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ON CERTAIN EULER PRODUCTS FOR SU(2, 1)

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Introduction

In a recent paper [12] Shintani investigated the Euler products associated with automorphic forms on SU(2, 1) which are eigenfunctions on the Hecke operators. He showed, in particular, how the action on the Hecke operators transforms the Fourier–Jacobi series of such forms and gave an integral representation of the associated Dirichlet series with Euler product. Such an integral representation with Euler product was also considered by Piatetski-Shapiro [7]. On the other hand, there is a lifting from ordinary elliptic modular forms to automorphic forms on SU(2, 1), defined by a certain theta function of two variables [4]. In this paper we show that the lift of a Hecke eigenform is again a Hecke eigenform and that the Dirichlet series of the lift has a nice expression in terms of that of the original form. In effect we show that the lifting in question is compatible with Langlands' principal of functoriality.

More precisely, let $K = \mathbf{Q}(i)$, let σ be the Galois automorphism of $K/\mathbf{Q}$, and let

$$R = \begin{pmatrix} & & 1 \\ & 2i & \\ -1 & & \end{pmatrix}.$$

Then $\frac{1}{2}iR$ defines a Hermitian form of signature $(1, 2)$ on $V = K^3$. Let

$$G(\mathbf{Q}) = \{g \in GL_3(K) \mid {}^t g^\sigma R g = \mu(g)R,\ \mu(g) \in \mathbf{Q}^\times\}$$

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be the group of similitudes of R, and let

$$D = \{\mathfrak{z} = (z, w) \in \mathbf{C}^2 \mid Im z - |w|^2 > 0\}$$

be the corresponding Hermitian symmetric space. Let $\mathcal{O} = \mathbf{Z}[i]$, and for the lattice $L = \mathcal{O}^3 \subset V$, let

$$G(L) = \{g \in G(\mathbf{Q}) \mid gL = L\}.$$

For $g \in G(\mathbf{R})$ and $\mathfrak{z} \in D$, set

$$J(g, z) = a_3 z + b_3 w + c_3$$

where (a_3, b_3, c_3) is the bottom row of g. Then for any $k \in \mathbf{Z}$, $k > 5$ and $k \equiv 0(4)$ there is a lifting

$$\mathcal{L}: S_{k-1}(\Gamma_0(4), (\frac{-4}{\quad})) \to A_k(G(L))$$

where $S_{k-1}(\Gamma_0(4), (\frac{-4}{\quad}))$ is the space of cusp forms of weight $k-1$ on $\mathfrak{H}$, the upper half plane, and $A_k(G(L))$ is the space of holomorphic functions F on D satisfying

$$F(g(\mathfrak{z})) = J(g, \mathfrak{z})^k F(\mathfrak{z})$$

for all $g \in G(L)$.

Our result is the following: If $f(\tau) = \Sigma_{n=1}^{\infty} a(n) e(n\tau) \in S_{k-1}(\Gamma_0(4), (\frac{-4}{\quad}))$ is a Hecke eigenform, then so is $\mathcal{L}(f)$, and

$$\zeta(s, \chi, L(f)) = L_K(s+1, \chi) L\left(s + \frac{k}{2}, f\right) L\left(s + \frac{k}{2}, f^\rho\right) \tag{0.1}$$

where $\zeta(s, \chi, F)$ is the Dirichlet series attached to a form on D, $L(s, f) = \Sigma_{n=1}^{\infty} a(n) n^{-s}$, $f^\rho(\tau) = \Sigma_{n=1}^{\infty} \overline{a(n)} e(n\tau)$, and χ is the grossencharacter of K defined by

$$\chi((\pi)) = \left(\frac{\pi}{|\pi|}\right)^k.$$

Now the function $\zeta(s, \chi, F)$ is the L-function of F associated, by Langlands [5], to a certain representation ρ of the L-group of G. If we let

$$H(\mathbf{Q}) = \{h \in GL_2(K) \mid {}^t h^\sigma R' h = \mu(h) R', \mu(h) \in \mathbf{Q}^\times\}$$

where

$$R' = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix},$$

then there is a homomorphism $\alpha : {}^L H \to {}^L G$ of L-groups such that, locally, with the exception of the Euler factor for $p = 2$, $\zeta_p(s, \chi, \mathscr{L}(f))$ is the Euler factor associated to f for the representation $\rho \circ \alpha$ of ${}^L H$.

Our proof of (0.1) is based on the classical technique used by Eichler [2, Satz 21.3] to relate the actions of Hecke operators and generalized Brandt matrices in spaces of θ-functions. On the other hand, Rallis [8] gave a representation theoretic proof of Eichler's result. In [9] Rallis applied his method to check local functoriality for the correspondences determined by the Weil representation for the dual reductive pair $(0_m, \mathrm{Sp}_n)$. It should be possible to extend Rallis' method to the dual reductive pairs $(U(V), U(V'))$.

The existence of a holomorphic analytic continuation and functional equation for $\zeta(s, \chi, \mathscr{L}(f))$ is an immediate consequence of (0.1). Eventually these properties will be obtainable for all F by the method of Shintani and Piatetski-Shapiro.

In §1 we specialize the machinery of [4] to the present case. We then recall the definition of the Hecke operators acting in $A_k(G(L))$ and state the theorem, which relates the action of the Hecke operators on the two variables of the theta function defining the lifting. This theorem has the above relation between Dirichlet series as an immediate consequence. The proof of the theorem is given in §2. In §3, we discuss functoriality.

I would like to thank Professor Shintani for a conversation which stimulated this paper, and Professor M. Karel for his advice about ${}^L G$.

§1. The lifting and Hecke operators

Let K, R, G, D, etc. be as in the introduction, and define a Hermitian form on V by

$$(X, Y) = \tfrac{1}{2} i \, {}^t \bar{X} R Y.$$

Note that this is the negative of Shintani's form. For $L \subset V$ as before, let

$$L^* = \{Y \in V \mid \mathrm{tr}(X, Y) \in \mathbf{Z}, \forall X \in L\}$$
$$= \mathcal{O} \oplus \frac{1}{2}\mathcal{O} \oplus \mathcal{O}$$

be the dual lattice. Here $\mathrm{tr}(\alpha) = \mathrm{tr}_{K/\mathbf{Q}}(\alpha)$, $\alpha \in K$. Also, for $\mathfrak{z} \in D$, let

$$P_+(\mathfrak{z}) = \begin{pmatrix} z \\ w \\ 1 \end{pmatrix}$$

and let

$$\eta(\mathfrak{z}) = (P(\mathfrak{z}), P(\mathfrak{z}))$$
$$= \mathrm{Im}(z) - |w|^2.$$

and let

$$(X, X)_{\tau,\mathfrak{z}} = x(X, X) + iy(X, X)_{\mathfrak{z}}.$$

Now for $\mathfrak{z} \in D$, the vector $P(\mathfrak{z}) \in V$ spans a positive line in V; and we may define the majorant $(,)_{\mathfrak{z}}$ of $(,)$ associated to $\mathfrak{z}$ by

$$(X, X)_{\mathfrak{z}} = 2\eta(\mathfrak{z})^{-1}|(X, P(\mathfrak{z}))|^2 - (X, X).$$

Then for $k \in \mathbf{Z} > 0$, $h \in L^*$ and $\tau = x + iy \in \mathfrak{H}$, set

$$\theta(\tau, \mathfrak{z}, h) = y^2 \sum_{X \equiv h(L)} (X, P_+(\mathfrak{z})\eta(\mathfrak{z})^{-1})^k e((X, X)_{\tau,\mathfrak{z}}). \tag{1.1}$$

According to Proposition 2.1 of [4], if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ then

$$\theta(\gamma\tau, \mathfrak{z}, h) = (c\tau + d)^{k-1}\psi(d)e(ab(h, h))\theta(\tau, \mathfrak{z}, ah) \tag{1.2}$$

where $\psi(d) = (sgnd)\left(\frac{-4}{|d|}\right)$ is the quadratic character associated to $K/\mathbf{Q}$. Also if $g \in G(L)$,

$$\theta(\tau, g(\mathfrak{z}), h) = \overline{J(g, \mathfrak{z})}^k \theta(\tau, \mathfrak{z}, g^{-1}(h)). \tag{1.3}$$

Set $\theta(\tau, \mathfrak{z}) = \theta(\tau, \mathfrak{z}, 0)$ and suppose that $k \equiv 0 \bmod 4$ so that $\theta(\tau, \mathfrak{z}) \not\equiv 0$. Then by Theorem 5.3 of [4], if $f \in S_{k-1}(\Gamma_0(4), \psi)$ and $k > 5$ the function

$$\mathscr{L}(f) = \int_{\Gamma_0(4)\backslash\mathfrak{H}} f(\tau)\overline{\theta(\tau, \mathfrak{z})}y^{k-3}\,\mathrm{d}x\,\mathrm{d}y$$

is a holomorphic form on D of weight k with respect to $G(L)$; $\mathscr{L}(f) \in A_k(G(L))$.

Note that the automorphy factor $J(g, \mathfrak{z})^k$ corresponds to the case $1 = k$, $m = 0$, $d = 0$ in Shintani p. 2 [12].

Now we recall the definition of the Hecke operators for G following Shintani [12]. For a prime P of K with $P = (\pi)$, $\pi \in K$, define a subset $S(P) \subset G(\mathbf{Q})$ as follows:

(1) If $P \neq P^\sigma$, $p = PP^\sigma$, let

$$S(P) = \{g \in M_3(\mathcal{O}) \cap G(\mathbf{Q}) \mid \mu(g) = p \text{ and } g \text{ has elementary divisors } \{1, \pi, p\}\};$$

(2) If $P = P^\sigma$, let

$$S(P) = \{g \in M_3(\mathcal{O}) \cap G(\mathbf{Q}) \mid \mu(g) = N(p)\}.$$

Then $S(P)$ is a union of $G(L)$ double cosets and may be written as a finite disjoint union

$$S(P) = \bigcup_j G(L)g_j.$$

For $F \in A_k(G(L))$ define

$$\{F \mid T(P)\}(\mathfrak{z}) = N(p)^{k/2} \sum_j J(g_j, \mathfrak{z})^{-k}F(g(\mathfrak{z})).$$

The $T(P)$'s generate a commutative algebra of operators on $A_k(G(L))$. If $F \in A_k(G(L))$ is a simultaneous eigenfunction of all the $T(P)$'s with

$$F \mid T(P) = \lambda(P)F,$$

and if ξ is a grossencharacter of K, then there is a Dirichlet series with Euler product associated to F and ξ in the following way:

(1) If $P \neq P^\sigma$, $P = (\pi)$, $p = \pi\pi^\sigma$ set

$$Q_1(X) = 1 - \lambda(P^\sigma)\bar{\chi}p^{-2}X + \lambda(P)\bar{\chi}p^{-3}X^2 - \bar{\chi}^2p^{-3}X^3$$

and

$$Q_2(Y) = 1 - \lambda(P)\chi p^{-2} Y + \lambda(P^\sigma)\chi p^{-3} Y^2 - \chi^2 p^{-3} Y^3$$

where $\chi = \chi(P) = \left(\frac{\pi}{|\pi|}\right)^k$. Define the Euler factor at p

$$\zeta_p(s, \xi, F) = Q_1(\xi(P)p^{-s})^{-1} Q_2(\xi(P^\sigma)p^{-s})^{-1}.$$

(2) If $P = P^\sigma = (p)$ is unramified in $K/\mathbf{Q}$, set

$$Q(X) = 1 + \{p - \lambda(P)\}p^{-4}X + p^{-4}X^2$$

and define the Euler factor at p

$$\zeta_p(s, \xi, F) = (1 - \xi(P)p^{-2-2s})Q(\xi(P)p^{-2s})^{-1}$$

(3) If $P = (1+i)$, $NP = 2$, set

$$Q(X) = 1 + \{2 - \epsilon\lambda(P)\}4^{-1}X + 4^{-1}X^2$$

where $\epsilon = (-1)^{k/4}$, and set

$$\zeta_2(s, \xi, F) = (1 - \xi(P)2^{-1-s})^{-1} Q(\xi(P)2^{-s})^{-1}.$$

Then define

$$\zeta(s, \xi, F) = \prod_p \zeta_p(s, \xi, F).$$

Also recall that if $f \in S_{k-1}(\Gamma_0(4), \psi)$, the Hecke operator $T'(p)_{k-1,\psi}$ acts on f via

$$f \mid T'(p)_{k-1,\psi} = p^{-1} \sum_{b=0}^{p-1} f(\tau + b)/p) + p^{k-2}\psi(p)f(p\tau)$$

where $\psi(2) = 0$. In the notation of [11; (3.4.1)], set

$$f \mid T''(2)_{k-1,\psi} = f \mid [\Gamma'(^2{}_1)\Gamma']_{k-1},$$

where $\Gamma' = \Gamma_0(4)$.

Now our main result, to be proved in the next section, describes the action the $T(P)$'s on the theta function $\theta(\tau, \mathfrak{z})$.

THEOREM: (1) *If* $P \neq P^\sigma$, $P = (\pi)$, *then*

$$p^{k/2}\bar{\theta}\,|\,T(P) = \overline{\theta\,|\,(p^2T'(p)_{k-1,\psi} + p(\pi)^k)}.$$

(2) *If $P = P^\sigma$ is unramified in $K/\mathbf{Q}$. Then*

$$p^k\bar{\theta}\,|\,T(P) = \overline{\theta\,|\,(p^4T'(p^2)_{k-1,\psi} + p^{k+2} + p^{k+1})}.$$

(3) *If $P = (1+i)$, then*

$$2^{k/2}\bar{\theta}\,|\,T(P) = \overline{\theta\,|\,(4T'(2)_{k-1,\psi} + 2(1+i)^k + 4T''(2)_{k-1,\psi})}.$$

REMARK: The theorem is an analogue of a classical result of Eichler [2, Satz 21.3] and the method of proof is similar to his. The thoerem is an explicit special case of the general results of Howe [3].

COROLLARY 1: *Suppose that $f \in S_{k-1}(\Gamma_0(4), \psi)$ is a normalized Hecke eigenform such that $\mathscr{L}(f) \neq 0$. Then $\mathscr{L}(f)$ is an eigenfunction of all the $T(P)$'s with eigenvalues given by:*

$$\lambda(P) = \begin{cases} p^{2-k/2}a_p + p\bar{\chi} & \text{if } P \neq P^\sigma \\ p^{4-k}a_p^2 + 2p^2 + p & \text{if } P = P^\sigma = (p) \text{ is unramified} \\ 2^{2-k/2}(a_2 + \bar{a}_2) + 2\epsilon & \text{if } P = (1+i) \end{cases}$$

where $\chi = \chi(P) = \left(\frac{\pi}{|\pi|}\right)^k$ in the first case, and $\epsilon = (-1)^{k/4}$.

PROOF: This follows immediately from the theorem and well known adjointness properties of the Hecke operators with respect to the Petersson inner product $\langle , \rangle$. Note that $\mathscr{L}(f) = \langle f, \theta \rangle$ and that $(1+i)^k = \epsilon 2^{k/2}$.

COROLLARY 2: *For a normalized Hecke eigenform $f \in S_{k-1}(\Gamma_0(4), \psi)$ such that $\mathscr{L}(f) \neq 0$, we have*

$$\zeta\left(s - \frac{k}{2}, \chi, \mathscr{L}(f)\right) = L_K\left(s + 1 - \frac{k}{2}, \chi\right) L(s, f) L(s, f^\rho)$$

where $L_K(s, \chi)$ is the Hecke L-series for the grossencharacter χ and $L(s, f)$ is as in the introduction. Thus the function

$$D(s, \chi, \mathscr{L}(f)) = 2^{3s} G_2\left(s + \frac{k}{2}\right) G_2\left(s + \frac{k}{2} - 1\right)^2 \zeta(s - 1, \chi, \mathscr{L}(f))$$

has an entire analytic continuation and satisfies the functional equation:

$$D(s, \chi, \mathscr{L}(f)) = -D(1-s, \chi, \mathscr{L}(f))$$

where $G_2(s) = (2\pi)^{1-s}\Gamma(s)$.

PROOF: If $P \neq P^\sigma$, we substitute the values given in Cor. 1 for $\lambda(P)$ and $\lambda(P^\sigma)$ into $Q_1(X)$ and obtain

$$\begin{aligned}(1 - (p^{2-k/2}a_p + p\chi)\bar{\chi}p^{-2}X + (p^{2-k/2}a_p + p\bar{\chi})\bar{\chi}p^{-3}X^2 - \bar{\chi}^2p^{-3}X^3) \\ = (1-p^{-1}X)(1-p^{-k/2}\bar{\chi}a_pX + \bar{\chi}^2p^{-2}X^2).\end{aligned}$$

Similarly for $Q_2(Y)$ we get

$$(1-p^{-1}Y)(1-p^{-k/2}\chi a_pY + \chi^2p^{-2}Y^2).$$

Now setting $X = \chi p^{k/2-s}$ and $Y = \bar{\chi}p^{k/2-s}$ yields

$$(1-\chi p^{k/2-1-s})(1-a_pp^{-s} + p^{k-2-2s})$$

and

$$(1-\bar{\chi}p^{k/2-1-s})(1-a_pp^{-s} + p^{k-2-2s}).$$

Since $\bar{a}_p = \psi(p)a_p = a_p$, we get, after a shift of $k/2$ in s, the required identity on Euler factors.

Similarly if $P = P^\sigma = (p)$ is unramified, the value for $\lambda(P)$ given in Cor. 1 yields

$$1 - (p^{4-k}a_p^2 + 2p^2)p^{-4}X + p^{-4}X^2$$

for $Q(X)$; and putting $X = p^{k-2s}$ gives

$$\begin{aligned}1 - (a_p^2 + 2p^{k-2})p^{-2s} + p^{2k-4-4s} \\ = (1-a_pp^{-s} - p^{k-2-2s})(1 + a_pp^{-s} - p^{k-2-2s}).\end{aligned}$$

Since $\bar{a}_p = \psi(p)a_p = -a_p$ this is the required identity, again with a shift of $k/2$ in s.

Finally, if $P = (1+i)$ we obtain

$$1 - (a_2 + \bar{a}_2)2^{-k/2}\epsilon X + 2^{-2}X^2.$$

Setting $X = \chi((1+i))2^{k/2-s}$ and noting that $\chi((1+i)) = \epsilon$, we get

$$1-(a_2+\bar{a}_2)2^{-s}+2^{k-2-2s}$$

which is just

$$(1-a_2 2^{-s})(1-\bar{a}_2 2^{-s})$$

since $a_2\bar{a}_2 = 2^{k-2}$, [6].

The functional equation of $D(s, \chi, \mathscr{L}(f))$ follows immediately from those of its factors.

§2. Proof of the Theorem

For $g \in G(\mathbf{Q})$, let $g^{\iota} = \mu(g)g^{-1}$. Then $g^{\iota} \in G(\mathbf{Q})$, $(gX, Y) = (X, g^{\iota}Y)$ and $\mu(g^{\iota}) = \mu(g)$. Also, $G(L)^{\iota} = G(L)$ and; since L is a maximal lattice in the sense of [10], the set $\{g \in G(\mathbf{Q}) \mid gL \subset L\}$ is also stable under ι.

Now let $S(P) = \bigcup_j G(L)g_j$ as before, and for $X \in L$ set

$$m(P, X) = |\{j \mid X \in g_j^{\iota}L\}|.$$

Note that the $g_j^{\iota}L$'s are certain sublattices of L independent of the choice of coset representatives; and that the multiplicities $m(P, X)$ only depend on the $G(L)$ orbit of X, since $G(L)$ simply permutes the sublattices in question.

For convenience write

$$f_{\tau,\mathfrak{z}}(X) = (X, P_+(\mathfrak{z})\eta(\mathfrak{z})^{-1})^k e((X, X)_{\tau,\mathfrak{z}}). \tag{2.1}$$

Then applying $T(P)$ to $\overline{\theta(\tau, \mathfrak{z})}$ yields:

$$\begin{aligned} \bar{\theta} \mid T(P) &= N(P)^{k/2} \sum_j J(g_j, \mathfrak{z})^{-k} y^2 \sum_{X\in L} \overline{f_{\tau, g_j(\mathfrak{z})}(X)} \\ &= N(P)^{k/2} \sum_j \mu(g_j)^{-k} y^2 \sum_{X\in L} \overline{f_{\mu(g_j)^{-1}\tau,\mathfrak{z}}(g_j^{\iota}X)} \\ &= N(P)^{-k/2} y^2 \sum_{X\in L} m(P, X) \overline{f_{N(P)^{-1}\tau,\mathfrak{z}}(X)} \end{aligned} \tag{2.2}$$

where we use the identity

$$P_+(g(\mathfrak{z}))\eta(g(\mathfrak{z}))^{-1} = \mu(g)^{-1}gP_+(\mathfrak{z})\eta(\mathfrak{z})^{-1}\overline{J(g, \mathfrak{z})}$$

and the fact that $\mu(g) = N(P)$ for all $g \in S(P)$.

Next we must compute the multiplicities $m(P, X)$.

For each rational prime p, let $K_p = K \otimes_{\mathbf{Q}} \mathbf{Q}_p$, $\mathcal{O}_p = \mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}_p$, $V_p = K_p^3$ and $L_p = \mathcal{O}_p^3$. Also let $G_p = G(\mathbf{Q}_p)$ and let

$$G(L)_p = \{g \in G_p \mid gL_p = L_p\}.$$

Then for a prime P of K with $p = P \cap \mathbf{Q}$, set

$$S(P)_p = \bigcup_j G(L)_p g_j$$

so that $S(P)_p \subset G_p$, $G(L)_p S(P)_p G(L)_p = S(P)_p$, and in fact the above union is still disjoint. For $p' \neq p$,

$$(g_j^{\iota} L)_{p'} = L_{p'}$$

so that

$$X \in g_j^{\iota} L \Leftrightarrow X \in g_j^{\iota} L_p;$$

and

$$m(p, X) = |\{j \mid X \in g_j^{\iota} L_p\}|$$

where now the g_j's can be any set of coset representatives for $G(L)_p$ in $S(P)_p$. Further observe that if $\operatorname{ord}_p \mu(g_j) = \alpha$, then

$$m(P, p^{\alpha} X) = |G(L)_p \backslash S(L)_p|$$

since

$$p^{\alpha} X \in g_j^{\iota} L_p \Leftrightarrow \mu(g_j)^{-1} g_j p^{\alpha} X \in L_p$$

and $\mu(g_j)^{-1} p^{\alpha}$ is a unit of $\mathcal{O}_p$. By the same argument $m(P, X)$ only depends on $X \bmod p^{\alpha} L_p$.

PROPOSITION: (1) *Suppose that $P \neq P^{\sigma}$, $P = (\pi)$. Let*

$$C_1 = \{X \in L_p \mid (X, X) \equiv 0(p), X \notin \pi L_p \cup \pi^{\sigma} L_p\}$$

$$C_{0+} = \{X \in L_p \mid (X, X) \equiv 0(p), X \in \pi L_p, X \notin pL_p\}$$
$$C_{0-} = \{X \in L_p \mid (X, X) \equiv 0(p), X \in \pi^\sigma L_p, X \notin pL_p\}$$
$$C_{00} = pL_p.$$

Then

$$m(P, X) = \begin{cases} 1 + p + p^2 & \textit{if } X \in C_{00} \\ 1 + p & \textit{if } X \in C_{0-} \\ 1 & \textit{if } X \in C_{0+} \cup C_1 \\ 0 & \textit{otherwise.} \end{cases}$$

(2) *Suppose that* $P = P^\sigma = (p)$ *is unramified. Let*

$$C_1 = \{X \in L_p \mid (X, X) \equiv 0 \bmod p^2, X \notin pL_p\}$$
$$C_0 = \{X \in L_p \mid (X, X) \equiv 0 \bmod p^2, X = pY, Y \in L_p, (Y, Y) \not\equiv 0 \bmod p\}$$
$$C_{00} = \{X \in L_p \mid (X, X) \equiv 0 \bmod p^2, X = pY, Y \in L_p, (Y, Y) \equiv 0 \bmod p, Y \notin pL_p\}$$

and

$$C_{000} = p^2 L_p.$$

Then

$$m(P, X) = \begin{cases} 1 + p + p^4 & \textit{if } X \in C_{000} \\ 1 + p & \textit{if } X \in C_{00} \\ 1 + p + p^2 & \textit{if } X \in C_0 \\ 1 & \textit{if } X \in C_1 \\ 0 & \textit{otherwise.} \end{cases}$$

(3) *Suppose that* $P = P^\sigma$ *is ramified, so* $P = (\pi)$, $\pi = (1 + i)$, *and* $p = 2$. *Let*

$$C_1 = \{X \in L_p \mid (X, X) \equiv 0(p), X \notin \pi L_p\}$$
$$C_0 = \{X \in L_p \mid (X, X) \equiv 0(p), X \in \pi L_p, p^{-1}X \notin L_p^*\}$$
$$C_{00} = \{X \in L_p \mid (X, X) \equiv 0(p), X \in \pi L_p, p^{-1}X \in L_p^*, X \notin pL_p\}$$

and

$$C_{000} = pL_p.$$

Then

$$m(P,X)=\begin{cases}1+p+p^2 & \text{if } X\in C_{000}\cup C_{00}\\ 1+p & \text{if } X\in C_0\\ 1 & \text{if } X\in C_1\\ 0 & \text{otherwise.}\end{cases}$$

Proof: By the above remarks, the function $m(P,X)$ is actually defined on $L_p/N(P)L_p$ and is constant on $G(L)_p$ orbits in this set. Therefore to compute $m(P,X)$ we need only find a set of $G(L)_p$ orbit representatives and a set of coset representatives g_j.

(1) If $P\neq P^\sigma$, $P=(\pi)$ with $\pi\in K_p$, and take idempotents e and e^σ in K_p. Then

$$V_p=eV_p+e^\sigma V_p\simeq \mathbf{Q}_p^3\times\mathbf{Q}_p^3$$

and

$$L_p=eL_p+e^\sigma L_p\simeq \mathbf{Z}_p^3\times\mathbf{Z}_p^3.$$

Note that we may assume that π is chosen so that the corresponding element of $\mathbf{Q}_p\times\mathbf{Q}_p$ is $[p,1]$. Set $R_1=eR$, $R_2=e^\sigma R$. Then

$$G_p=G(\mathbf{Q}_p)\simeq\{(g_1,g_2)\in GL_3(\mathbf{Q}_p)\times GL_3(\mathbf{Q}_p)\mid {}^tg_1R_2g_2=cR_2,\ c\in\mathbf{Q}_p^x\}$$

$$\simeq GL_3(\mathbf{Q}_p)\times\mathbf{Q}_p^x$$

since $g_2=cg_1^*$ with $g_1^*=R_2^{-1}\check{g}_1R_2$, $\check{g}_1={}^tg_1^{-1}$. The action of $[g,c]\in GL_3(\mathbf{Q}_p)\times\mathbf{Q}_p^x$ on $[X,Y]\in\mathbf{Q}_p^3\times\mathbf{Q}_p^3$ is given by

$$[g,c][X,Y]=[gX,cg^*Y].$$

Since $R_2\in GL_3(\mathbf{Z}_p)$,

$$G(L)_p\simeq GL_3(\mathbf{Z}_p)\times\mathbf{Z}_p^x.$$

The above action is equivalent to the "standard" action

$$[g,c]'[X,Y]=[gX,c\check{g}Y]$$

under the isomorphism $1\times R_2:\mathbf{Q}_p^3\times\mathbf{Q}_p^3\to\mathbf{Q}_p^3\times\mathbf{Q}_p^3$. It is then easy to determine orbit representatives for $GL_3(\mathbf{F}_p)\times\mathbf{F}_p^x$, and hence for

$G(L)_p$, in

$$L_p/pL_p \simeq \mathbf{F}_p^3 \times \mathbf{F}_p^3.$$

For example, we may take

$$[{}^t(1,0,0), {}^t(0,1,0)], [0, {}^t(1,0,0)], [{}^t(1,0,0), 0], [0,0],$$
$$\text{and } [{}^t(1,0,0), {}^t(1,0,0)];$$

and these correspond to the cases C_1, C_{0+}, C_{0-}, C_{00}, and 'otherwise' above.

Now Shintani [12, Lemma 2], found a nice set of coset representatives for $S(P^\sigma)_p/G(L)_p$. For $w \in K_p$ and $u \in \mathbf{Q}_p$, set

$$h(w,u) = \begin{pmatrix} 1 & 2iw^\sigma & u + iww^\sigma \\ & 1 & w \\ & & 1 \end{pmatrix},$$

and let

$$H(L)_p = \{h(u,w) \mid w \in \mathcal{O}_p, u \in \mathbf{Z}_p\}.$$

Then $H(L)_p \subset G(L)_p$, and Shintani proved that

$$S(P^\sigma)_p = \begin{pmatrix} 1 & & \\ & \pi^\sigma & \\ & & p \end{pmatrix} G(L)_p \cup H(L)_p \begin{pmatrix} \pi^\sigma & & \\ & \pi & \\ & & \pi^\sigma \end{pmatrix} G(L)_p$$
$$\cup\, H(L)_p \begin{pmatrix} p & & \\ & \pi^\sigma & \\ & & 1 \end{pmatrix} G(L)_p.$$

It follows easily that

$$\begin{pmatrix} 1 & & \\ & \pi^\sigma & \\ & & p \end{pmatrix},\ h(w,0) \begin{pmatrix} \pi^\sigma & & \\ & \pi & \\ & & \pi^\sigma \end{pmatrix},\ \text{and } h(w^\sigma,u) \begin{pmatrix} p & & \\ & \pi^\sigma & \\ & & 1 \end{pmatrix}$$

give a set of coset representatives for $S(P^\sigma)_p/G(L)_p$ where w runs over representatives for $\mathcal{O}_p/\pi\mathcal{O}_p$ and u runs over representatives for $\mathbf{Z}_p/p\mathbf{Z}_p$. Since $S(P^\sigma)_p = S(P)_p^\iota$, we may take the above representatives as our g_j^{ι}'s. The images of these g_j^{ι}'s in $GL_3(\mathbf{Q}_p)\times GL_3(\mathbf{Q}_p)$ under the isomorphism corresponding to the above "standard" action will be

$$\left[\begin{pmatrix}1&&\\&1&\\&&p\end{pmatrix},\begin{pmatrix}p&&\\&p&\\&&1\end{pmatrix}\right],\ \left[\begin{pmatrix}1&&\\&p&b\\&&1\end{pmatrix},\begin{pmatrix}p&&\\&1&\\&-b&p\end{pmatrix}\right]$$

and

$$\left[\begin{pmatrix}p&\omega b&a\\&1&0\\&&1\end{pmatrix},\begin{pmatrix}1&&\\-\omega b&p&\\-a&&p\end{pmatrix}\right]$$

where a and b run over a set of representatives for $\mathbf{Z}_p/p\mathbf{Z}_p$ and the image of $2i$ in $\mathbf{Q}_p\times\mathbf{Q}_p$ is $[\omega,-\omega]$.

The images of these elements in $M_3(\mathbf{F}_p)\times M_3(\mathbf{F}_p)$ have cokernels of dimension 3, and the multiplicities $m(P,X)$ given in (1) of the proposition are obtained by counting the number of these cokernels which contain X. We may, of course, take X to be one of our orbit representatives above.

If $P=P^{\sigma}=(p)$ is unramified, we must determine $G(L)_p$ orbits in L_p/p^2L_p. First observe that there are precisely three $G(L)_p$ orbits in L_p/pL_p represented by 0, ${}^t(0,1,0)$, and ${}^t(0,0,1)$. In fact

$$L_p/pL_p\simeq\mathbf{F}_{p^2}^3$$

and R determines a non-degenerate Hermitian form on this space. If $\tilde{G}_p$ is the group of similitudes of this form, then

$$\mu:\tilde{G}_p\to\mathbf{F}_p^{x}\to 1$$

and so $\tilde{G}_p$, and hence $G(L)_p$, acts transitively on $\{X\in L_p/pL_p\mid(X,X)\not\equiv 0 \bmod p\}$. If $X\in\mathbf{F}_{p^2}^3$ with $(X,X)=0$ but $X\neq 0$, then we may complete X to a Witt basis X,Y,Z for $\mathbf{F}_{p^2}^3$ with $(X,Y)=1$, $(Y,Y)=(X,Z)=(Y,Z)=0$ and $(Z,Z)=1$. Thus such X form a single $G(L)_p$ orbit.

Now suppose that $X\in L_p/p^2L_p$ with $(X,X)\equiv 0 \bmod p^2$ but $X\not\in pL_p$. Then there exists $g\in G(L)_p$ such that

$$gX={}^t(px_1,px_2,1+px_3)$$

and $a=1+px_3\in\mathcal{O}_p^{x}$. Then

$$Y = \begin{pmatrix} a^\sigma & & \\ & 1 & \\ & & a^{-1} \end{pmatrix} X = \begin{pmatrix} py_1 \\ py_2 \\ 1 \end{pmatrix}$$

where

$$\begin{pmatrix} a^\sigma & & \\ & 1 & \\ & & a^{-1} \end{pmatrix} \in G(L)_p.$$

The condition $(Y, Y) \equiv 0 \bmod p^2$ is equivalent to

$$-(2i)^{-1}p(y_1^\sigma - y_1) - p^2 y_2 y_2^\sigma \equiv 0 \bmod p^2$$

and hence

$$y_1^\sigma - y_1 \equiv 0 \bmod p.$$

Therefore

$$py_1 \equiv pb \bmod p^2\mathcal{O}_p$$

with $b \in \mathbf{Z}_p$, and so

$$h(-py_2, -pb)Y \equiv {}^t(0, 0, 1) \bmod p^2 L_p.$$

This shows that the set C_1 in (2) of the proposition corresponds to a single $G(L)_p$ orbit in L_p/p^2L_p; and by the previous remarks, the set

$$\{X \in L_p/p^2L_p \mid (X, X) \equiv 0 \bmod p^2, X \in pL_p\}$$

breaks up into three orbits corresponding to C_0, C_{00} and C_{000} above.

(2) Let $H(L)_p$ be as before. Then Shintani shows that

$$S(P)_p = \begin{pmatrix} 1 & & \\ & p & \\ & & p^2 \end{pmatrix} G(L)_p \cup \bigcup_c \begin{pmatrix} p & & \\ & p & \\ & & p \end{pmatrix} G(L)_p$$

$$\cup\, H(L)_p \begin{pmatrix} p^2 & & \\ & p & \\ & & 1 \end{pmatrix} G(L)_p$$

where c runs over a set of representatives for $\mathbf{Z}_p/p\mathbf{Z}_p$. It follows easily that

$$\begin{pmatrix}1&&\\&p&\\&&p^2\end{pmatrix},\ \begin{pmatrix}p&&c\\&p&\\&&p\end{pmatrix}\text{ and }\begin{pmatrix}p^2&2ipb^\sigma&a+ibb^\sigma\\&p&b\\&&1\end{pmatrix}$$

are a set of coset representatives for $S(P)_p/G(L)_p$, where a runs over $\mathbf{Z}_p/p^2\mathbf{Z}_p$, b runs over $\mathcal{O}_p/p\mathcal{O}_p$, and c is as before. Since $S(P)_p^\iota = S(P)_p$ in this case, we may take these coset representatives as the g_j^ι's. Again if we consider the cokernels of these g_j^ι's acting in L_p/p^2L_p, we find the claimed values for $m(P, X)$ as the number of such cokernels which contain X.

(3) Finally in the case $P = P^\sigma$ is ramified, so $P = (\pi)$, $\pi = (1+i)$, $p = 2$; there are in fact 11 $G(L)_p$ orbits in L_p/pL_p for which we may choose representatives ${}^t(1+\pi, 0, 0)$, ${}^t(1+\pi, \pi, 0)$, ${}^t(1, \pi, 0)$, and ${}^t(1, 0, 0)$ corresponding to C_1; ${}^t(\pi, 0, 0)$ corresponding to C_0; ${}^t(0, \pi, 0)$ corresponding to C_{00}; 0 corresponding to C_{000}, and ${}^t(1, 0, \pi)$, ${}^t(\pi, 1, 0)$, ${}^t(0, 1+\pi, 0)$ and ${}^t(0, 1, 0)$ corresponding to 'otherwise'. Applying Lemma 2 of Shintani, we find coset representatives

$$\begin{pmatrix}1&&\\&\pi&\\&&p\end{pmatrix},\ \begin{pmatrix}\pi&&\\&\pi&\\&&\pi\end{pmatrix},\ \begin{pmatrix}\pi&i\pi^2&-1\\&\pi&1\\&&\pi\end{pmatrix},\ \begin{pmatrix}p&&\\&\pi^\sigma&\\&&1\end{pmatrix},$$

$$\begin{pmatrix}p&ip\pi^\sigma&i\\&\pi^\sigma&1\\&&1\end{pmatrix},\ \begin{pmatrix}p&ip\pi^\sigma&1+i\\&\pi^\sigma&1\\&&1\end{pmatrix},$$

and

$$\begin{pmatrix}p&0&1\\&\pi^\sigma&0\\&&1\end{pmatrix},$$

for $S(P)_p/G(L)_p$. Using these as our g_j^ι's – again $S(P)_p^\iota = S(P)_p$ – we can compute the $m(P, X)$'s as in the previous cases, and the proposition is proved.

REMARK: Although we have considered only the case $K = \mathbf{Q}(i)$, the proof of the previous proposition, at least in cases (1) and (2), is identical for an arbitrary quadratic extension K/k. Therefore both the theorem of §1 and the resulting relation between the L-series of the lift and that of the original form should hold in the general case.

PROOF OF THE THEOREM: We use the values for $m(P, X)$ just determined.

If $P \neq P^\sigma$, $P = (\pi)$ with $\pi \in K$, (2.2) becomes

$$(2.3)\quad p^{k/2}\bar{\theta} \mid T(P) = y^2 \sum_{X \in C_1 \cup C_{0+} \cup C_{0-} \cup C_{00}} \overline{f_{p^{-1}\tau,\mathfrak{z}}(X)} + py^2 \sum_{X \in C_{0-} \cup C_{00}} \overline{f_{p^{-1}\tau,\mathfrak{z}}(X)} + p^2y^2 \sum_{X \in C_{00}} \overline{f_{p^{-1}\tau,\mathfrak{z}}(X)}.$$

On the other hand,

$$(2.4)\quad \overline{\theta \mid T'(p)_{k-1,\psi}} = p^{-1} \sum_{b=0}^{p-1} \overline{\theta(p^{-1}(\tau+b), \mathfrak{z})} + p^{k-2}\overline{\theta(p\tau, \mathfrak{z})} = p^{-2}y^2 \sum_{\substack{X \in L \\ (X, X) \equiv 0(p)}} \overline{f_{p^{-1}\tau,\mathfrak{z}}(X)} + p^k y^2 \sum_{X \in L} \overline{f_{p\tau,\mathfrak{z}}(X)}.$$

Since

$$C_1 \cup C_{0+} \cup C_{0-} \cup C_{00} = \{X \in L \mid (X, X) \equiv 0 \bmod p\},$$

the first term on the right side of (2.3) is just p^2 times the first term on the right side of (2.4). Also we have

$$p^2y^2 \sum_{X \in C_{00}} \overline{f_{p^{-1}\tau,\mathfrak{z}}(X)} = p^{k+2}y^2 \sum_{X \in L} \overline{f_{p\tau,\mathfrak{z}}(X)}$$

so that this term is just p^2 times the last term in (2.4). Now

$$C_{0-} \cup C_{00} = \pi^\sigma L.$$

So that

$$py^2 \sum_{X \in C_{0-} \cup C_{00}} \overline{f_{p^{-1}\tau,\mathfrak{z}}(X)} = p(\pi^\sigma)^k y^2 \sum_{X \in L} \overline{f_{\tau,\mathfrak{z}}(X)} = p(\pi^\sigma)^k \overline{\theta(\tau, \mathfrak{z})},$$

and the first part of the theorem is proved.

If $P = P^\sigma = (p)$ is unramified, (2.2) becomes:

(2.5)
$$p^k\bar{\theta}\mid T(P) = y^2 \sum_{X\in C_1\cup C_0\cup C_{00}\cup C_{000}} \overline{f_{p^{-2}\tau,\mathfrak{z}}(X)} + py^2 \sum_{X\in C_0\cup C_{00}\cup C_{000}} \overline{f_{p^{-2}\tau,\mathfrak{z}}(X)}$$
$$+p^2y^2 \sum_{X\in C_0} \overline{f_{p^{-2}\tau,\mathfrak{z}}(X)} + p^4y^2 \sum_{X\in C_{000}} \overline{f_{p^{-2}\tau,\mathfrak{z}}(X)},$$

while

(2.6)
$$\overline{\theta\mid T'(p^2)_{k-1,\psi}} = p^{-4}y^2 \sum_{\substack{X\in L\\ (X,X)\equiv 0(p^2)}} \overline{f_{p^{-2}\tau,\mathfrak{z}}(X)}$$
$$-p^{k-2}y^2 \sum_{\substack{X\in L\\ (X,X)\equiv 0(p)}} \overline{f_{\tau,\mathfrak{z}}(X)} + p^{2k}y^2 \sum_{X\in L} \overline{f_{p^2\tau,\mathfrak{z}}(X)}.$$

If we write A_1, A_2, A_3 and A_4 for the four terms on the right side of (2.5) and B_1, B_2 and B_3 for the terms on the right side of (2.6); then, as in the previous case;

$$p^4B_1 = A_1 \text{ and } p^4B_3 = A_4.$$

At the same time

$$A_3 = p^{k+2}y^2 \sum_{\substack{X\in L\\ (X,X)\not\equiv 0(p)}} \overline{f_{\tau,\mathfrak{z}}(X)}$$
$$p^4B_2 = -p^{k+2}y^2 \sum_{\substack{X\in L\\ (X,X)\equiv 0(p)}} \overline{f_{\tau,\mathfrak{z}}(X)},$$

so that

$$A_3 - p^4B_2 = p^{k+2}\theta(\tau, \mathfrak{z}).$$

Finally

$$A_2 = p^{k+1}y^2 \sum_{X\in L} \overline{f_{\tau,\mathfrak{z}}(X)}$$
$$= p^{k+1}\overline{\theta(\tau, \mathfrak{z})},$$

and so

$$p^k\bar{\theta}\mid T(P) = \overline{\theta\mid (p^4T'(p^2)_{k-1,\psi} + p^{k+2} + p^{k+1})}$$

as claimed.

If $P=(1+i)$, $p=2$; (2.2) becomes:

$$p^{k/2}\bar{\theta}\mid T(P)=y^2\sum_{X\in C_1\cup C_0\cup C_{00}\cup C_{000}}\overline{f_{p^{-1}\tau,\mathfrak{z}}(X)}$$

$$+py^2\sum_{X\in C_0\cup C_{00}\cup C_{000}}\overline{f_{p^{-1}\tau,\mathfrak{z}}(X)}+p^2y^2\sum_{X\in C_{00}\cup C_{000}}\overline{f_{p^{-1}\tau,\mathfrak{z}}(X)}, \tag{2.7}$$

while

$$\overline{\theta\mid T'(p)_{k-1,\psi}}=p^{-2}y^2\sum_{\substack{X\in L\\(X,X)\equiv 0(p)}}\overline{f_{p^{-1}\tau,\mathfrak{z}}(X)}. \tag{2.8}$$

Thus the first term on the right side of (2.7) is $p^2\overline{\theta\mid T'(p)_{k-1,\cdot}}$. The second term on the right side of (2.7) is

$$p(1+i)^k y^2\sum_{X\in L}\overline{f_{\tau,\mathfrak{z}}(X)}=2(1+i)^k\overline{\theta(\tau,\mathfrak{z})}.$$

Finally, we note that if $\Gamma'=\Gamma_0(4)$,

$$\Gamma'\begin{pmatrix}2&\\&1\end{pmatrix}\Gamma'=\Gamma'\begin{pmatrix}2&\\&1\end{pmatrix}\cup\Gamma'\begin{pmatrix}1&\\-2&1\end{pmatrix}\begin{pmatrix}2&\\&1\end{pmatrix}$$

so that

$$\theta\mid T''(2)_{k-1,\psi}=2^{k-2}\theta(2\tau,\mathfrak{z})+2^{(k-1)/2-1}\theta\mid\begin{pmatrix}1&\\-2&1\end{pmatrix}\begin{pmatrix}2&\\&1\end{pmatrix}. \tag{2.9}$$

Since for $h\in L^*$,

$$\theta(\tau+1,\mathfrak{z},h)=e((h,h))\theta(\tau,\mathfrak{z},h)$$

and

$$\theta(-1/\tau,\mathfrak{z},h)=-\tfrac{1}{2}i(-\tau)^{k-1}\sum_{h'\in L^*/L}e((h,h'))\theta(\tau,\mathfrak{z},h'),$$

we obtain

$$\theta\mid\begin{pmatrix}1&\\-2&1\end{pmatrix}=\theta(\tau,\mathfrak{z},h_0),$$

where

$$h_0=\tfrac{1}{2}\,{}^t(0,1+i;0).$$

Then the second term on the right side of (2.9) is just

$$2^{k-2}\theta(2\tau, \mathfrak{z}, h_0).$$

On the other hand, in the last term on the right side of (2.7) we have:

$$p^2y^2 \sum_{X\in C_{000}} \overline{f_{p^{-2}\tau,\mathfrak{z}}(X)} = p^{k+2}\overline{\theta(2\tau, \mathfrak{z})}.$$

Finally, setting $\pi = 1+i$ and writing $X = \pi Y$, we have:

(2.10) $$p^2y^2 \sum_{X\in C_{00}} \overline{f_{p^{-1}\tau,\mathfrak{z}}(X)} = \pi^k p^2 y^2 \sum_{\substack{Y\in L\\ Y\notin \pi^\sigma L\\ p^{-1}\pi Y\in L^*}} \overline{f_{\tau,\mathfrak{z}}(Y)}.$$

Setting $Y' = p^{-1}\pi Y$ we have:

$$Y' \in L^*,\ Y' \notin L \text{ and } \pi^\sigma Y' \in L$$

hence

$$Y' \in h_0 + L,$$

and (2.10) becomes

$$p^{k+2}y^2 \sum_{Y'\in h_0+L} \overline{f_{p\tau,\mathfrak{z}}(Y')} = p^{k-2}\overline{\theta(2\tau, \mathfrak{z}, h_0)}.$$

This completes the proof of the theorem.

§3. Functoriality

Let G and H be the reductive groups over $\mathbf{Q}$ defined in the introduction. Following the notation of [1] we find that

$$^LG^0 \simeq GL_3(\mathbf{C}) \times \mathbf{C}^x$$

and that the action of σ is given by

$$(g, t)^\sigma = (w\check{g}wt, t)$$

where $g = {}^t g^{-1}$ and

$$w = \begin{pmatrix} & & -1 \\ & 1 & \\ -1 & & \end{pmatrix}.$$

Then we take ${}^L G = {}^L G^0 \rtimes \langle \sigma \rangle$. Similarly

$${}^L H^0 \simeq GL_2(\mathbf{C}) \times \mathbf{C}^x$$

and the action of σ is given by:

$$(h, s)^\sigma = (h, s^{-1} \det h).$$

Let ${}^L H = {}^L H^0 \rtimes \langle \sigma \rangle$.

Now the Euler product considered by Shintani arises in the following way. Let ρ^0 by the 3-dimensional representation of ${}^L G^0$ given by

$$\rho^0(g, t) = gt \det g^{-1}$$

and let

$$\rho = \mathrm{Ind}_{{}^L G^0}^{{}^L G} \rho^0.$$

Let

$${}^L T^0 = \left\{ t = \left(\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}, t_0 \right) \in {}^L G^0 \right\}$$

If $F \in A_k(G(L))$ is a Hecke eigenform with eigenvalues $\lambda(P)$ and central character χ as in §1, representatives in ${}^L T^0 \times \sigma_p$ of the corresponding semi-simple ${}^L G^0$-conjugacy classes in ${}^L G^0 \times \sigma_p$ are determined as follows. If $p = PP^\sigma$, with $P \neq P^\sigma$ in $K/\mathbf{Q}$,

$$\lambda(P^\sigma) = p(t_1 + t_2 + t_3)$$
$$\lambda(P) = p(t_1^{-1} + t_2^{-1} + t_3^{-1})$$
$$\chi(P) = t_1 t_2 t_3$$

and

$$t_0 = 1.$$

If $(p) = P$ is inert in $K/\mathbf{Q}$,

$$\lambda(P) - p = p^2(t_1 t_3^{-1} + t_1^{-1} t_3)$$

and

$$\chi(P) = t_0.$$

Note that, in the latter case, the condition that t be σ-invariant forces $t_2^2 = 1$ and $t_3^{-1} = t_1$ as well.

We then find that, for $p \neq 2$, Shintani's Euler factor is

$$\zeta_p(s-1, \chi, F) = \det(1 - \rho(t \rtimes \sigma_p) p^{-s})^{-1}$$

where σ_p is the Frobenius in $\mathrm{Gal}(K/\mathbf{Q})$ at p.

We now construct an L-homomorphism. First define $\alpha^0 : {}^L H^0 \to {}^L G^0$ by

$$\alpha^0 : (h, s) \to \left[\begin{pmatrix} a & & b \\ & s & \\ c & & d \end{pmatrix}, \det h \right]$$

for $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbf{C})$. This can be extended to a homomorphism $\alpha : {}^L H \to {}^L G$ by taking

$$\alpha : 1 \rtimes \sigma \to \left[\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, -1 \right] \rtimes \sigma.$$

Finally, to a normalized Hecke eigenform $f \in S_{k-1}(\Gamma_0(4), \psi)$ we associate, for each $p, p \neq 2$, the semi-simple ${}^L H^0$-conjugacy class in ${}^L H^0 \rtimes \sigma_p$ represented by the element

$$s \rtimes \sigma_p$$

where

$$s = \left[\begin{pmatrix} s_1 & \\ & s_2 \end{pmatrix}, s_0 \right] \in {}^L H^0$$

and σ_p is the Frobenius at p. Here

$$p^{1/2}(s_1+s_2) = p^{1-(1/2)(k-1)}a_p$$
$$s_1 s_2 = \psi(p)$$

and

$$s_0 = \begin{cases} \chi(P) & \text{if } p = PP^\sigma, P \neq P^\sigma \\ \psi(p) & \text{if } (p) = P. \end{cases}$$

We then find that, for $p \neq 2$, the eigenvalue $\lambda(P)$ associated to $\alpha(s \rtimes \sigma_p) \in {}^LT^0 \rtimes \sigma_p$ is just as in Corollary 1, and that

$$\det(1 - \rho \circ \alpha(s \times \sigma_p)p^{-s}) = L_p(s, \chi)L_p\left(s - 1 + \frac{k}{2}, f\right)L_p\left(s - 1 + \frac{k}{2}, f^\rho\right).$$

This shows that, in the notation of [1, 7.2],

$$L_p(s, \mathscr{L}(f), \rho) = L_p(s, f, \rho \circ \alpha)$$

for classical Hecke eigenforms $f \in S_{k-1}(\Gamma_0(4), \psi)$, and hence that, for such forms the Weil representation defines a global lifting compatible locally at almost all p with the L-homomorphism α.

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