COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 42, nº 3 (1980), p. 361-373
<http://www.numdam.org/item?id=CM_1980__42_3_361_0>

COMPOSITIO MATHEMATICA Vol. 42, Fasc. 3, 1981, pag. 361–373

Printed in the Netherlands

CHARACTERIZATIONS OF CERTAIN SINGULARITIES OF A BRANCHED COVERING*

Nadia Chiarli

Introduction

Let X, Y be locally noetherian schemes, where Y is normal irreducible and X is reduced, and let $\phi : X \to Y$ be a finite covering of degree n (see definition 1.4). The problem is: how much ramification is allowed in order for X to have nice singularities, in particular in order for X to be seminormal or normal?

We studied essentially the seminormality of X in [4] when $n = 2$, and in [5] when ϕ is locally monogenic of arbitrary degree and X is integral. The purpose of this paper is to give a more general answer to the problem, studying the normal case and generalizing the seminormal case in a way leading also to the unification of the results of [4] and [5].

All the results are obtained by assuming Y to be the spectrum of a discrete valuation ring (see sections 1, 2, 3): they can be globalized (see section 4) in the same way shown in [4] and [5].

In section 1 and 2 we study respectively the normality and the seminormality of X, giving characterizations for both of them in terms of the value of the discriminant sheaf at the points of Y of codimension 1, and showing the relations with the tame ramification over Y of the normalization of X (see 1.2, 1.8, 2.2, 2.7).

In section 3 we study the particular case when X is Gorenstein, and finally in section 4 we discuss the globalization of the previous results.

* This work was done during a stay at the Mathematics Department, Poona University, and at the Bhaskaracharya Pratishthana, Poona (India), under financial support of a N.A.T.O. fellowship, in the ambit of G.N.S.A.G.A. of C.N.R.

The author wishes to thank prof. S.S. Abhyankar for helpful advice.

0010-437X/81/03361-13$00.20/0

Conventions and notations

Rings are assumed to be noetherian, commutative, with identity. In the remainder of this paper, unless stated to the contrary, we make the following assumptions:

(a) A is a discrete valuation ring, with uniformizing parameter t, residue field k and valuation v;

(b) K is the fraction field of A and L is a reduced K-algebra such that $[L: K] = n$;

(c) B' is a finite A-algebra, with L as total quotient ring;

(d) B is the integral closure of A in L, finite over A;

(e) for all $\mathfrak{m}_i \in \operatorname{Max} B$, the extensions $k(\mathfrak{m}_i)/k$ are separable;

(f) if M is a sub-A-module of B, free of rank n, then $D_{M/A}$ denotes the discriminant of M over A ([19] §3, p. 59);

(g) l_A denotes the length of an A-module;

(h) for every $\mathfrak{m}_i \in \operatorname{Max} B$, put $f_i = [k(\mathfrak{m}_i): k]$ and $\Sigma_i f_i = f = l_A(B/\operatorname{rad} B)$; for every $\mathfrak{b} \in \operatorname{Max} B'$, put $g_j = [k(\mathfrak{p}_j): k]$ and $\Sigma_j g_j = g = l_A(B'/\operatorname{rad} B')$. Obviously $g \leqq f$.

Remark that from (a), (b), (c), (d) it follows that B and B' are sub-A-modules of B, free of rank n.

For general facts on ramification theory see [1], [8], [9].

1. Normality

PROPOSITION 1.1: *Let A, K, B, L be as above. Suppose M and N are two sub-A-modules of B, free of rank n. Then, for $M \subseteq N$:*

$$v(D_{M/A}) = 2\ell_A(N/M) + v(D_{N/A}).$$

PROOF. By [8] th. 1, p. 26 there exists a basis $\{b_1, \ldots, b_n\}$ of N such that $\{t^{r_1} b_1, \ldots, t^{r_n} b_n\}$ $(r_h \leqq r_{h+1})$ is a basis of M. Therefore ([8] prop. 1, p. 46):

$$\det(\mathrm{T}r_{M/A}(t^{r_i} b_i t^{r_j} b_j)) = (\det(a_{ij}))^2 \det(\mathrm{T}r_{N/A}(b_i b_j))$$

where (a_{ij}) is the matrix associated with the A-linear mapping

$$(b_1, \ldots, b_n) \rightarrow (t^{r_1} b_1, \ldots, t^{r_n} b_n).$$

Then $(\det(a_{ij}))^2 = t^{2 \Sigma_h r_h}$, and $v(D_{M/A}) = 2 \Sigma_h r_h + v(D_{N/A})$. Moreover it is easy to prove, by induction over $\ell_A(N/M)$, that $\Sigma_h r_h = \ell_A(N/M)$ and this concludes the proof.

We recall that B is said to be *tamely ramified* over A if for every $\mathfrak{m}_i \in \operatorname{Max} B$ the characteristic of k does not divide the ramification index e_i of $\mathfrak{m}_i$.

THEOREM 1.2: *If B' is normal, then $v(D_{B'/A}) \geqq n - f$.*
Moreover the following are equivalent:
(i) *B' is normal and tamely ramified over A.*
(ii) $v(D_{B'/A}) = n - f$.
(iii) $v(D_{B'/A}) \leqq n - g$.
(iv) $v(D_{B'/A}) = n - g$.

PROOF: Suppose $B' = B$. We have: $B = \Pi_l B_l$, where the B_l's are normal domains. Therefore ([19] prop. 6, p. 60 and prop. 13, p. 67):

$$D_{B_l/A} = N(\delta_{B_l/A}) = N[\Pi_s(\mathfrak{q}_{ls}^{h_{ls}})] \text{ with } h_{ls} \geqq e_{ls} - 1,$$

where $\mathfrak{q}_{ls} \in \operatorname{Max} B_1$ for every s, e_{ls} is the ramification index of q_{ls}, N is the norm and $\delta_{B_l/A}$ is the different of B_l over A.

So: $v(D_{B_l/A}) = \Sigma_s f_{ls} h_{ls} \geqq \Sigma_s f_{ls}(e_{ls} - 1)$, where $f_{ls} = [k(\mathfrak{q}_{ls}) : k]$. Now, since $D_{B/A} = \Pi_l D_{B_l/A}$ ([8] lemma 1, p. 87) we have: $v(D_{B/A}) = \Sigma_l v(D_{B_l/A}) \geqq \Sigma_l (\Sigma_s f_{ls}(e_{ls} - 1) = \Sigma_i f_i(e_i - 1) = n - f$, where the last equality follows from [3] th. 2, p. 147.

(i) → (ii) Follows by the previous arguments, after observing that, due to the tame ramification of B over A, we have $h_{ls} = e_{ls} - 1$ ([9] prop. 13, p. 67) for every l and s.

(ii) → (iii) Follows from $g \leqq f$ (see (h) above).

(iii) → (iv) By 1.1 since $v(D_{B/A}) \geqq n - f$, we have $2\ell_A(B/B') + n - f \leqq n - g$, which implies $2\ell_A(B/B') \leqq f - g$. But $\ell_A(B/B') \geqq \ell_A(B/\operatorname{rad} B) - \ell_A(B'/\operatorname{rad} B') = f - g$, so $f = g$ and $B' = B$. Moreover from $n - g = n - f \leqq v(D_{B'/A}) \leqq n - g$, it follows $v(D_{B'/A}) = n - g$.

(iv) → (i) By 1.1 and the first part of this theorem, we have $n - g \geqq 2\ell_A(B/B') + n - f$, so $2\ell_A(B/B') \leqq f - g$, which implies, by the same arguments as in (iii) → (iv), $f = g$ and B' normal. Moreover B' is tamely ramified over A: in fact (with the same notations as in the first part of this proof) we get $h_{ls} = e_{ls} - 1$ for every l and s.

COROLLARY 1.3: (i) $v(D_{B'/A}) \geqq n - g$.
(ii) $v(D_{B'/A}) = n - g$ *iff B' is normal and tamely ramified over A.*
(The lower bound $n - g$ for $v(D_{B'/A})$ shall be improved in 2.3 (i): see also remark 2.4).

PROOF: By 1.1 and 1.2 we have: $v(D_{B'/A}) \geqq 2\ell_A(B/B') + n - f \geqq 2(f-g) + n - f = n + f - 2g \geqq n - g$.

(ii) Follows from 1.2.

We will give now a geometrical interpretation of 1.2.

DEFINITION 1.4: Let X, Y be two locally noetherian schemes, with X reduced and Y integral, and denote by $X_i (i = 1, \ldots, s)$ the irreducible components of X: let $\phi : X \to Y$ be a morphism. We say that ϕ is a *finite covering* if ϕ is finite and $\phi_{|X_i} : X_i \to Y$ is surjective for every i: in this case $\phi_{|X_i}$ induces a natural embedding $k(Y) \hookrightarrow k(X_i)$ for every i. We call *degree* of ϕ the integer $\Sigma_i\ [k(X_i): K(Y)]$.

Let now $\phi : X \to Y$ be a finite covering of degree n between two schemes locally of finite type over an algebraically closed field k: assume that Y is normal and irreducible, and let $\mathfrak{D}$ be the discriminant sheaf of ϕ (see e.g. [4]). Let $Z \subset Y$ be an irreducible closed subscheme of codimension 1, with generic point $\mathfrak{q}$: assume that $Z \not\subset \operatorname{Sing} Y$ and denote by v_z the valuation associated with the discrete valuation ring $\mathfrak{D}_Z$. Let $Z_1, \ldots, Z_r$ be the irreducible components of $\phi^{-1}(Z)$ and, for each i, denote by z_i the generic point of Z_i.

PROPOSITION 1.5: *Assume that for every i we have*: $k(z_i)/k(z)$ *is separable and* $\mathcal{O}_{z_i}$ *is tamely ramified over* $\mathcal{O}_z$ (e.g. k has characteristic zero). *Then the following are equivalent*:

(i) $Z_i \not\subset \operatorname{Sing} X$ *for all* i's.

(ii) *There is a non-empty open* $U \subset Z$ *such that for every closed point* $\zeta \in U$ *the cardinality of the set* $\phi^{-1}(\zeta)$ *is equal to* $n - v_z(\mathfrak{D}_z)$.

PROOF: For every i the morphism $\phi_i = \phi_{|Z_i} : Z_i \to Z$ is a finite covering of degree $d_i = [k(z_i): k(z)]$: by [10] th. 7, p. 117 there is a non-empty open set $U_i \subset Z$ such that $d_i = \#$points of $\phi^{-1}(\alpha)$, for all closed points $\alpha \in U_i$.

Hence $\Sigma_i d_i = \#$points of $\phi^{-1}(\zeta)$, where ζ is closed and belongs to the open set $(\cap_i U_i) - [\cup_{i \neq j} \phi(Z_i \cap Z_j)]$ which is non-empty because Z as well as the Z_i's are irreducible. Now, if we denote by A the local ring of Y at z and by B' the semilocal ring of X at $z_1, \ldots, z_r$, we have by 1.3, $f = \Sigma_i d_i \leqq n - v_z(\mathfrak{D}_z)$, where the equality holds iff B' is normal, i.e. iff $Z_i \not\subset \operatorname{Sing} X$ for all i's.

COROLLARY 1.6: *Let X, Y be two algebraic curves over an algebraically closed field k of characteristic zero, and let* $\phi : X \to Y$ *be a finite covering of degree n. Let* $P \in Y$ *be a non-singular (closed) point* *and let* $\phi^{-1}(P) = \{P_1, .., P_s\}$ *(as a set). Then*:

(i) $s \geqq n - v_P(\mathfrak{D}_P)$.

(ii) $s = n - v_P(\mathfrak{D}_P)$ *iff* $P_1, .., P_s$ *are non-singular.*

LEMMA 1.7: $\hat{B} = \overline{\hat{B}'}$ *and* $[\hat{L}:\hat{K}] = [L:K]$.

PROOF: Since B is semilocal $\hat{B} = \Pi_i \hat{B}_{\mathfrak{m}_i}$, and since dim $B' = 1$ dim $B = 1$ too. Moreover, since $B_{\mathfrak{m}_i}$ is a discrete valuation ring, $\hat{B}_{m_i}$ is also a discrete valuation ring and therefore $\hat{B}$ is normal. From $B' \subset B \subset L$ it follows: $\hat{B}' = \hat{B}' \otimes_{B'} B' \subset B \otimes_{B'} \hat{B}' \subset L \otimes_{B'} \hat{B}'$ and then $\hat{B}' \subset B \subset L \otimes_{B'} \hat{B}'$, and $\hat{B}$ is finite over $\hat{B}'$. We have: $L = B'_f$ where $f \in B'$ is a non zero-divisor belonging to rad B'. Therefore: $L \otimes_{B'} \hat{B}' = B'_f \otimes_{B'} \hat{B}' = \hat{B}'_f$, where, by flatness f is a non zero-divisor in rad $\hat{B}'$. So $L \otimes_{B'} \hat{B}'$ is the total quotient ring of $\hat{B}'$. But $L \otimes_B \hat{B}' = L \otimes_B B \otimes_{B'} \hat{B}' = L \otimes_{B'} \hat{B}'$ and then $L \otimes_{B'} \hat{B}'$ is the total quotient ring of $\hat{B}$, which implies $\hat{B} = \hat{B}'$. Moreover $[\hat{L}:\hat{K}] = [(L \otimes_B \hat{B}):(K \otimes_A \hat{A})] = [(L \otimes_A \hat{A}):(K \otimes_A \hat{A})] = [L:K]$.

THEOREM 1.8: *If* B' *is normal and tamely ramified over* A, *then* $v(D_{B'/A}) \leqq n-1$.

The converse holds if either:

(i) $n = 2$, *or*

(ii) B' *is local, or*

(iii) *there exists a finite group* G *of automorphisms of* B' *such that* $B'^G = A$.

PROOF: The first claim follows from 1.2.

(i) We have either $f = 2$ or $f = 1$, so the claim follows from 1.1.

(ii) Claim first that $f = g$. By 1.1 and 1.2 we have $n - f + 2(f-g) \leqq v(D_{B/A}) + 2\ell_A(B/B') \leqq n-1$ and so $f - 2g \leqq -1$. Now, if k' is the residue field of B' we have: $f = \dim_k(B/\text{rad } B) = g \dim_{k'}(B/\text{rad } B)$. If $f \neq g$, then $\dim_{k'}(B/\text{rad } B) > 1$, which implies $f \geqq 2g$; a contradiction. So $f = g$. On the other hand, by 1.1 and 1.2 we have: $n - 1 \geqq v(D_{B'/A}) \geqq 2\ell_A(B/B') + n - f$ and $\ell_A(B/B') = g\ell_{B'}(B/B')$.

Therefore: $2g\ell_{B'}(B/B') \leqq f - 1 = g - 1$, so $g[2\ell_{B'}(B/B') - 1] \leqq -1$, which implies $B = B'$.

Moreover B is tamely ramified over A. Indeed, denoting by $\mathfrak{m}$ the unique maximal ideal of B and by e its ramification index, we have: $\delta_{B/A} = \mathfrak{m}^h$ with $h \geqq e - 1$. Now, $v(D_{B/A}) = v(N(\delta_{B/A})) = fh$; then $fh \leqq n - 1 = ef - 1$ ([3] th. 2, p. 147), so $f(h-e) \leqq -1$. This implies $h \leqq e - 1$, which concludes the proof ([9] prop. 13, p. 67).

(iii) We have: $B' = \bigoplus_{j=1}^{r} B'_j$ where all the B'_j's are local. Moreover

([3] th. 2, p. 42) $[k(\mathfrak{p}_j): k]$, $[k(\mathfrak{m}_{jl}): k]$, $[L'_j: \hat{K}]$ do not depend on j, for all $\mathfrak{p}_j \in \operatorname{Max} B'$ and all $\mathfrak{m}_{jl} \in \operatorname{Max} B$ over $\mathfrak{p}_j$: and also $[L'_j: \hat{K}] = n/r$ by 1.7.

Since $v(D_{B'/A}) = v(D_{\hat{B}'/\hat{A}})$ ([9] prop. 10, p. 61), we have $v(D_{B'/A}) \leqq n-1$. Therefore: $v(D_{B'_j/\hat{A}}) = (1/r)v(D_{B'/A}) \leqq [(n-1)/r] \leqq [L'_j: \hat{K}] - 1$. Since for every B'_j condition (e) is verified because the residue fields do not change by completion, B'_j is normal and tamely ramified over A for every j by (ii): then B' itself is normal and tamely ramified over A.

COROLLARY 1.9: ([4] prop. 1.6). *Suppose that $n = 2$ and that A contains a field of characteristic $\neq 2$. Then B' is normal iff $v(D_{B'/A}) \leqq 1$.*

PROOF: B' is tamely ramified over A and the claim follows from 1.8.

REMARK 1.10: In general $v(D_{B'/A}) \leqq n-1$ does not imply B' normal even if B is tamely ramified over A, as shown by the following:

COUNTEREXAMPLE 1.11: Suppose char $k = 0$, and let $A = k[X]_{(X)}$, $B' = A[Y]/(Y^4 - Y^2 - X^3)$ (B' is the semilocal ring of the points of the curve $F = Y^4 - Y^2 - X^3 = 0$ which are contained in the line $X = 0$). We have: $v(D_{B'/A}) = v(\operatorname{Res}_Y(F, F')) = 3$ (by direct computation), where $\operatorname{Res}_Y(F, F')$ is the resultant of F and its derivative F' with respect to Y.

Therefore $v(D_{B'/A}) \leqq 4 - 1$; but B' is not normal.

Moreover we can show, by the following counterexample, that $n-1$ is the best upper-bound for $v(D_{B'/A})$ in order to grant, under the assumptions of 1.8, the normality of B' and its tame ramification over A.

COUNTEREXAMPLE 1.12: Put $A = \mathbb{R}[T]_{(T)}$, $B = \mathbb{C}[X]_{(X)}$ with the ring homomorphism given by $T \to X^2$ (so that $n = 4$), and let $B' = \mathbb{R}[X, iX]_{(X, iX)}$. B' is local and moreover there is a finite group G of automorphisms of B' such that $B'^G = A$, given by: $G = \{\sigma_1, .., \sigma_4\}$ where $\sigma_1 = id_{B'}$, $\sigma_2(X, iX) = (X, -iX)$, $\sigma_3(X, iX) = (-X, -iX)$, $\sigma_4 = \sigma_2 \circ \sigma_3$. We have $v(D_{B'/A}) = 4 = n$ (see 3.5), but B' is not normal.

The following example shows that for every $n \in \mathbb{N}$ there exists B' normal and tamely ramified over A such that $v(D_{B'/A}) = 0, 1, .., n-1$; therefore in particular the maximum $n-1$ is attained.

EXAMPLE 1.13: Let $q_1, .., q_u \in k[X]$ be irreducible, $q_i \neq q_j$ and non-associate whenever $i \neq j$; assume char $k = 0$ and let $S =$

$\{q \in k[X] \mid q_i$ does not divide q for every $i\}$. Put $A = k[T]_{(T)}$ and $B' = k[X]_S$ with the ring homomorphism given by $T \to \Pi_i q_i^{a_i}$ where the a_i's are positive integers and $\Sigma_i a_i = n$. The maximal ideals of B' are the $\mathfrak{m}_i = (q_i)k[X]_{(q_i)}$ and $[k(\mathfrak{m}):k] = \deg q_i$ $(i = 1, \ldots, u)$. We have: $K = k(T)$, $L = k(X)$; so $[L:K] = n$. B' is normal and tamely ramified over A, therefore by 1.2 $v(D_{B'/A}) = n - \Sigma_i \deg q_i$. Now, for a suitable choice of the q_i's it is possible to obtain every value of $v(D_{B'/A})$ between 0 and $n - 1$ (compare with 1.6).

2. Seminormality

For general facts on seminormality see [6] or [11].

LEMMA 2.1: *The following are equivalent*:
(i) *B' is seminormal.*
(ii) rad B' = rad B.
(iii) $\ell_A(B/B') = f - g$.
If moreover $f = g$, then (i), (ii), (iii) *are also equivalent to*:
(iv) *B' is normal.*

PROOF: Let $\mathfrak{b}$ be the conductor of B.

(i)→(ii) If B' is seminormal, then $B/\mathfrak{b}$ is reduced ([11] lemma 1.3, p. 588); so, after renumbering the $\mathfrak{m}_i$'s, we have: $\mathfrak{b} = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_s \supset \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_r = \text{rad}\, B$. But $\mathfrak{b} \subset B'$, so rad $B \subset$ rad B' and we are done.

(ii)→(i) $\mathfrak{b} \supset$ rad B' = rad B, therefore $B/\mathfrak{b} = (B/\text{rad}\, B)/(\mathfrak{b}/\text{rad}\, B) = (k_1 \times \ldots \times k_r)/I$ (I a suitable ideal). Thus $B/\mathfrak{b} = k_{i_1} \times \ldots \times k_{i_r}$ which implies $B/\mathfrak{b}$ reduced and B' seminormal ([6] cor. 2.7, p. 10).

(ii)↔(iii) $\ell_A(B/B') = \ell_A(B/\text{rad}\, B) - \ell_A(B'/\text{rad}\, B) = f - g$ iff rad B = rad B'.

The rest is obvious.

THEOREM 2.2: *Consider the following conditions*:
(i) *B' is seminormal.*
(ii) $v(D_{B'/A}) \geqq n + f - 2g$.
(iii) $v(D_{B'/A}) = n + f - 2g$.
(iv) $v(D_{B'/A}) \leqq n + f - 2g$.
(v) *B' is seminormal and B is tamely ramified over A.*
Then: (i)→(ii) *and* (iii), (iv), (v) *are equivalent.*

PROOF: (i)→(ii) By 2.1 we have $\ell_A(B/B') = f - g$, and therefore by 1.1 and 1.2 $v(D_{B'/A}) \geqq n + f - 2(f - g) = n + f - 2g$.

(iii)$\to$(iv) Trivial.

(iv)$\to$(v) Let C be the seminormalization of B' ([11] pp. 585–586): by 1.1 we have $v(D_{C/A}) + 2\ell_A(C/B') = v(D_{B'/A}) \leqq n + f - 2g$. But since C is seminormal and the sum of the degrees over k of its residue fields equals g, we have: $n + f - 2g + 2\ell_A(C/B') \leqq n + f - 2g$, which implies $\ell_A(C/B') = 0$ and $C = B'$.

Moreover B is tamely ramified over A. In fact: $v(D_{B/A}) + 2\ell_A(B/B') \leqq n + f - 2g$ implies, by 2.1, $v(D_{B/A}) + 2(f - g) \leqq n + f - 2g$; so $v(D_{B/A}) \leqq n - f$ and the claim follows by 1.2.

(v)$\to$(iii) Follows from 1.2 and from (i)$\to$(ii).

COROLLARY 2.3: (i) $v(D_{B'/A}) \geqq n + f - 2g$.

(ii) $v(D_{B'/A}) = n + f - 2g$ *iff* B' *is seminormal and* B *is tamely ramified over* A.

PROOF: (i) Let C be the seminormalization of B'; by 1.1 we have $v(D_{B'/A}) \geqq v(D_{C/A}) \geqq n + f - 2g$.

(ii) Follows from 2.2.

REMARK 2.4: Since $f \geqq g$, then $n + f - 2g \geqq n - g$, with strict inequality whenever $f \neq g$. Therefore 2.3 (i) is an improvement of 1.3 (i).

REMARK 2.5: From 2.2 it follows that if B' is seminormal and B is tamely ramified over A, then $v(D_{B'/A}) \leqq 2n - 2$ (the upper bound is obtained when $f = n$ and $g = 1$).

Counterexample 1.11 shows that the converse is false, in general: in fact B' is not seminormal and still $v(D_{B'/A}) \leqq 2 \cdot 4 - 2$.

The following example shows that for every $n \in \mathbb{N}$ there exists B' seminormal, with B tamely ramified over A, such that $v(D_{B'/A}) = n, n+1, \ldots, 2n-2$; therefore, in particular, the maximum $2n - 2$ is attained.

EXAMPLE 2.6: With the same notations as in 1.13 put: $A = k[T]_{(T)}$, $B = k[X]_S$, $B' = k + \operatorname{rad} B$.

B' is seminormal by 2.1 and since B is tamely ramified over A, from 2.2 it follows $v(D_{B'/A}) = n + \Sigma_i \deg q_i - 2$. Therefore, for a suitable choice of the q_i's, it is possible to obtain every value of $v(D_{B'/A})$ between n and $2n - 2$.

PROPOSITION 2.7: (i) *B' is seminormal iff $\hat{B}'$ is seminormal.*

(ii) *If* $B' = C \oplus D$ (direct sum of rings), *then B' is seminormal iff C and D are seminormal.*

PROOF: (i) B' is seminormal iff rad $B' = $ rad B (see 2.1) iff rad $\hat{B}' =$ rad $\hat{B}$ iff rad $\hat{B}' = $ rad $\bar{\hat{B}}'$ (see 1.7) iff $\hat{B}'$ is seminormal (see 2.1).

(ii) If B' is seminormal, then B' is the largest subring of B such that spec $B \to$ spec B' is a homeomorphism with trivial residue field extension. Therefore for C and D the same property holds; and conversely.

In remark 2.5 we pointed out that $v(D_{B'/A}) \leqq 2n-2$ is not a sufficient condition in order for B' to be seminormal. Now we want to find a function $F(n, f, g)$ such that $v(D_{B'/A}) \leqq F(n, f, g)$ gives such a sufficient condition (under suitable hypotheses). In the next theorem we show that $F(n, f, g) = n + f - 1$ is the required function.

THEOREM 2.8: *Assume that B is tamely ramified over A. If B' is seminormal, then $v(D_{B'/A}) \leqq n + f - 1$.*
The converse holds if either:
(i) $n = 2$, *or*
(ii) *B' is local, or*
(iii) *there exists a finite group G of automorphisms of B' such that $B'^G = A$.*

PROOF. By 2.2 $v(D_{B'/A}) = n + f - 2g$: moreover $2g \geqq 1$, therefore $v(D_{B'/A}) \leqq n + f - 1$, which proves the first part of the theorem.

(i) If B' is local, we can apply (ii). If B' is not local we have $f = g = 2$, which implies B' normal and then seminormal.

(ii) Let C be the seminormalization of B'; by 2.3 and 1.1 we have: $n + f - 2g \leqq v(D_{C/A}) \leqq v(D_{B'/A}) \leqq n + f - 1$, which implies $v(D_{B'/A}) - v(D_{C/A}) \leqq 2g - 1$ and so $2\ell_A(C/B') \leqq 2g - 1$. But, since B' is local, $\ell_A(C/B') = g\ell_{B'}(C/B')$: then we have $2g\ell_{B'}(C/B') \leqq 2g - 1$, which implies $\ell_{B'}(C/B') = 0$ and B' is seminormal.

(iii) With the same notations as in the proof of 1.8 (ii) we have: $v(D_{B'_j/\hat{A}}) = (1/r)v(D_{B'/A}) \leqq [(n + f - 1)/r] \leqq n/r + f/r - 1 = [L'_j : \hat{K}] + [k(\mathfrak{p}_j) : k] - 1$, which implies B'_j seminormal for every j, by (ii). Therefore B' itself is seminormal by 2.7.

REMARK 2.9: In general $v(D_{B'/A}) \leqq n + f - 1$ does not imply B' seminormal, as shown by counterexample 1.2 of [5].

Moreover we can show, by the following counterexample, that $n + f - 1$ is the best upper bound for $v(D_{B'/A})$ in order to grant, under the assumptions of 2.8, the seminormality of B'.

COUNTEREXAMPLE 2.10: Let $A = \mathbb{R}[T^2]_{(T^2)}$, $B' = \mathbb{C}[T^2, T^3]_{(T^2,T^3)}$ and $B = \mathbb{C}[T]_{(T)}$. We have $n = 4$ and $f = g = 2$. Moreover $\ell_{B'}(B/B') = 1$, so $\ell_A(B/B') = 2$. Since B is tamely ramified over A, from 1.3 it follows $v(D_{B/A}) = 4 - 2 = 2$ and by 1.1 $v(D_{B'/A}) = 4 + 2 = n + f$; but B' is not seminormal, though it is local.

REMARK 2.11: We do not know if, in theorem 2.8, when (i) or (ii) or (iii) are verified and $v(D_{B'/A}) \leqq n + f - 1$, B happens to be tamely ramified over A.

3. The Gorenstein case

THEOREM 3.1: *Suppose B' is Gorenstein and B is tamely ramified over A.*
If B' is seminormal, then $v(D_{B'/A}) \leqq n$.
The converse holds if either:
(i) *$n = 2$, or*
(ii) *B' is local, or*
(iii) *there exists a finite group G of automorphisms of B' such that $B'^G = A$.*

PROOF: For every $\mathfrak{p}_j \in \text{Max } B'$, let $p_j = \ell_A(\overline{B}'_{\mathfrak{p}_j}/\text{rad } \overline{B}'_{\mathfrak{p}_j})$: from [6] th. 8.1, p. 46 it follows $2g_j \geqq p_j$ and since $f = \Sigma_j p_j$ we get $2g = 2 \Sigma_j g_j \geqq f$, and the claim follows from 2.2.
The converse follows from 2.8.

COROLLARY 3.2 ([5] 1.1 and 1.3): *Let $B' = A[x]$ be a domain, and suppose either* char $k = 0$ *or* char $k > n$.
If B' is seminormal, then $v(D_{B'/A}) \leqq n$.
The converse holds if B' is local.

PROOF: If G is the characteristic polynomial of x, we have $B' = A[X]/(G)$ and, since $A[X]$ is Gorenstein, B' is also Gorenstein. Moreover from the formula $\Sigma_{\mathfrak{p}}\ e_{\mathfrak{p}} f_{\mathfrak{p}} = n$, it follows $e_{\mathfrak{p}} \leqq n$ for every $\mathfrak{p} \in \text{spec } B$, which implies that B is tamely ramified over A (obviously $e_{\mathfrak{p}}$ denotes the ramification index of $\mathfrak{p}$, and $f_{\mathfrak{p}} = [k(\mathfrak{p}) : k]$). Then the claim follows from 3.1.

COROLLARY 3.3 ([4] 1.7): *Assume that $n = 2$, that A contains a field of characteristic $\neq 2$, and that B' is a domain.*
Then B' is seminormal iff $v(D_{B'/A}) \leqq 2$.

PROOF: B' is monogenic over A([4] 1.1), then it is Gorenstein (see proof of 3.2) and B is tamely ramified over A. Then the claim follows from 3.1.

REMARK 3.4: In general $v(D_{B'/A}) \leqq n$ does not imply that B' is seminormal, even when B' is Gorenstein and B tamely ramified over A (see counterexample 1.2 of [5]).

In [5] we proved that if B' and k are as in 3.2, and if the characteristic polynomial of x is $X^n - a$ ($a \in A$), then the following are equivalent:

(i) B' is seminormal.

(ii) $v(D_{B'/A}) \leqq n$.

(iii) $a = ut^q$, where u is a unit in A and $q \leqq n/(n-1)$. Recently S.S. Abhyankar made us to notice that when $n \geqq 3$ (i), (ii), (iii) are also equivalent to:

(iv) B' is normal.

In fact, when $n \geqq 3$, (iii) implies $v(a) \leqq 1$: now, if $v(a) = 0$, then $v(D_{B'/A}) = v(n^n a^{n-1}) = 0$ and B' is normal by 1.1; if $v(a) = 1$, then B' is local and B' is normal by 1.8.
Moreover (iv) $\rightarrow$ (i).

Now, supposing that A contains the n^{th} roots of 1, then there is the group G of automorphisms of B', $G = \{\sigma_1, \ldots, \sigma_n\}$, where $\sigma_{i|A} = id_A$ and $\sigma_i(x) = x \cdot \xi^i$ (ξ a fixed primitive n^{th} root of 1) for every i; for this group obviously $B'^G = A$. Therefore it is natural to conjecture that when $n \geqq 3$, B' is Gorenstein and either B' is local or there is a finite group G of automorphisms of B' such that $B'^G = A$, then $v(D_{B'/A}) \leqq n$ implies B' normal (notice that by 3.1 B' is seminormal). The following counterexample gives a negative answer to the conjecture.

COUNTEREXAMPLE 3.5: Let A, B, B' be as in 1.12. B' is Gorenstein and seminormal; moreover by 2.2 we have $v(D_{B'/A}) = n + f - 2g = 4 + 2 - 2 \leqq 4$. But B' is not normal.

4. Globalization

Suppose X, Y are locally noetherian schemes, with Y integral normal, and let $\phi : X \rightarrow Y$ be a finite covering of degree n. From the going-up and going-down theorems ([3] cor. 2, p. 38 and th. 3, p. 56) it follows that if $x \in X$ is a point of codimension 1, then $y = \phi(x) \in Y$ is a point of codimension 1, which implies that $\mathcal{O}_y$ is a discrete valuation ring.

Now, when X is S_2 the seminormality and normality of X can be checked in codimension 1 (see [6] th. 2.6, p. 9 and the Krull-Serre criterion, [7] th. 39, p. 125) i.e. it is enough to look at $v_y(\mathfrak{D}_y)$ for all $y \in Y$ of codimension 1.

LEMMA 4.1: (i) *If B' is Gorenstein, then $B'$is S_2.*

(ii) *If E is any normal domain, and $B' = E[x]$ is a domain integral over A, then B' is S_2 and Gorenstein in codimension* 1.

PROOF: (i) By definition B' is Cohen-Macaulay, hence S_r for all r.

(ii) Since E is normal $\{1, x, .., x^{n-1}\}$ is a free basis of B' as an E-module. Now E is S_2 and the fibers of the canonical embedding $E \hookrightarrow B'$ are also S_2, being 0-dimensional: therefore since B' is faithfully flat over E, B' is S_2 ([7] cor. 2, p. 154).

Moreover for every $\mathfrak{q} \in \operatorname{spec} B'$ of height 1, we have $B'_{\mathfrak{q}} = (E_{\mathfrak{Q}}[x])_{\mathfrak{q}}$ where $\mathfrak{Q} = \mathfrak{q} \cap E$: now $E_{\mathfrak{Q}}[x]$ is Gorenstein because $E_{\mathfrak{Q}}$ is a discrete valuation ring and $E_{\mathfrak{Q}}[x]$ is a domain (see proof of 3.2), then $B'_{\mathfrak{q}}$ is Gorenstein and we are done.

By assuming X to be S_2 we can globalize 1.2, 1.8, 2.2, 2.8: by assuming X to be S_2 and Gorenstein in codimension 1, we can globalize in particular 3.1.

We wish to remark explicitly that when we assume X to be integral and locally monogenic over Y, then by 4.1 (ii) X is both S_2 and Gorenstein in codimension 1: which shows that the result obtained by globalizing 3.1 generalizes the analogous results of [4] and [5].

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(Oblatum 23–XI–1979 & 23–VI–1980)

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