathfrak{n}^-$, $\mathfrak{b}$.
1.2 R_λ, R_λ^+, B_λ, $W_{B'}$, $w_{B'}$, W_λ, w_λ, $M(\lambda)$, $\overline{M(\lambda)}$, $L(\lambda)$, $J(\lambda)$, e_λ.
1.3 $\check{u}$, ${}^tu$, U, j, $\mathfrak{k}$, $\mathfrak{k}^\wedge$.
1.4 $\mathcal{O}$, $Z(\mathfrak{g})$, $\hat{\lambda}$, $\mathcal{O}_{\hat{\lambda}}$, $p_{\hat{\lambda}}$.
1.5 $L(M, N)$, $L(M \otimes N)^*$, $L(\lambda, \mu)$.
1.7 $\mathcal{H}$.
1.8 τ, M^τ, $\delta(M)$.
1.11 $[M]$, $[M(\lambda): L(\mu)]$.
1.12 $P(R)$, $P(R)^+$, $P(R)^{++}$, $E(\nu)$.

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O. Gabber
Department of Mathematical Sciences
Tel Aviv University
Ramat-Aviv, Tel-Aviv
Israel

A. Joseph
Université de Paris VI
U.E.R. de Mathématiques
4, Place Jussieu
75230 Paris Cedex 05
France