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ON THE BERNSTEIN-GELFAND-GELFAND RESOLUTION AND THE DUFLO SUM FORMULA

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Abstract

Let \( \mathfrak{g} \) be a complex semisimple Lie algebra. In ([8], Prop. 12) Duflo gave a remarkable sum formula interrelating induced ideals. The main result of this paper provides a natural generalization of this formula and more precisely gives a resolution for certain primitive quotients of the enveloping algebra \( U(\mathfrak{g}) \). The proof has three distinct steps. One, the extension of the Bernstein-Gelfand-Gelfand (in short, B.G.G.) resolution of a simple finite dimensional \( U(\mathfrak{g}) \) module to certain simple highest weight modules. Two, the description of the so-called \( \ell \)-finite part of the space of homomorphisms of any one Verma module to any other. Three, the proof of exactness of a certain functor. The last can be viewed as a non-trivial generalization of the fact that a Verma module with dominant highest weight is projective in the so-called \( \mathcal{O} \) category. A by-product gives some results on a problem of Kostant relating \( U(\mathfrak{g}) \) to the \( \ell \)-finite part of the space of endomorphisms of a simple highest weight module.

1. Preliminaries

1.1: Let \( \mathfrak{g} \) be a complex semisimple Lie algebra, \( \mathfrak{h} \) a Cartan subalgebra for \( \mathfrak{g} \), \( \mathcal{R} \) the set of non-zero roots, \( \mathcal{R}^+ \subset \mathcal{R} \) a system of positive roots, \( \mathcal{B} \subset \mathcal{R}^+ \) the set of simple roots, \( \rho \) the half sum of the positive roots, \( s_\alpha \in \text{Aut}(\mathfrak{h}^*) \) the reflection corresponding to the root \( \alpha \in \mathcal{R} \), and \( W \) the group generated by the \( s_\alpha : \alpha \in \mathcal{B} \). Let \( X_\alpha \) be the element of a
Chevalley basis for \( g \) corresponding to the root \( \alpha \) and set

\[
\pi^+ = \sum_{\alpha \in R^+} C \chi_{\alpha}, \quad \pi^- = \sum_{\alpha \in R^+} C \chi_{-\alpha}, \quad b = \mathfrak{h} \oplus \mathfrak{p}^+.
\]

1.2: For each \( \lambda \in \mathfrak{h}^* \), set \( R_\lambda = \{ \alpha \in R : 2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z} \} \) (which is itself a root system) and \( R_\lambda^+ = R_\lambda \cap R^+ \), with \( B_\lambda \subset R_\lambda^+ \) the corresponding set of simple roots. Call \( \lambda \) regular (resp. dominant) if \( (\lambda, \alpha) \neq 0 \) (resp. \( (\lambda, \alpha) \geq 0 \)) for all \( \alpha \in R^+ \). For each \( B' \subset B_\lambda \), let \( W_{B'} \) be the subgroup of \( W \) generated by the \( s_a : a \in B' \) and \( w_{B'} \) the largest element of \( W_{B'} \) with respect to its Bruhat order \( \leq \) (as defined in [7], 7.7.3). If \( B' = B_\lambda \) we write \( W_{B'} = W_\lambda \), \( w_{B'} = w_\lambda \). Let \( M(\lambda) \) denote the Verma module with highest weight \( \lambda \)-\( \rho \) associated to the quadruplet \( g, \mathfrak{h}, B, \lambda \) (see [7], 7.1.4), \( M(\lambda) \) the unique maximal submodule of \( M(\lambda) \), and set \( L(\lambda) = M(\lambda)/M(\lambda), J(\lambda) = \text{Ann } L(\lambda) \). For each \( \mathfrak{h} \)-module \( V \) we set \( V_{\lambda} = \{ v \in V : Hv = (\lambda, H)v, \text{ for all } H \in \mathfrak{h} \} \). Let \( e_\lambda \) denote the canonical generator of \( M(\lambda) \) (which has weight \( \lambda - \rho \)). Set \( R_0^\lambda = \{ \alpha \in R : (\alpha, \lambda) = 0 \} \).

1.3: Let \( u \mapsto \hat{u} \) (resp. \( u \mapsto u' \)) denote the involutory antiautomorphism of \( U(\mathfrak{g}) \) defined by \( \hat{X} = -X : X \in \mathfrak{g} \) (resp. \( \hat{X}_\alpha = X_{-\alpha} : \alpha \in R \), \( \hat{H} = H : H \in \mathfrak{h} \)). Identify \( U = U(\mathfrak{g}) \otimes U(\mathfrak{g}) \) canonically with \( U(\mathfrak{g} \times \mathfrak{g}) \).

Define \( j : \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g} \) through \( j(X) = (X, -X) \), set \( \mathfrak{t} = j(\mathfrak{g}) \), so \( U(\mathfrak{t}) \) may be regarded as a subalgebra of \( U \). Let \( \mathfrak{t}^\ast \) denote the set of equivalence classes of finite dimensional irreducible representations of \( \mathfrak{t} \). For each locally finite \( \mathfrak{t} \)-module \( L \) and each \( \sigma \in \mathfrak{t}^\ast \), we let \( L_{\sigma} \) denote the isotypical component of type \( \sigma \) of \( L \). Let \( \iota : U(\mathfrak{t}) \to U(\mathfrak{g}) \) be the \( \mathbb{C} \) algebra isomorphism sending \( -X, X \) to \( X \) for every \( X \in \mathfrak{g} \). If \( R \to \mathbb{S} \) is a ring homomorphism and \( M \) is a left \( S \) module, we let \( M^\mathfrak{g} \) denote the left \( R \) module which consists of the underlying abelian group \( |M| \) of \( M \) together with the operation \( (r, m) \mapsto \varphi(r) \cdot m \) of \( R \) on \( |M| \).

1.4: Let \( \mathcal{O} \) denote the category of finitely generated \( U(\mathfrak{g}) \) modules which are \( \mathfrak{h} \)-simple and \( \mathfrak{b} \) locally finite (see [1-3, 6]). Each \( M \in 0b\mathcal{O} \) has finite length [2]. This category has enough projectives and so the extension groups \( \text{Ext}^i(\cdot, \cdot) \) relative to \( \mathcal{O} \) are thereby defined. Let \( Z(\mathfrak{g}) \) denote the centre of \( U(\mathfrak{g}) \). Then \( \text{Max } Z(\mathfrak{g}) \) is isomorphic to \( \mathfrak{h}^*/W \) such that for each \( \lambda \in \mathfrak{h}^*, \hat{\lambda} := W\lambda \) corresponds to the element \( Z(\mathfrak{g}) \cap J(\lambda) \) of \( \text{Max } Z(\mathfrak{g}) \). Let \( \mathcal{O}_i \) denote the subcategory of \( \mathcal{O} \) of all modules annihilated by a power of this maximal ideal. Each \( M \in 0b\mathcal{O} \) admits a primary decomposition and we denote by \( p_\lambda : 0b\mathcal{O} \to 0b\mathcal{O}_i \) the projection onto the primary component defined by \( \hat{\lambda} \). It is an exact functor on \( \mathcal{O} \).

1.5: Given \( M, N \in Ob\mathcal{O} \), consider \( \text{Hom}_c(M, N) \) (resp. \( (M \otimes N)^* \)) as a \( U \) module through \( ((a \otimes b) \cdot x)m = ((a \otimes \hat{b} \cdot x)m \) (resp. \( ((a \otimes b) \cdot y, \hat{a} \otimes \hat{b} \cdot x)m \)).
\[ m \otimes n = (y, \, a m \otimes b n) \] where \( a, b \in U(g), \, m \in M, \, n \in N, \, x \in \text{Hom}(M, N), \, y \in (M \otimes N)^*. \) We remark that \((M(-\lambda) \otimes M(-\mu))^*\) is isomorphic to the \( \mathfrak{g} \times \mathfrak{g} \) module co-induced from the \( \mathfrak{b} \times \mathfrak{b} \) module \( C_{\lambda+\mu+\rho} \). Let \( L(M, N) \) (resp. \( L(M \otimes N)^* \)) denote the set of all \( \mathfrak{t} \)-finite elements of \( \text{Hom}(M, N) \) (resp. \( (M \otimes N)^* \)) which we remark is again a \( U \) module. For \( \lambda, \mu \in \mathfrak{h}^* \), we set \( L(\lambda, \mu) = L(M(-\lambda) \otimes M(-\mu))^* \).

1.6: Let \( E \) be a finite dimensional \( U(g) \) module and given \( M \in 0b \mathcal{C} \), consider \( E \rightarrow M \) as a \( U(g) \) module through the diagonal action. One has \( E \otimes M \in 0b \mathcal{C} \) and the functor \( M \mapsto E \otimes M \) is exact. Again one has the natural isomorphisms

\[ \text{Hom}_d(E, \text{Hom}_c(M, N)) \cong \text{Hom}_d(E \otimes M, N) \cong \text{Hom}_d(M, E^* \otimes N). \]

The latter gives on taking projective resolutions natural isomorphisms

\[ \text{Ext}^k(E \otimes M, N) \cong \text{Ext}^k(M, E^* \otimes N) : k \in \mathbb{N}. \]

1.7: Let \( \mathcal{H} \) denote the category of all \( U \) modules which satisfy the following properties. One, each \( L \in 0b \mathcal{H} \) is locally finite as a \( \mathfrak{t} \) module. Two, \( \dim L_{\sigma} < \infty \) for each \( \sigma \in \mathfrak{t}^\ast \). Three, each \( L \in 0b \mathcal{H} \) admits a finite filtration such that the centre of \( U \) acts by scalars on each subquotient. Clearly \( \mathcal{H} \) is stable under tensoring with finite dimensional \( U \) modules. It follows from the classification ([21], I, Sect. 4) of the simple modules in \( \mathcal{H} \) that each \( L \in 0b \mathcal{H} \) has finite length (for example, as shown in ([2], 4.2)). For each \( M, N \in 0b \mathcal{C} \), one has \( L(M, N) \in \mathcal{H} \). Indeed, the first property holds by construction. The second obtains from the isomorphism valid for any simple finite dimensional \( U(g) \) module \( E \), namely \( \text{Hom}_d(E, L(M, N)) \cong \text{Hom}_d(E \otimes M, N) \), the last space being finite dimensional (since \( E \otimes M, N \) have finite length). The third obtains by taking composition series for \( M, N \). We have shown that

**Lemma:** For each \( M, N \in 0b \mathcal{C} \), the \( U \) module \( L(M, N) \) has finite length.

1.8: Observe that \( \tau: a \mapsto 'a \) is an involutory automorphism of \( U(g) \). Given \( M \in 0b \mathcal{C} \), we let \( \delta(M) \) denote the submodule of \((M^*)'\) of all \( \mathfrak{h} \) finite elements. Through the existence of a non-degenerate contravariant form on \( L(\lambda) \) (see [11], 1.6), one has \( L(\lambda) \cong \delta(L(\lambda)) \). In particular \( E^* \cong E^* \) for any finite dimensional module \( E \). Again each \( M \in 0b \mathcal{C} \) has finite length, so \( \delta(M) \in 0b \mathcal{C} \) and \( \delta(M) \) has the same composition factors as \( M \) (with the same multiplicities).
1.9: For each $M, N \in \mathfrak{Ob}$, define $\sigma : \text{Hom}_c(M, (N^*)^*) \to (N \otimes M)^*$ through $(\sigma(x), m \otimes n) = (xm, n)$. From $(\sigma((a \otimes b) \cdot x), m \otimes n) = (((a \otimes b) \cdot x)m, n) = (\sigma(\sigma(x), \sigma(b)m) = (a \otimes b) \cdot x, m \otimes n)$, it follows that $\sigma$ is a $U$ module homomorphism. Again $\sigma$ is obviously injective. Given $y \in (N \otimes M)^*$, then for each $m \in M$ the map $g(y, m) : n \mapsto (y, n \otimes m)$ of $N$ to $C$ is $C$-linear. It follows that the map $\eta(y) : m \mapsto g(y, m) of M$ to $(N^*)^*$ is $C$-linear and the map $\eta : y \mapsto \eta(y)$ is inverse to $\sigma$.

**Lemma:** The map $\sigma$ restricts to a $U$ module isomorphism of $L(M, \delta(N))$ onto $L(N \otimes M)^*$. In particular $L(N \otimes M)^*$ has finite length as a $U$ module.

If $x \in L(M, \delta(N))$, then $\sigma(x)$ is obviously $l$-finite. Conversely for each $y \in L(N \otimes M)^*$, $m \in M, x \in \mathfrak{a}$, we have $X(\eta(y)m) = \eta(j(X)y)m + \eta(y)Xm$, and so the local finiteness of $\mathfrak{b}$ on $M$ implies that $\eta(y)m \in \delta(N)$. Hence the surjectivity of the restriction of $\sigma$. The last part follows from 1.7.

1.10: Define an ordering on $\mathbb{Z}B$ through $\mu \geq \nu$ if $\mu - \nu \in \mathbb{N}B$. Given $M \in \mathfrak{Ob}$, set $\Omega(M) = \{\lambda \in \mathfrak{b}^*: M_{\lambda} \neq 0\}$. If $M \neq 0$, then $\Omega(M)$ admits at least one maximal element. Note that $H_0(\mathfrak{n}^-, M) = M/\mathfrak{n}^-M$ is a locally finite semisimple $\mathfrak{b}$ module.

**Lemma:** Suppose $M, N \in \mathfrak{Ob}$ with $N$ a submodule of $M$. If $H_0(\mathfrak{n}^-, M), H_0(\mathfrak{n}^-, N)$ are isomorphic as $\mathfrak{b}$ modules, then $M = N$.

Assume $Q := M/N \neq 0$. Let $\mu \in \Omega(Q)$ be maximal. Through the maximality of $\mu$ one has $(\mathfrak{n}^-M)_\mu = \Sigma X_{-\lambda}M_{\lambda + \alpha} = \Sigma X_{-\lambda}N_{\lambda + \alpha} = (\mathfrak{n}^-N)_\mu$. Yet $\dim N_{\mu}/(\mathfrak{n}^-N)_\mu = \dim M_{\mu}/(\mathfrak{n}^-M)_\mu$, by hypothesis. This gives $M_{\mu} = N_{\mu}$ which is a contradiction.

1.11: For each $M \in \mathfrak{Ob}$, let $[M]$ denote the corresponding element in the Grothendieck group $G$ of $\mathfrak{Ob}$. For each $\lambda \in \mathfrak{b}^*/W$, let $G_\lambda$ denote the subgroup of $G$ corresponding to $G_\lambda$. It is well-known that $\{L(\lambda) : \mu \in \lambda\}$ is a basis for $G_\lambda$. Again each $M(\lambda) : \lambda \in \mathfrak{b}^*$ has finite length with simple factors amongst the $L(\lambda) : \mu \in \lambda$ and we denote by $[M(\lambda) : L(\mu)]$ the number of times $L(\mu)$ occurs in $M(\lambda)$. The resulting matrix is invertible (by [7], 7.6.23) and (by [7], 7.6.14) one has

$$[E \otimes M(\lambda)] = \sum_{\nu \in \mathcal{H}(E)} [M(\lambda + \nu)] \dim E_{\nu}$$

for any finite dimensional $U(\mathfrak{a})$ module $E$. 

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1.12: Let $P(R)$ denote the lattice of integral weights. Let $P(R)^+$ (resp. $P(R)^{++}$) denote the dominant (resp. dominant and regular) elements of $P(R)$. For each $\nu \in P(R)$, let $E(\nu)$ denote a (unique up to isomorphism) simple finite dimensional $U(\mathfrak{g})$ module with extreme weight $\nu$. The map $\nu \mapsto E(\nu)^*$ identifies the $\mathfrak{t}^*$ of classes of finite dimensional simple $U(\mathfrak{t})$ modules with $P(R)/W$ and hence with $P(R)^+$. Frobenius reciprocity gives $\dim \text{Hom}(E(\nu)^*, L(\lambda, \mu)) = \dim E(\nu)_{\mu-\lambda}$ for all $\lambda, \mu \in \mathfrak{h}^*$, $\nu \in P(R)$. In particular, $L(\lambda, \mu) \neq 0$ if and only if $\lambda - \mu \in P(R)$. Now assume $\lambda - \mu \in P(R)$. Then by 1.9, $L(\lambda, \mu)$ has finite length. Since $\dim E(\lambda - \mu)_{\lambda - \mu} = 1$, it follows that $L(\lambda, \mu)$ admits a unique simple subquotient, which we denote by $V(\lambda, \mu)$, satisfying $\dim \text{Hom}(E(\lambda - \mu), V(\lambda, \mu)) = 1$. We shall need the following

**Theorem:**

(i) Every simple module in $\mathcal{H}$ is isomorphic to some $V(\lambda, \mu)$.

(ii) $V(\lambda, \mu)$ is isomorphic to $V(\lambda', \mu')$ if and only if there exists $w \in W$ such that $\lambda' = w\lambda, \mu' = w\mu$.

(iii) Suppose $\lambda \in \mathfrak{h}^*$ is dominant. Then if $L(M(\lambda), L(\mu)) \neq 0$ (which holds in particular if $\lambda$ is regular), it is isomorphic to $V(-\mu, -\lambda)$. Furthermore every simple $V \in \text{Ob} \mathcal{H}$ is so obtained.

(i), (ii) are just ([9], I, 4.1, 4.5) and (iii) follows from ([14], 4.7) and (i), (ii).

1.13: Given $-\lambda \in \mathfrak{h}^*$ dominant, then for each $-\mu \in -\lambda + P(R)$ dominant we define following Jantzen ([11], Sect. 2) a translation operator $T^{\lambda}_{\mu} : \mathcal{H} \rightarrow \mathcal{H}$ through $T^{\lambda}_{\mu} M = p_{\mu}(E(\mu - \lambda) \otimes p_{\lambda}(M))$. If $R^0_{\mu} \subset R^0_{\mu}$, then for all $w \in W$ we have $T^{\lambda}_{\mu} M(w\lambda) \equiv M(w\mu)$ (see [10], 2.10). Let $E$ be a finite dimensional $U(\mathfrak{g})$ module. Through the natural $U$ module isomorphisms $L(M, N) \otimes (C \otimes E) \rightarrow L(M \otimes E^*, N)$, $L(M, N) \otimes (E \otimes C) \rightarrow L(M, N \otimes E^*)$, it is obvious how to define exact functors on $\mathcal{H}$ satisfying $R^\mu_{\lambda} L(M, N) \equiv L(T^\mu_{\lambda} M, N)$, $S^\mu_{\lambda} L(M, N) \equiv L(M, T^\mu_{\lambda} N)$ for $M, N \in \text{Ob} \mathcal{H}$. Again by 1.6, $T^{\lambda}_{\mu}$ is both left and right adjoint to $T^{\lambda}_{\mu}$.

1.14: For each $j \in \mathbb{N}$, $\mu \in \mathfrak{h}^*$, $N \in \text{Ob} \mathcal{H}$, one has $\text{Ext}^j(M(\mu), N) \equiv H^j(\lambda^+(N), N)_{\mu} \equiv ( H^j(\lambda^+(N), \delta(N))_{\mu} \otimes p_{\lambda}(M))$, the first isomorphism being due to Delorme ([6], Thm. 2), the second a formal consequence of the appropriate standard complexes.

1.15: Take $\lambda, \mu \in \mathfrak{h}^*$ and let us note the almost obvious fact that $L(M(\lambda), M(\mu)) = 0$ unless $\lambda - \mu \in P(R)$. This latter condition further implies that $W^{\lambda} = W^{\mu}$. 


**Lemma:** Fix $-\lambda, -\mu \in \mathfrak{h}^*$ dominant with $\lambda - \mu \in P(R)$. Then for each $w \in W_\lambda$ and each finite dimensional $U(\mathfrak{g})$ module $E$ one has
\[
\dim \text{Hom}_t(E^t, L(M(w_\lambda \lambda), M(w_\mu))) = \dim \text{Hom}_t(E^t, L(M(w_\lambda \lambda), M(\mu))).
\]

We show that both sides equal $\dim E_{w_\lambda - w_\mu \lambda}$. For the right hand side this follows from the fact that $M(\mu)$ is simple (and so isomorphic to $\delta M(\mu)$), 1.9 and 1.12, noting that $\Omega(E)$ is $W$ stable. The left hand side equals (by 1.7) $\dim \text{Hom}_t(M(w_\lambda \lambda), E^* \otimes M(w_\mu))$; since $M(w_\lambda \lambda)$ is projective in $\mathcal{O}$, we have by 1.11 that the latter equals $\dim(E^* )_{w_\lambda - w_\mu} = \dim E_{w_\lambda - w_\mu \lambda}$.

**Remarks:** Although this also follows from ([14], 4.10) the above proof is much simpler. It is not difficult to extend the above to a further proof of ([14], 4.3) and hence of Duflot's theorem ([8], Thm. 1); but then this becomes essentially the proof given in ([3], 4.4).

1.16: Take $\lambda \in \mathfrak{h}^*$ dominant. $Z(\mathfrak{g})$ acts on $M(\lambda)$ by a homomorphism $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$. Let $C = \lambda + P(R)$, and let $\mathcal{O}_C$ be the full subcategory of $\mathcal{O}$ consisting of those modules $M$ that satisfy $\Omega(M) \subset C$. Define a functor $T : \mathcal{O}_C \to \mathcal{H}$ by $T(N) = L(M(\lambda), N)$ (cf. 1.7). $T$ is exact since any $M(\lambda) \otimes E$ ($E$ being a finite dimensional $U(\mathfrak{g})$ module) is projective in $\mathcal{O}$. Let $\mathcal{H}$ consisting of those $M \in 0b(\mathcal{H})$ on which $1 \otimes Z(\mathfrak{g})$ acts through $1 \otimes z \mapsto \chi_\lambda(z)$. The image of $T$ lies in $\mathcal{H}_\lambda$, and in the following theorem we view $\mathcal{H}_\lambda$ as the target category of $T$.

**Theorem:**
(i) $T$ has a left adjoint $T'$.
(ii) The unit map $\eta : 1d_{\mathcal{H}_\lambda} \to TT'$ is an isomorphism of functors.
(iii) If $\lambda$ is regular, then $T$ is an equivalence of categories.

We indicate a proof for the theorem, which has also been proved by Bernstein and Gelfand ([3], 6.3, 6.1 (ii), 5.9 (i)).

(i). If $M \in 0b(\mathcal{H}_\lambda)$, we make $M$ into a two-sided $U(\mathfrak{g})$ module by $amb = (\bar{a} \otimes \bar{b}) \cdot m$ for all $m \in M$, $a, b \in U(\mathfrak{g})$. Define $T'(M) = M \otimes_A M(\lambda)$, where $A = U(\mathfrak{g})/U(\mathfrak{g})\ker(\chi_\lambda)$. Now $T'(M) \in 0b(\mathcal{O}_C)$ because if $E \subset M$ is a finite dimensional $\mathfrak{g}$ stable generating subspace (so $M = EU(\mathfrak{g})$), then we get a surjective $\mathfrak{g}$ linear map $E^{*-1} \otimes M(\lambda) \to T'(M)$. If $M \in 0b(\mathcal{H}_\lambda)$ and $N \in 0b(\mathcal{O}_C)$, one defines an isomorphism $\zeta(M, N) : \text{Hom}_A(M \otimes_A M(\lambda), N) \to \text{Hom}_U(M, L(M(\lambda), N))$ by $\zeta(\varphi) = (m \mapsto \varphi(m \otimes u))$. This makes $T'$ a left adjoint to $T$.

(ii). We have to show that for any $M \in 0b(\mathcal{H}_\lambda)$ the map $\eta(M) : M \to 0b(\mathcal{H}_\lambda)$ is an isomorphism.
L(M(\lambda), M \otimes A M(\lambda)) \text{ (given by } m \mapsto (n \mapsto m \otimes n)) \text{ is bijective. We make } A \text{ into a } U \text{ module by } (a \otimes b) \cdot x = 'a x b, \text{ for all } x \in A, \ a, \ b \in U(g). \text{ Then } \eta(A) \text{ is an isomorphism by } ([13], 6.4).

If } E \text{ is a finite dimensional } U(g) \text{ module we have natural isomorphisms }
\begin{align*}
T(E \otimes N) & \leftarrow (E' \otimes C) \otimes T(n), \ N \in 0b(\mathcal{C}) \\
T'((E' \otimes C) \otimes M) & \leftarrow E \otimes T'(M), \ M \in 0b(\mathcal{H}_\lambda).
\end{align*}

Using these isomorphisms, one shows that if } \eta(M) \text{ is an isomorphism then so is } \eta((E' \otimes C) \otimes M). \text{ In particular, } \eta((E' \otimes C) \otimes A) \text{ is an isomorphism. This implies that } \eta(M) \text{ is an isomorphism for any } M \in 0b(\mathcal{H}_\lambda), \text{ by observing that } TT' \text{ is right exact and that for suitable finite dimensional } U(g) \text{ modules } E_1, E_2 \text{ there exists an exact sequence } (E_1 \otimes C) \otimes A \rightarrow (E_2 \otimes C) \otimes A \rightarrow M \rightarrow 0 \text{ in } \mathcal{H}_\lambda.

(iii). \text{ We have to show that the counit map } \varepsilon : T'T \rightarrow Id_{\mathcal{C}} \text{ is also an isomorphism of functors. The composition } T \xrightarrow{\eta} TT'T \xrightarrow{\varepsilon} T \text{ is } Id_T, \text{ so by (ii) } \varepsilon_T \text{ is an isomorphism. Thus, as } T \text{ is exact, } 0 = T(\ker(\varepsilon(N))) = T(\coker(\varepsilon(N))) \text{ for any } N \in 0b(\mathcal{C}). \text{ So it remains to show that if } N \in 0b(\mathcal{C}) \text{ and } TN = 0 \text{ then } N = 0. \text{ Indeed, if } N \neq 0, \text{ then } N \text{ contains a simple submodule } L(\mu) : \mu \in C, \text{ so } TN \supset TL(\mu); \text{ but by } ([14], 4.7) TL(\mu) \neq 0, \text{ and we get a contradiction.}

2. The generalized B.G.G. resolution

Throughout this section we fix } -\lambda \in \mathfrak{h}^* \text{ dominant and regular.}

2.1: Given } \alpha \in B_\lambda, \text{ one can choose } \nu_\alpha \in P(R) \text{ such that } -\lambda_\alpha := -\lambda + \nu_\alpha \text{ is dominant and } (\beta, \lambda_\alpha) = 0; \beta \in R^+ \text{ is equivalent to } \beta = \alpha. \text{ Following Vogan ([22]) we set } \theta_\alpha = T^\lambda_{\alpha} \circ T^\lambda_{\alpha} : C \rightarrow C. \text{ Using 1.13, } \theta_\alpha \text{ is left adjoint to } \theta_\alpha. \text{ So we obtain natural isomorphisms }
\begin{align*}
\Ext^j(\theta_\alpha M, N) & \rightarrow \Ext^j(M, \theta_\alpha N) : j \in N, \ M, N \in 0b \mathcal{C}.
\end{align*}

2.2: For each } w \in W_\lambda, \text{ let } l(w) \text{ denote the reduced length of } w \text{ with respect to } B_\lambda. \text{ For each } w, w' \in W_\lambda, \text{ we define an expression } P_{w, w'} \text{ in the indeterminate } q \text{ through }
P_{w, w}(q) = \sum_{k=0}^{\infty} q^{l(w)-l(w')-2k} \text{ dim } \Ext^k(M(w\lambda), L(w'\lambda)).

A result of Casselman and Schmid (proved also in [6], Thm. 4)
implies that $P_{w,w'}(q)$ is polynomial in $q^{1/2}$. Kazhdan and Lusztig ([19], Conj. 1.5) have further conjectured that $P_{w,w'}(q)$ is polynomial in $q$ and that this polynomial is determined by a particular purely combinatorial procedure which uses only the description of $W_\lambda$ as a Coxeter group. This has been shown to follow from certain other conjectures ([10], [23]); but for the moment remains an open problem. Here we just establish one of the identities which would follow from the Kazhdan-Lusztig conjecture.

**Lemma:** For each $w, w' \in W_\lambda$, $\alpha \in B_\lambda$, such that $w'\alpha < w'$, one has that $P_{w\alpha,w}(q) = P_{w,w'}(q)$. In particular, for each $B' \subset B_\lambda$, $w \in W_{B'}$, one has that $P_{w,w'}(q) = 1$.

We can assume $w_\alpha < w$, without loss of generality. Then the conclusion of the lemma is equivalent to the identity

\[ \dim \text{Ext}^{i+1}(M(w_\alpha \lambda), L(w' \lambda)) = \dim \text{Ext}^i(M(w \lambda), L(w' \lambda)), j \in \mathbb{N}. \]

Under the hypothesis $w'\alpha < w'$, it follows that $M(w'\alpha \lambda)$ is a submodule of $M(w' \lambda)$ and by ([10], 2.10a) that $\theta_\alpha M(w' \lambda) \not\subset \theta_\alpha M(w'\alpha \lambda)$. Hence $\theta_\alpha (M(w' \lambda)/M(w'\alpha \lambda)) = 0$. Since $L(w' \lambda)$ is a quotient of $M(w' \lambda)/M(w'\alpha \lambda)$, it follows that $\theta_\alpha L(w' \lambda) = 0$, and so $\text{Ext}^i(\theta_\alpha M(w \lambda), L(w' \lambda)) = 0$ by 2.1. In particular $L(w \lambda)$ is not a quotient of $\theta_\alpha M(w \lambda)$. Then from ([11], 2.17) we obtain an exact sequence

\[ 0 \to M(w \lambda) \to \theta_\alpha M(w \lambda) \to M(w_\alpha \lambda) \to 0 \]

from which the corresponding long exact sequence for $\text{Ext}^*(\cdots, L(w' \lambda))$ gives (\*).

2.3: From 1.8, 1.14 and 2.2 we obtain

**Corollary:** For each $B' \subset B_\lambda$, $w \in W_\lambda$, one has

\[ \dim H_j(n^-, L(w_B \lambda))_{w\alpha - p} = \begin{cases} 1: w \in W_{B'}, j = l(w_B) - l(w), \\ 0: \text{otherwise}. \end{cases} \]

**Remarks.** As is well-known the remaining weight spaces of $H_j(n^-, L(w_B \lambda))$ are null. This follows from the action of $Z(g)$ and the fact that $w\lambda - \lambda \in Z B$ implies $w \in W_\lambda$. This result then generalizes the Bott-Kostant formula established for finite dimensional simple modules (i.e. when $-\lambda \in P(R)^{++}$ and $B' = B$).

2.4: Fix $B' \subset B_\lambda$ and set $s = l(w_B)$. Then for each $j \in \mathbb{N}$, set $W_{B'}^j = \ldots$
\{w \in W_B : l(w) = j\}, and

\[ C_j = \bigoplus_{w \in W_B^j} M(w\lambda). \]

As \( M(w\lambda) \) is \( U(n^-) \) free, we have for each \( y \in W_B \) that

\[ \dim H_t(n^-, C_j)_{\lambda - \rho} = \begin{cases} 1 : t = 0, j = l(y), \\ 0 : \text{otherwise}. \end{cases} \]

2.5: For each \( w \in W_B \), fix a \( U(g) \) module embedding \( i_w : M(w\lambda) \to M(w_B \lambda) \). For \( w, w' \in W_B \) such that \( w \leq w' \), let \( i_{w,w'} : M(w\lambda) \to M(w'\lambda) \) be the embedding such that \( i_w \circ i_{w,w'} = i_w \).

Fix \( j \in \{1, 2, \ldots, s\} \) and consider a \( U(g) \) module map \( \partial_j : C_{j-1} \to C_j \)

\[ \text{defined by } (x_w)_{w \in W_{j-1}} \mapsto (y_w)_{w \in W_j} \text{ when } \]

\[ y_{w'} = \sum_{w \preceq w'} c^j_{w,w'} i_{w,w'}(x_w), \quad x_w \in M(w\lambda), \]

where \( c^j_{w,w'} \in \mathbb{Z} \) is non-zero and defined whenever \( w \leq w' \), \( w \in W_{j-1} \), \( w' \in W_j \).

**Lemma:** The natural surjection \( H_0(n^-, C_{j-1}) \to H_0(n^-, \text{Im} \partial_j) \) is bijective.

Set \( K = \ker \partial_j, V = W_{j-1}^- \). We have an exact sequence \( 0 \to K/K \cap n^- C_{j-1} \to C_{j-1}/n^- C_{j-1} \to \partial_j C_{j-1}/n^- (\partial_j C_{j-1}) \to 0 \), so the lemma is equivalent to \( K \subset \bigoplus_{w \in V} n^- M(w\lambda) \), or to \( K \subset \bigoplus_{w \in V} \overline{M(w\lambda)} \), that is to

\[ \overline{K} : = \text{Im}(K \to \bigoplus_{w \in V} L(w\lambda)) = 0. \]

If \( \overline{K} \neq 0 \), there exists \( w \in V \) such that \( [\overline{K} : L(w\lambda)] > 0 \), and so \( [K : L(w\lambda)] > 0 \). Yet equality of lengths in \( V \) implies through ([7], 7.6.23) that \( [C_j : L(w\lambda)] = 1 \), so \( 0 = [C_{j-1}/K : L(w\lambda)] = [\partial_j C_{j-1} : L(w\lambda)] \). On the other hand since there exists \( w' \in W_{j-1}^+ \) such that \( w \leq w' \) and by hypothesis we then have \( c^j_{w,w'} \neq 0 \), it follows that \( \partial_j \) is injective on the summand \( M(w\lambda) \) of \( C_{j-1} \). Thus \( \partial_j C_{j-1} \) contains a copy of \( M(w\lambda) \), which implies \( [\partial_j C_{j-1} : L(w\lambda)] \geq 1 \). This contradiction proves the lemma.

2.6: An appropriate combinatorial property of the Bruhat ordering enables one to choose the \( c^j_{w,w'} \) of 2.5 such that \( \partial_j \partial_{j-1} = 0 \), for all \( j = 2, \ldots, s \). (See [2], Sect. 11 or [7], 7.8.14). Furthermore
PROPOSITION: The sequence

\[ 0 \to C_0 \to C_1 \to \cdots \to C_{j-1} \to C_j \to \cdots \to C_s \to L(w_B \lambda) \to 0 \]

is exact.

Set \( X_{s+1} = Y_{s+1} = L(w_B \lambda), \ Y_s = \ker(C_s \to X_{s+1}), \) and for each \( j \in \{1, 2, \ldots, s\}, \) set \( X_j = \text{Im } \partial_j, \ Y_{j-1} = \ker \partial_j. \) For each \( j \in \{1, 2, \ldots, s+1\}, \) \( X_j \) is a submodule of \( Y_j \) and we show that \( X_j = Y_j. \) Fix \( r \geq 1 \) and assume that this has been established for all \( j > r. \) This means that we have the short exact sequences

\[ 0 \to X_j \to C_j \to X_{j+1} \to 0 : r < j \leq s. \]

By 2.4, the associated long exact sequence for homology implies for all \( \mu \in \mathfrak{h}^* \) and \( r < j \leq s \) that

\[
\dim H_t(n^-, X_j)_\mu = \begin{cases} 
\dim H_{t+1}(n^-, X_{j+1})_\mu & : t > 0 \\
\dim H_t(n^-, X_{j+1})_\mu - \dim H_0(n^-, X_{j+1})_\mu + \dim H_0(n^-, C)_\mu & : t = 0.
\end{cases}
\]

Then from 2.3 and 2.4 we obtain

\[
\dim H_t(n^-, X_j)_\mu = \begin{cases} 
1 : \mu = w \lambda, \ w \in W_B, \ l(w) = j - t - 1, \\
0 : \text{otherwise.}
\end{cases}
\]

for all \( j > r \) and in particular for \( j = r + 1. \)

Finally from the long exact sequence associated to \( 0 \to Y_r \to C_r \to X_{r+1} \to 0, \) 2.4 and the above we eventually obtain

\[
\dim H_0(n^-, Y_r)_\mu = \dim H_0(n^-, C_{r-1})_\mu
\]

for all \( \mu \in \mathfrak{h}^*. \) Then by 2.5, \( H_0(n^-, Y_r) \) and \( H_0(n^-, X_r) \) are isomorphic as \( \mathfrak{h} \)-modules and so \( X_r = Y_r \) by 1.10. Noting that \( \partial_1 \) is injective completes the proof of the proposition.

Remark. This generalizes the B.G.G. resolution originally established [2] for the case \( -\lambda \in P(R)^{++}, \ B' = B. \) The original proof is different to ours and can only be generalized to the case when \( B' \subset B \) (see [20] for this). The present proof was found following conversations with M. Duflo and P. Delorme.
3. Mappings of Verma modules

3.1: Take $-\lambda \in \mathfrak{h}^*$ dominant. Then $M(\lambda)$ is a simple module and so isomorphic to $\delta(M(\lambda))$. Then by 1.9 one has for all $\mu \in \mathfrak{h}^*$ that

$$L(M(\mu), M(\lambda)) = L(M(\mu), \delta M(\lambda)) = L(-\lambda, -\mu),$$

up to isomorphisms. This relationship of mappings of Verma modules to the principal series has been known for some time. Here we consider the most general form this takes when $-\lambda$ is not necessarily dominant. Some results in this direction were already obtained in ([5], 5.5) and in ([14], 4.10).

3.2: Fix $-\lambda, -\mu \in \mathfrak{h}^*$ dominant with $\lambda - \mu \in P(R)$ (recall 1.15). Choose $w_1, w_2 \in W_\lambda$ and $\alpha \in B_\lambda$ such that $s_\alpha w_1 > w_1$, $s_\alpha w_2 < w_2$. The second relation implies that $M(s_\alpha w_2 \lambda)$ is a submodule of $M(w_2 \lambda)$.

**Lemma:** Under the above hypotheses, one has $L(M(w_1 \mu), M(w_2 \lambda)/M(s_\alpha w_2 \lambda)) = 0$. Equivalently for any finite dimensional $U(\mathfrak{g})$ module $E$ one has $\text{Hom}_u(M(w_1 \mu), (E \otimes (M(w_2 \lambda)/M(s_\alpha w_2 \lambda)))) = 0$. To establish this it is enough to show that $L(w_1 \mu)$ is not a subquotient of $p_\mu(E \otimes M(w_2 \lambda)/M(s_\alpha w_2 \lambda))$. Now by 1.11 and the invariance of $\Omega(E)$ under $W$ one has

$$[E \otimes (M(w_2 \lambda)/M(s_\alpha w_2 \lambda))] = \sum_{\nu \in \Omega(E)} ([M(w_2(\lambda + \nu))] - [M(s_\alpha w_2(\lambda + \nu))])(\text{dim } E,\nu),$$

and so

$$(*) \quad [p_\mu(E \otimes M(w_2 \lambda)/M(s_\alpha w_2 \lambda))] =$$

$$= \sum_{w \in W_\lambda} (\text{dim } E_{w \mu - \lambda})([M(w_2 w \mu)] - [M(s_\alpha w_2 w \mu))].$$

Through the hypothesis $s_\alpha w_1 > w_1$, one has by ([10], 5.19) that

$$(**) \quad [M(w_2 w \mu): L(w_1 \mu)] = [M(s_\alpha w_2 w \mu): L(w_1 \mu)].$$

Combined with (*) this establishes the assertion of the lemma.

**Remarks.** A technically easier proof of (**) follows from ([11],
2.16) and ([3], 4.5 (6)). Again the analysis of ([14], 5.4) can be combined with the operators of coherent continuation to give an alternative proof of the fact that $L(w_1\mu)$ is not a subquotient of $E \otimes (M(w_2\lambda)/M(s_0w_2\lambda))$.

3.3: Let $W$ be a Coxeter group with $S$ the corresponding set of simple reflections and length function $l(\cdot)$. It is well-known that there exists an associative product $*$ on $W$ uniquely defined through

$$w*w' = ww' \quad \text{if} \quad l(w) + l(w') = l(ww'),$$

$$s*s = s \quad \text{if} \quad s \in S.$$

(Up to a sign, these are the defining relations for the generators of the “singular Hecke algebra” obtained say from ([19], Sect. 1) by putting $q = 0$.)

**Lemma:** For all $w, y \in W$, $s \in S$, one has

(i) $s*w = \begin{cases} sw : sw > w, \\ w : sw < w. \end{cases}$

(ii) $w*s = \begin{cases} ws : ws > w, \\ w : ws < w. \end{cases}$

(iii) $(w*y)^{-1} = y^{-1}*w^{-1}.$

(iv) $w*w' \succeq w', w.$

The top lines of (i), (ii) are immediate from the definition of $*$. For the bottom line in say (ii), set $w' = sw$. Then $sw' > w'$ and so $s*w = s*(s*w') = (s*s)*w' = s*w' = w$.

We prove (iii) by induction on $l(w)$. For $l(w) = 0, 1$, it follows from (i), (ii). Otherwise write $w = s*z : l(z) \leq l(w)$. Then $(w*y)^{-1} = ((s*z)*y)^{-1} = (s*(z*y))^{-1} = ((z*y)^{-1}*s) = (y^{-1}*z^{-1}*s) = y^{-1}*z^{-1}*s = y^{-1}*w^{-1}$. (iv) follows from (i), (ii).

3.4: Fix $-\lambda, -\mu \in \mathfrak{h}^*$ dominant. For all $w_1, w_2 \in W_\lambda$, one has from 3.3 (iv) that $w_2^{-1}w_1w_\lambda \geq w_1w_\lambda$ and so $w_3 := (w_2^{-1}w_1w_\lambda)w_\lambda \leq w_1$.

**Proposition:** Assume $-\lambda, -\mu \in \mathfrak{h}^*$ dominant with $\lambda - \mu \in P(R)$. Given $w_1, w_2 \in W_\lambda$, define $w_3 \in W_\lambda$ as above. Then the $U$-module
homomorphism of $L(M(w_1\lambda), M(w_2\mu))$ into $L(M(w_3\lambda), M(w_2\mu))$ defined by restriction is injective with image $L(M(w_3\lambda), M(\mu))$.

The assertion is clear for $w_2 = 1$. If $w_2 \neq 1$, choose $\alpha \in B_\lambda$ such that $s_\alpha w_2 < w_2$. If $s_\alpha w_1 > w_1$, then by 3.2 the natural embedding $L(M(w_1\lambda), M(s_\alpha w_2\mu)) \hookrightarrow L(M(w_1\lambda), M(w_2\mu))$ is surjective. If $s_\alpha w_1 < w_1$, then by ([13], 6.1) the map of $L(M(w_1\lambda), M(w_2\mu))$ into $L(M(s_\alpha w_1\lambda), M(w_2\mu))$ defined by restriction is injective and so, by 3.2 again, we obtain an embedding of $L(M(s_\alpha w_1\lambda), M(w_2\mu))$ into $L(M(s_\alpha w_1\lambda), M(s_\alpha w_2\mu))$. In either case we obtain an embedding of $L(M(w_1\lambda), M(w_2\mu))$ into $L(M((s_\alpha w_1)w_1\lambda), M(s_\alpha w_2\mu))$, and so by induction an embedding into $L(M(w_3\lambda), M(\mu))$. On the other hand we can take $\alpha \in B_\lambda$ such that $s_\alpha w_1 < w_1$. Then a similar argument gives an embedding of $L(M(s_\alpha w_1\lambda), M((s_\alpha w_2)\mu))$ into $L(M(w_1\lambda), M(w_2\mu))$. By induction this gives an embedding of $L(M(w_1\lambda), M((w_1 w_2^{-1} w_2)\mu))$ into $L(M(w_1\lambda), M(w_2\mu))$ which we saw above further embeds in $L(M(w_3\lambda), M(\mu))$, both maps having been defined by restriction.

Now by 3.3, we have $(w_2^{-1} w_1 \lambda)^{-1} = w_\lambda w_1^{-1} w_2$ and so by 1.15 the combined map is surjective. Consequently the second map must also be surjective, proving the assertion.

3.5: Assume $-\lambda, -\mu \in \mathfrak{h}^*$ dominant with $-\lambda - \mu \in P(R)$ and fix $B' \subset B_\lambda$.

**COROLLARY:** For each $w \in W_B$ and each finite dimensional $U(\mathfrak{g})$ module $E$, one has

$$\dim \text{Hom}_E(Q, M(w\mu)) = \dim \text{Hom}_E(Q, \delta M(w\mu)),$$

where $Q = E \otimes M(w_B\lambda)$.

From $l(ww_\lambda) = l(w_\lambda) - l(w)$ for all $w \in W_\lambda$ and an analogous assertion for $W_B'$, we obtain $l(w^{-1} w_B' w_\lambda) = l(w^{-1}) + l(w_B' w_\lambda)$. Since $w_\lambda^2 = 1$, it follows from the definition of $*$ that $(w^{-1} w_B' w_\lambda) w_\lambda = w^{-1} w_B$, so by 3.4, 3.1 one has the isomorphisms $L(M(w_B'\lambda), M(\mu)) \cong L(M(w^{-1} w_B' w_\lambda), M(\mu)) \cong L(-\mu, -w^{-1} w_B' \lambda)$. On the other hand, by 1.9 we have $L(M(w_B'\lambda), \delta(M(w\mu))) \cong L(-w_\mu, -w_B' \lambda)$. Combined with 1.7 and 1.12, these isomorphisms imply the assertion of the corollary.

3.6: Take $\lambda, \mu, w_1, w_2, \alpha$ as in 3.2.

**LEMMA:**

(i) $L(M(w_2\lambda)/M(s_\alpha w_2\lambda), \delta M(w_1\mu)) = 0$. 
(ii) \( L(L(w_2\lambda), L(w_1\mu)) = 0. \)

(iii) The map of \( L(-w_1\mu, -w_2\lambda) \) into \( L(-w_1\mu, -s_n w_2\lambda) \) defined by restriction is injective.

For (i), observe that \( L(w_1\mu) \) is the unique simple submodule of \( \delta M(w_1\mu) \), so it suffices to show for any finite dimensional module \( E \) that

\[
[E \otimes (M(w_2\lambda)/M(s_n w_2\lambda)) : L(w_1\mu)] = 0.
\]

This obtains by an argument parallel to 3.2. Hence (i). Through the embedding \( \text{Hom}_g(E \otimes L(w_2\lambda), L(w_1\mu)) \subseteq \text{Hom}_g(E \otimes (M(w_2\lambda)/M(s_n w_2\lambda)), L(w_1\mu)) \) and (*) we obtain (ii). Recalling 1.9, (i) gives (iii).

Remark. When \( \alpha \in B \), the result in (iii) is due to Zelobenko (see [8], Lemmes 4, 5).

3.7: We conclude this section with a result of obvious importance which by virtue of ([4], 2.14) is a far reaching generalization of 3.6 (ii). We start with the following

**Lemma:** For all \( \lambda, \mu, \nu \in \mathfrak{h}^* \) one has

(i) \( L(L(\mu), L(\lambda)) \neq 0 \Leftrightarrow L(L(\lambda), L(\mu)) \neq 0. \)

(ii) \( L(L(\mu), L(\lambda)) L(L(\nu), L(\mu)) = 0 \) implies that one of these modules must vanish.

(i) follows from the isomorphism \( \delta (L(\mu)) \cong L(\mu) \). (ii) follows from the simplicity of \( L(\mu) \).

3.8: **Proposition:** Let \( \lambda \in \mathfrak{h}^* \) be dominant and regular. Then for each \( w, y \in W_\lambda \), one has

\( L(L(w\lambda), L(y\lambda)) \neq 0 \Leftrightarrow J(w^{-1}\lambda) = J(y^{-1}\lambda). \)

Suppose \( L(L(w\lambda), L(y\lambda)) \neq 0 \). Then there exists a finite dimensional \( U(\mathfrak{g}) \) module \( E \) such that \( \text{Hom}_g(L(w\lambda), L(y\lambda) \otimes E) \neq 0 \) and so \( L(w\lambda) \) is a submodule of \( L(y\lambda) \otimes E \). It follows that \( L(M(\lambda), L(w\lambda)) \) is a submodule of \( L(M(\lambda), L(y\lambda) \otimes E) \). Hence the right annihilator of \( L(M(\lambda), L(w\lambda)) \) contains the right annihilator \( J \) of \( L = L(M(\lambda), L(y\lambda) \otimes E) \). Since \( L \) is isomorphic to \( L(M(\lambda), L(y\lambda)) \otimes (E^+ \otimes \mathbb{C}) \), it
follows that \( J \) coincides with the right annihilator of \( L(M(\lambda), L(y\lambda)) \). By ([14], 4.7, 4.12) this gives \( J(w^{-1}\lambda) \supseteq J(y^{-1}\lambda) \). By 3.7 (i), interchange of \( w, y \) gives the reverse inclusion.

Suppose \( J(w^{-1}\lambda) = J(y^{-1}\lambda) \). By ([8], Prop. 8) \( U(g)/J(w^{-1}\lambda) \) has a unique \( U \) submodule which is furthermore isomorphic to some \( V(-\sigma\lambda, -\lambda) \) with \( \sigma \) an involution of \( W_\lambda \). By ([14], 4.12) it is clear that \( J(\sigma\lambda) = J(w^{-1}\lambda) \). After Vogan ([24], 3.5) there exists a finite dimensional \( U(g) \) module \( E \) such that \( U(g)/J(w^{-1}\lambda) \) (and hence \( V(-\sigma\lambda, -\lambda) \)) is a submodule of \( V(-w^{-1}\lambda, -\lambda) \otimes (C \otimes E) \). From 1.12(i), we have \( V(-w^{-1}\lambda, -\lambda) \cong V(-\lambda, -w\lambda) \), and so \( V(-\sigma\lambda, -\lambda) \) is a submodule of \( V(-w\lambda, -\lambda) \otimes (E \otimes C) \). Then by 1.12 (iii), \( L(M(\lambda), L(\sigma\lambda)) \) is a submodule of \( L(M(\lambda), L(w\lambda)) \otimes (E \otimes C) \) which is isomorphic to \( L(M(\lambda), L(w\lambda) \otimes E') \). The resulting injection \( i : L(M(\lambda), L(\sigma\lambda)) \to L(M(\lambda), L(w\lambda) \otimes E') \) must come by 1.16(iii) by applying \( T \) to an injection \( L(\sigma\lambda) \to L(w\lambda) \otimes E' \). Hence \( L(L(\sigma\lambda), L(w\lambda)) \neq 0 \). Interchanging \( w, y \) and using 3.7 gives \( L(L(w\lambda), L(y\lambda)) \neq 0 \), as required.

4. Exactness of the functor \( L(M(w_B\lambda), \cdot) \).

In this section we fix \(-\lambda \in h^* \) dominant and \( B' \subset B_\lambda \). Set \( \Lambda = \{ \mu \in \lambda + P(R) : -\mu \text{ is dominant} \} \).

4.1: Let \( \mathcal{O}^B_\Lambda \) denote the subcategory of \( \mathcal{O} \) consisting of all those modules (necessarily of finite length) whose simple factors are amongst the \( L(w\mu) : \mu \in \Lambda, w \in W_B \). By ([6], Thm. 4(iv)) it follows that the \( M(w_B\mu) : \mu \in \Lambda \) are projective in \( \mathcal{O}^B_\Lambda \). On the other hand \( \mathcal{O}^B_\Lambda \) is not closed under tensoring with finite dimensional \( U(g) \) modules. Nevertheless we have the

**Proposition:** Suppose \( M_1, M_2, M_3 \in \mathcal{O} \), with

\[
0 \to M_1 \to M_2 \to M_3 \to 0
\]

exact. Then

\[
0 \to L(M(w_B\lambda), M_1) \to L(M(w_B\lambda), M_2) \to L(M(w_B\lambda), M_3) \to 0
\]

is exact.

This is proved in sections 4.2, 4.3.

4.2: A module \( M \in \mathcal{O} \) is said to admit a \( p \)-filtration if it has a finite filtration with factors isomorphic to Verma modules. For
example, by ([7], 7.6.14) $E \otimes M(\mu)$ ($E$ finite dimensional, $\mu \in \mathfrak{h}^*$) has a $p$-filtration.

**Lemma**: Suppose $Q \in 0\mathfrak{h} \mathfrak{o}$ admits a $p$-filtration. Then for all $\mu \in \mathfrak{h}^*$, $k > 0$, one has

$$\text{Ext}^k(Q, \delta M(\mu)) = 0.$$

It is enough to prove the assertion for $Q$ a Verma module, say $M(\nu): \nu \in \mathfrak{h}^*$. By 1.14, $\text{Ext}^k(M(\nu), \delta M(\mu)) = (H_k(\mathfrak{n}^-, M(\mu))_{\nu^\alpha})^*$, up to isomorphism, so the assertion follows from the fact that $M(\mu)$ is $U(\mathfrak{n}^-)$ free.

4.3: Let $E$ be a finite dimensional $U(\mathfrak{g})$ module and set $Q = E \otimes M(w_\theta \lambda)$ and fix $\mu \in \Lambda$. We show that $\text{Ext}^1(Q, L(\mu)) = 0: y \in W_{\beta'}$ by induction on $l(y)$. This will establish 4.1. When $l(y) = 0$, that is $y = 1$, we have $L(\mu) \cong M(\mu) \cong \delta M(\mu)$ and so the assertion follows from 4.2. Now fix $w \in W_{\beta'}$ and suppose the assertion proved for all $y \in W_{\beta'}$ such that $l(y) < l(w)$. In particular this gives

(1) $$\text{Ext}^1(Q, \overline{M(w\mu)}) = 0.$$

From the exact sequence

$$0 \to L(w\mu) \to \delta M(w\mu) \to \overline{\delta M(w\mu)} \to 0$$

and 4.2 we obtain an exact sequence

(2) $$0 \to \text{Hom}(Q, L(w\mu)) \to \text{Hom}(Q, \delta M(w\mu)) \to \text{Hom}(Q, \overline{\delta M(w\mu)}) \to$$

$$\to \text{Ext}^1(Q, L(w\mu)) \to 0.$$

From the exact sequence

$$0 \to \overline{M(w\mu)} \to M(w\mu) \to L(w\mu) \to 0,$$

and (1) we obtain an exact sequence

(3) $$0 \to \text{Hom}(Q, \overline{M(w\mu)}) \to \text{Hom}(Q, M(w\mu)) \to \text{Hom}(Q, L(w\mu)) \to 0.$$

Combining (2) and (3) gives
The first term in curly brackets vanishes by the induction hypothesis and the fact that $\delta M(w_\mu)$ and $M(w_\mu)$ have the same composition factors which are amongst the $L(y \mu) : y < w$. The second term vanishes by 3.5.

4.4: Let $M$ be a simple $U(g)$ module. The natural action of $U(g)$ in $M$ defines an embedding of $U(g)/\text{Ann}M$ into $\text{Hom}(M, M)$ and in fact the image lies in the $f$-finite part $L(M, M)$. Kostant has asked if the image is exactly $L(M, M)$. This is generally false ([5], 6.5; [13], 9.3, 9.4); yet it is quite important to ascertain when it does hold, especially for highest weight modules.

**THEOREM:** For each $-\lambda \in \mathfrak{h}^*$ dominant and $B' \subset B$, one has

$$U(g)/J(w_B \lambda) = L(L(w_B \lambda), L(w_B \lambda)).$$

By 4.1, $L(M(w_B \lambda), L(w_B \lambda))$ is a quotient of $L(M(w_B \lambda), M(w_B \lambda))$ and the latter by ([14], 3.6) identifies with $U(g)/\text{Ann}M(w_B \lambda)$. Since $L(L(w_B \lambda), L(w_B \lambda))$ is a submodule of $L(M(w_B \lambda), L(w_B \lambda))$, this proves the theorem.

**Remark.** In the special case when $B' \subset B$ the above result is due to Conze-Berline and Duflo ([5], 2.12, 6.3). Their proof does not admit further generalization since it uses induction from the parabolic subalgebra defined by $B'$. When $B' = B_\lambda$ with $\lambda$ regular, the result is noted in ([12], 5.7).

4.5: For $\mu \in \mathfrak{h}^*$, we write $A_\mu : = U(g)/J(\mu)$, $A'_\mu : = L(L(\mu), L(\mu))$. The embedding of $A_\mu$ into $A'_\mu$ extends ([13], 4.3) to an embedding of Fract $A_\mu$ into Fract $A'_\mu$. In order to compute the scale factors in the Goldie polynomial defined by the Goldie rank of $A_\mu$ (see [15], 5.12) it is useful to know when Fract $A_\mu = \text{Fract} A'_\mu$.

Since $J(\mu)$ is a prime ideal, $A_\mu$ admits a unique simple submodule $V_\mu$ which furthermore ([8], Prop. 4) has annihilator $J(\mu) = J(\mu) \otimes U(g) + U(g) \otimes J(\mu)$. We let $l_0(A'_\mu)$ denote the number of factors in a $U$ composition series of $A'_\mu$ having annihilator $J(\mu)$.

**Lemma:** $l_0(A'_\mu) = 1 \Leftrightarrow \text{Fract} A_\mu = \text{Fract} A'_\mu$. 

\[
\dim \text{Ext}^1(Q, L(w_\mu)) = \{ \dim \text{Hom}(Q, \delta M(w_\mu)) - \\
\quad - \dim \text{Hom}(Q, M(w_\mu)) \} \\
- \{ \dim \text{Hom}(Q, \delta M(w_\mu)) \\
- \dim \text{Hom}(Q, M(w_\mu)) \}. 
\]
If $M$ is a finitely generated left $U(g)$ module, let $\text{Dim } M$ denote its Gelfand-Kirillov dimension over $U(g)$ as defined in ([17], 2.1). Now let $M$ be a simple $U$ subquotient of $A_\mu'$, which by $k$-finiteness is a finitely generated left $U(g)$ module. By ([17], 1.4, 3.1 and 3.3 Remark) we have $\text{Dim } M = \text{Dim} (U(g)/\text{Ann} U(g)M)$. Since $\text{Ann} U(g)M \supseteq \tilde{J}(\mu)$, it follows from the primeness of $\tilde{J}(\mu)$ that $\text{Ann} U(g)M = \tilde{J}(\mu)$ if and only if $\text{Dim } M = \text{Dim } V_\mu$. A similar argument on the right, taking account of ([8], Prop. 4), shows that $\text{Ann } M = J_\mu$ if and only if $\text{Dim } M = \text{dim } V_\mu$.

Let $S$ denote the set of regular elements of $A_\mu$. Since $A_\mu'$ is $\mathfrak{t}$-finite and has finite length as a $U$ module, it follows from ([18], 3.7) that $S$ is an Ore subset of the regular elements of $A_\mu'$ and $S^{-1}A_\mu' = \text{Fract } A_\mu'$. Hence it remains to show that $S^{-1}M = 0$ if and only if $\text{Dim } M < \text{Dim } V_\mu = \text{Dim } U(g)/J(\mu)$. This follows from ([16], 5.1, 5.2(ii)).

4.6: Retain the above notation and take $v \in \mu + P(R)$ in the upper closure of the $W_\mu$ facette containing $\mu$ (for this see [11], 2.6).

**LEMMA:** Set $H^*_\mu = R^*_\mu S^*_\mu$ (notation 1.13). Then

(i) $H^*_\mu A^{'}_\mu = A^{'}_\mu$.

(ii) $H^*_\mu A_v = A_v$.

(iii) $I_0(A^{'}_\mu) = I_0(A_v)$.

(iv) $H^*_\mu V_\mu = V_v$.

(v) $\text{Fract } A_\mu = \text{Fract } A^{'}_\mu \Leftrightarrow \text{Fract } A_v = \text{Fract } A_v$.

By ([11], 2.10, 2.11) we have under the hypothesis of the lemma the isomorphisms $T^{*}_\mu L(\mu) \cong L(v)$ (resp. $T^{*}_\mu M(\mu) \cong M(\nu)$) and so by 1.13 the isomorphisms $H^*_\mu A^{'}_\mu = A^{'}_\mu$ (resp. $H^*_\mu L(M(\mu), M(\mu)) = L(M(\nu), M(\nu))$). Hence (i). Since $L(M(\mu), M(\mu)) \subseteq (U(g)/\text{Ann } M(\mu))$ by ([13], 6.4) and $A_\mu$ is the image of $U(\mu)/\text{Ann } M(\mu)$ in $A^{'}_\mu$, exactness of $H^*_\mu$ gives (ii). Now let $K$ be a simple $U$ subquotient of $A_\mu'$. Then by 1.12, $K$ is isomorphic to some $L(M(\lambda_1), L(\lambda_2)) : \lambda_1, \lambda_2 \in \mathfrak{h}^*$ with $\lambda_1$ dominant. Furthermore from the action of the centre of $U$ it easily follows that $\lambda_1, \lambda_2 \in W_\mu$. Then from ([11], 2.10, 2.11) and 1.12, 1.13, it follows that either $H^*_\mu K = 0$ or is a simple subquotient of $A_v$; then, by an argument similar to that given in ([4], 2.11), $H^*_\mu K$ has the same Gelfand-Kirillov dimension as $K$. Moreover by a trivial extension of ([4], 2.4), whether or not $H^*_\mu K = 0$ depends only on $\text{Ann } K$. Hence (iii), (iv). Finally (v) follows from (iii).

4.7: **COROLLARY:** Fix $-\lambda \in \mathfrak{h}^*$ dominant, regular and take $B' \subseteq B_\lambda$. Then for each $\alpha \in B'$, one has

$$\text{Fract } U(g)/J(w_B s_\alpha \lambda) = \text{Fract } L(L(w_B s_\alpha \lambda), L(w_B s_\alpha \lambda)).$$
With respect to $\lambda$, $\alpha$ define $\nu_\alpha$ as in 2.1. Then apply 4.6(v) to 4.4 with $\mu = w_B s_\alpha \lambda$, $\nu = w_B s_\alpha (\lambda - \nu_\alpha) = w_B (\lambda - \nu_\alpha)$.

4.8: For each $w \in W_\lambda$, set $S(w) = \{\alpha \in R^+_\lambda : w \alpha \in R_\lambda^+\}$. Define an ordering $\subseteq$ on $W_\lambda$ through $y \subseteq w$ given $S(y^{-1}) \subseteq S(w^{-1})$. One checks that $y \subseteq w$ implies $y \leq w$ and that $y \subseteq w \iff (y^{-1} w w_\lambda) w_\lambda = y^{-1} w$. Thus the obvious generalization of 4.1 shows that $L(M(w_\lambda), \cdot)$ is exact when restricted to the subcategory of $\mathcal{C}_\lambda$ of all modules with simple factors $L(y_\lambda) : y \in W_\lambda$ where $y$ satisfies $y' \leq y \Rightarrow y' \subseteq w$. Since $s_\alpha \leq y$, $\forall \alpha \in \supp y$ it follows that $\supp y \subseteq S(w^{-1})$, that is $y \in W_{B'}$ where $B' = B_\lambda \cap S(w^{-1})$. Though this rather weak generalization is probably not the best the corresponding assertion with $\subseteq$ replaced by $\leq$ is false for it implies that Kostant's problem has always a positive answer (which is false by ([5], 6.5)) for simple highest weight modules. This is in spite of the fact that $\Ext^1(M(w_\mu), L(y_\lambda)) = 0$ if $w \geq y$.

5. Main theorem

5.1: Fix $-\lambda$, $-\mu \in h^*$ dominant, $\mu$ regular, with $\lambda - \mu \in P(R)$. Take $B' \subset B_\lambda$. Let $s = l(w_B)$, and for each $j \in \{0, 1, 2, \ldots, s\}$ set $D_j = \bigoplus_{w \in W_B} L(M(w_B \lambda), M(w_\mu))$. Finally put

$$ L = L(M(w_B \lambda), L(w_B \mu)). $$

**Theorem:** There is a long exact sequence

$$ 0 \rightarrow D_0 \rightarrow D_1 \rightarrow \cdots \rightarrow D_3 \rightarrow L \rightarrow 0. $$

Apply 4.1 to 2.6.

5.2: When $\lambda = \mu$ in 5.1, we have that $L = U(\mathfrak{g})/J(w_B \lambda)$ by 4.4. Again by 3.4 one has that

$$ D_3 = L(M(w_B \lambda), M(w_B \lambda)) = U(\mathfrak{g})/J(\lambda), $$

$$ D_{3-1} = \bigoplus_{\alpha \in B'} L(M(w_B \lambda), M(w_B s_\alpha \lambda)) = \bigoplus_{\alpha \in B'} L(M(s_\alpha \lambda), M(\lambda)) = \bigoplus_{\alpha \in B'} J(s_\alpha \lambda). $$

In view of the definition of the maps in 5.1 this gives the
**Corollary:** For each $B' \subset B$, one has

$$\sum_{\alpha \in B'} J(s_\alpha \lambda) = J(w_{B'} \lambda).$$

**Remark.** When $B' \subset B$, this result is due to Duflo ([8], Prop. 12). When $B' = B$, it is just ([12], 4.4, 4.5). By ([12], 4.5) it implies that $J(w_{B'} \lambda)/J(\lambda)$ is an idempotent ideal and has exactly $\text{card } B'$ distinct maximal submodules.

5.3: Again take $\lambda = \mu$ in 5.1. Then by 3.1, 3.4

$$D_l = \bigoplus_{w \in W_k} L(M(w_{B'} \lambda), M(w \lambda)) = \bigoplus_{w \in W_k} L(\lambda, -w^{-1} w_{B'} \lambda).$$

Combined with 1.12 this gives the following multiplicity formula for simple $\mathfrak{t}$-submodules of $U(g)/J(w_{B'} \lambda)$.

**Corollary:** Fix $-\lambda \in \mathfrak{h}^*$ dominant and regular. Then for each $\nu \in P(R)$ one has

$$\dim \text{Hom}_\mathfrak{t}(E(\nu), U(g)/J(w_{B'} \lambda)) = \sum_{w \in W_{B'}} (\det w) \dim E(\nu)_{\lambda - w\lambda}.$$

**Remarks.** When $B' \subset B$, Conze-Berline and Duflo ([5], 2.12, 6.3) gave a formula for the left hand side above. Their formula obtains from 4.4 and Frobenius reciprocity with respect to induction from the parabolic subalgebra defined by $B'$. The equivalence of these two formulae imply a combinatorial statement concerning weight subspaces of finite dimensional $U(g)$ modules.

6. Duality

6.1: Some of our results can be given a dual form with the help of the following. Fix $\lambda, \mu \in \mathfrak{h}^*$ with $\lambda - \mu \in P(R)$. Then (see 6.3) $L(\lambda, \mu) \times L(-\lambda, -\mu)$ admits a bilinear form $\langle, \rangle$ satisfying

$$\langle (a \otimes b)x, y \rangle = \langle x, (\hat{a} \otimes \hat{b})y \rangle,$$

for all $x \in L(\lambda, \mu)$, $y \in L(-\lambda, -\mu)$, $a, b \in U(g)$. For each $\sigma, \tau \in k^*$, $\langle, \rangle$ restricts to a $\mathfrak{t}$-invariant bilinear form on $L(\lambda, \mu)_{\sigma} \times L(-\lambda, -\mu)$, which is non-degenerate if $\tau$ is contra-gradient to $\sigma$ and zero otherwise.
6.2: To apply 6.1 to the comparison of mappings of principal series modules we start with the following observation. Suppose $\lambda, \lambda' \in \mathfrak{h}^*$ are chosen so that we have an embedding of $M(\lambda')$ into $M(\lambda)$. Then there exists $a \in U(\mathfrak{g})_{\lambda' - \lambda}$ such that $ae_\lambda = e_{\lambda'}$. (Furthermore $a$ is unique up to a non-zero scalar which can be fixed canonically as follows. First, under the above hypothesis, $\lambda - \lambda'$ is a non-negative integral linear combination of the $\alpha \in B$ (with say coefficients $k_\alpha$) and second, with respect to the canonical filtration of $U(\mathfrak{g})$, the leading term of $a$ is just

$$\prod_{\alpha \in B} X^{k_\alpha}$$

up to a non-zero scalar ([21], Lemma 1). Fix this scalar to be one.)

**Lemma:** There exists an embedding of $M(-\lambda)$ into $M(-\lambda')$ and $ae_{-\lambda'} = se_{-\lambda}$, with $s = \pm 1$.

Fix $\alpha \in B$. Then $[X_\alpha, a]e_\lambda = 0$ and so $[X_\alpha, a] \in \text{Ann } e_\lambda = U(\mathfrak{g})\mathfrak{n}^+ + \sum_{\beta \in B} U(\mathfrak{g})(H_\beta - (\lambda - \rho, H_\beta))$. Since $a \in U(\mathfrak{g})_{\lambda' - \lambda}$ and $\alpha$ is simple, we have in fact the more precise result, namely

$$[X_\alpha, a] \in U(\mathfrak{g})_\eta(H_\alpha - (\lambda - \rho, H_\alpha)),$$

where $\eta = \lambda' - \lambda + \alpha$. Hence

$$[X_\alpha, \tilde{a}] \in (H_\alpha + (\lambda - \rho, H_\alpha))U(\mathfrak{g})_\eta = U(\mathfrak{g})_\eta(H_\alpha + (\lambda + \eta - \rho, H_\alpha)).$$

Yet $-(\lambda + \eta - \rho, H_\alpha) = -(\lambda' + \rho, H_\alpha)$, and so $X_\alpha e_{-\lambda'} = [X_\alpha, \tilde{a}] e_{-\lambda'} = 0$. Since $\alpha$ was arbitrary, it follows that $\tilde{a}e_{-\lambda'}$ is a highest weight vector (necessarily non-zero) of weight $(\lambda' - \lambda) - (\lambda' + \rho) = -(\lambda + \rho)$ and hence proportional to the canonical generator $e_{-\lambda}$ of $M(-\lambda)$ embedded in $M(-\lambda')$ “canonically” as above. Comparison of leading terms shows that the constant of proportionality is just $(-1)^{\sum k_\alpha}$.

**Remark.** Of course the first part also obtains from ([7], 7.6.23). When $B_\lambda \subset B$, the second part can also be derived from ([7], 7.8.8).

6.3: The bilinear form referred to in 6.1 has been defined purely algebraically in ([7], 9.6.9) for the case $\lambda = \mu$. We describe the modifications needed in the general case. In this we denote by $t, u, v$ elements of $U(\mathfrak{t})$, $a, b$ elements of $U(\mathfrak{g})$, $\theta$ an element of $U(\mathfrak{t})^*$, $f$ an element of $L := L(M(\lambda) \otimes M(\mu))^*$.
Define an action of $U(t) \otimes U(t)$ on $U(t)^*$ through $((u \otimes v) \cdot \theta(t)) = \theta(\tilde{u}tv)$ and set

$$U(t)_l = U(t) \otimes \mathbb{C}, \quad U(t)_r = \mathbb{C} \otimes U(t).$$

By ([7], 2.7.12) the sum of the simple finite dimensional $U(t)_l$ submodules of $U(t)^*$ coincides with the sum of the simple finite dimensional $U(t)_r$ submodules of $U(t)^*$, and we denote this subspace by $L(U(t)^*)$. Let $\epsilon : U(t) \to \mathbb{C}$ be the augmentation. $Ce$ occurs as the unique one dimensional subrepresentation of $L(U(t)^*)$. Let $\varphi_0 : L(U(t)^*) \to \mathbb{C}$ be the linear form on $L(U(t)^*)$ which takes the value 1 on $\epsilon$ and zero on the $U(t) \otimes U(t)$ stable complement of $Ce$ in $L(U(t)^*)$.

Now for each $v \in \mathfrak{b}^*$, define $T_v = \{ \theta \in U(t)^* : \theta(u_j(H)) = (v, H)\theta(u) \}$ for all $H \in \mathfrak{h}$, $u \in U(t)$ which is a $U(t)_l$ module. With $\nu = \lambda - \mu$, $f \in L$, we define $\theta_f \in T_v$ through $\theta_f(u) = f(u(e_\lambda \otimes e_\mu))$. Then the map $f \mapsto \theta_f$ is a $U(t)$ module isomorphism of $L$ onto the $U(t)_l$ finite part $L(T_v)$. (For this see [7], 5.5.8 or [8], Sect. I.2). Now take $\lambda' \in \mathfrak{b}^*$ such that $M(\lambda') \subset M(\lambda)$ and $a \in U(\mathfrak{n}_-)_{\lambda' - \lambda}$ as in 6.2. Then for all $f \in L$, we have $((1 \otimes j(a)) \cdot \theta_f)(u) = \theta_f(u_j(a)) = f(u_j(a)(e_\lambda \otimes e_\mu)) = f(u(ae_\lambda \otimes e_\mu))$, since $a \in U(\mathfrak{n}_-)$ and $'Xe_\mu = 0$ for all $X \in \mathfrak{n}_-$. Let $\psi : L \to L': = L(M(\lambda') \otimes M(\mu))$ be defined by restriction. Set $\nu' = \lambda' - \mu$, and define for any $f' \in L'$ the element $\theta_{f'} \in T_{\nu'}$ as above. Then for all $f \in L$, we have $\theta_{\psi(f)}(u) = \psi(f)(u(e_\lambda \otimes e_\mu)) = f(u(e_\lambda \otimes e_\mu))$. Since $ae_\lambda = e_{\nu'}$, this gives

(*) \quad (1 \otimes j(a)) \cdot \theta_f = \theta_{\psi(f)}.

Similarly let $\psi' : L(M(-\lambda') \otimes M(-\mu))^* \to L(M(-\lambda) \otimes M(-\mu))^*$ be defined by restriction. Then for each $g' \in L(M(-\lambda') \otimes M(-\mu))^*$, we have $\theta_{g'} \in T_{-\nu'}$, $\theta_{\psi'(g')} \in L(T_{-\nu'})$ and by (*) and 6.2 we get

$$((1 \otimes j(a)) \cdot \theta_{g'}) = s \theta_{\psi'(g')}.$$
one checks that the form \( (g, f) \mapsto \varphi_0(\theta_g \theta_f) \) on \( L(\lambda, \mu) \times L(-\lambda, -\mu) \) has the properties claimed in 6.1. Furthermore with respect to the above maps we have the

**Lemma:** The diagram

\[
\begin{array}{ccc}
L(\lambda', \mu) \times L(-\lambda, -\mu) & \xrightarrow{1 \times \psi} & L(\lambda', \mu) \times L(-\lambda', -\mu) \\
\downarrow s\psi' \times 1 & & \downarrow \\
L(\lambda, \mu) \times L(-\lambda, -\mu) & \longrightarrow & C
\end{array}
\]

commutes. That is \( s(\psi'(g'), f) = (g', \psi(f)) \).

Indeed

\[
s(\psi'(g'), f) = \varphi_0(s\theta_{\psi'(g')} \theta_f) = \\
\varphi_0(((1 \otimes j(\hat{a})) \cdot \theta_g) \theta_f) = \varphi_0(\theta_{\xi}(1 \otimes j(a)) \cdot \theta_f) = \\
\varphi_0(\theta_g \theta_{\psi(f)}) = (g', \psi(f)).
\]

**Remark.** A similar result holds for the second variable.

6.4: Take \( \lambda, \mu, w_1, w_2, \alpha \) as in 3.2. Under the hypothesis of 3.2, it follows that \( M(-w_2\lambda) \) is a submodule of \( M(-s_{w_2}\lambda) \). Applying the analogue of 6.3 with respect to second variable to 3.6 we obtain

**Corollary:** The map \( L(w_1\mu, s_{w_2}\lambda) \rightarrow L(w_1\mu, w_2\lambda) \) defined by restriction is surjective.

**Remark.** When \( \alpha \in B \), this was given in ([4], V, 1.11).

6.5: Both 3.6 and 6.4 admit analogous assertions for the first variable. This gives the commutative diagram of restriction maps
which implies an isomorphism of $L(-w_1\mu, -w_2\lambda)$ with $L(-s_\alpha w_1\mu, -s_\alpha w_2\lambda)$. The intertwining operators of (§8, Sect. I, 2) also give an isomorphism between those modules.

Index of notation

Symbols frequently used are given below in order of appearance.

1.1 $\mathfrak{g}, \mathfrak{h}, R, R^+, B, \rho, s_\alpha, W, X_\lambda, u^+, u^-, \mathfrak{b}$.
1.2 $R_\lambda, R^{\lambda\ast}, B_\lambda, W_B, w_B, W_\lambda, w_\lambda, M(\lambda), \overline{M(\lambda)}, L(\lambda), J(\lambda), e_\lambda$.
1.3 $u, \, ^t u, U, j, t, t^\ast$.
1.4 $\mathcal{O}, Z(\mathfrak{g}), \hat{\lambda}, \mathcal{O}_\lambda, p_\lambda$.
1.5 $L(M, N), L(M \otimes N)^*, L(\lambda, \mu)$.
1.7 $\mathcal{H}$.
1.8 $\tau, M^*, \delta(M)$.
1.11 $[M], [M(\lambda) : L(\mu)]$.
1.12 $P(R), P(R)^+, P(R)^{++}, E(v)$.

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