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EACH SCHWARTZ FRÉCHET SPACE IS A SUBSPACE OF A SCHWARTZ FRÉCHET SPACE WITH AN UNCONDITIONAL BASIS

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Abstract

It is shown that for $1 \leq p \leq \infty$, {Schwartz Fréchet spaces embeddable in a product of ℓ_p-spaces} = {subspaces of Schwartz Fréchet Köthe ℓ_p-sequence spaces}. As a corollary, we obtain the fact that every Schwartz Fréchet space is a subspace of Schwartz Fréchet space with an unconditional basis. Furthermore, the Schwartz variety has a universal generator which is separable, barreled and bornological. These results are shown to be best possible.

It is well known that there are Banach spaces which are not isomorphic to any subspace of a Banach space with an unconditional basis. The author, in [4], has produced examples of Fréchet Montel spaces which are not isomorphic to any subspace of a Fréchet space (Montel or otherwise) with an unconditional basis. On the other hand, each nuclear Fréchet space is isomorphic to a subspace of s^N (a countable product of spaces isomorphic to the space of rapidly decreasing sequences) which is a nuclear Fréchet and it has an unconditional basis [10]. As the title indicates, the results of this paper show how to embed any Schwartz Fréchet space into a Schwartz Fréchet space with an unconditional basis.

The Theorem (below) actually yields more. For $1 \leq p \leq \infty$, let S_p = {Schwartz spaces that are isomorphic to a subspace of a product of ℓ_p-spaces} and let $\mathcal{B}_p$ be the set of Fréchet spaces in S_p. Then each space in $\mathcal{B}_p$ is isomorphic to a subspace of some Schwartz Fréchet Köthe ℓ_p-sequence space. There are two weaker results in the literature of this kind. Randtke [13] has shown that every space in $\mathcal{B}_\infty$ is isomorphic to a subspace of a compact projective limit of a sequence

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of spaces, each isomorphic to c_0, and in [2] there is a similar result for $\mathscr{B}_2$. Both of these results depend on special properties of c_0 or ℓ_2, and their proofs do not generalize to $\mathscr{B}_p$. The Theorem is a much stronger result whose proof is surprisingly easy.

As a corollary, we obtain a separable barreled and bornological universal generator for S_p. The Proposition (below) shows that this is best possible, in the sense that there are no Fréchet universal generators for S_p. This generalizes a result of Moscatelli [9], who proved it for $p = \infty$. Moreover, the proof below avoids the use of bornologies in [9] by the use of approximate dimension rather than diametral dimension (see [10]). The previously known universal generators for S_p (see [6], [12], [13], [9], [2] and [3]) are not even σ-barreled (= bounded sequences in the dual are equicontinuous).

1. Terminology

A LCS is a locally convex topological vector space which is Hausdorff. If ρ is a continuous semi-norm on the LCS E, then E_ρ is the normed space $(E/\ker \rho, \rho)$. The LCS E is *Schwartz* (respectively, *nuclear*) if for each continuous semi-norm ρ, there is a continuous semi-norm μ, with $\rho \leq \mu$ and so that the identity on E induces a precompact (respectively nuclear) map $E_\mu \to E_\rho$. A *Fréchet* space is a complete metrizable LCS. A Fréchet space is *Montel* if each closed bounded set is compact. For Fréchet spaces, nuclear implies Schwartz implies Montel, but neither implication can be reversed.

Let N denote the set of natural numbers. For $1 \leq p \leq \infty$, ℓ_p denotes the Banach space of all p-sumable scalar sequences (α_n) with norm $\|(\alpha_n)\| = [\Sigma|\alpha_n|^p]^{1/p}$. We will use ℓ_∞ to denote the Banach space of bounded scalar sequences (α_n) with norm $\|(\alpha_n)\| = \sup|\alpha_n|$ and c_0 will be its subspace of null-scalar sequences.

A *prevariety* [1] is a collection of LCS's (locally convex spaces) closed with respect to the formation of subspaces and arbitrary products. A *variety* [5] is a prevariety which is also closed under the formation of quotients by closed subspaces. The collections S_p, $1 \leq p \leq \infty$ are prevarieties [3], while the collections (nuclear spaces), {Schwartz spaces} = S_∞, and S_2 are varieties [5], [2]. A LCS X is a *universal generator* for the prevariety $\mathscr{P}$ if $x \in \mathscr{P}$ and each $Y \in \mathscr{P}$ is isomorphic to a subspace of X^I, for some index set I.

Suppose $1 \leq p \leq \infty$ and let $[a_n^k]_{k,n=1}^\infty$ be a matrix of non-negative reals. Define for each k and scalar sequence (α_n), $\|(\alpha_n)\|_k = \|\Sigma_n a_n^k \alpha_n\|_p$, where $\|\cdot\|_p$ is the ℓ_p-norm. The set of sequences (α_n), with $\|(\alpha_n)\|_k < \infty$, for each k along with the semi-norms $\{\|\cdot\|_k\}$ defines a Fréchet space which is called a *Köthe ℓ_p-sequence space* [8, p. 419]. Let $1 \leq p \leq \infty$ and $[a_n^k]$ be as above with $a_n^{k+1} \geq a_n^k$ and Y be the resulting Köthe ℓ_p-sequence space. It is well known that Y is Schwartz if and only if, for each k, there is a j, with $\lim_n(a_n^k/a_n^j) = 0$. (Here $0/0 = 0$.)

Finally we note that we use the notion of approximate dimension of LCS's without definition. The definition and its basic properties may be found in Pietsch [10].

2. Results

THEOREM: *Each Fréchet space in S_p is isomorphic to a subspace of a Schwartz Fréchet Köthe ℓ_p-sequence space.*

PROOF: Let $X \in \mathscr{B}_p$. Since ℓ_p is isomorphic to its square, there is a neighborhood basis $\{U(n)\}$ of the origin in X with the following properties:

(1) For each n, X_n (the quotient space $X/ker\ \rho_{U(n)}$ with norm $\rho_{U(n)}$) is isomorphic to a subspace of ℓ_p, say by the map i_n: $X_n \to \ell_p$.

(2) For each n, the canonical map j_n: $X_{n+1} \to X_n$ is precompact.

(3) X is isomorphic to a subspace of the projective limit of

$$\cdots \to X_{n+1} \xrightarrow[j_n]{} X_n \to \cdots \xrightarrow[j_1]{} X_1.$$

For each m, we construct a Köthe ℓ_p-sequence space K_m as follows. We define R_k, S_k, T_k by induction. Let $S_1: X_{m+1} \to \ell_p$ be the composite $i_m j_m$, Since S_1 is compact, by Lemma 2.1 of [3] there are compact maps $R_1: X_{m+1} \to \ell_p$ and $T_1: \ell_p \to \ell_p$ with $S_1 = T_1 R_1$ and T_1 is a diagonal map. Suppose $R_k: X_{m+k} \to \ell_p$, $S_k: X_{m+k} \to \ell_p$ and $T_k: \ell_p \to \ell_p$ have been chosen. Let $S_{k+1} = R_k j_{m+k+1}$, and again factor $S_{k+1} = R_{k+1} T_{k+1}$ so that T_{k+1} is a diagonal compact map: $\ell_p \to \ell_p$ and $R_{k+1}: X_{m+k+1} \to \ell_p$ is compact. Let K_m be the projective limit of

$$\cdots \to \ell_p \xrightarrow[T_k]{} \ell_p \to \cdots \xrightarrow[T_1]{} \ell_p$$

and let ϕ_m be the induced map: $X \to K_m$ (see Figure 1).

By construction, K_m is a Schwartz Fréchet Köthe ℓ_p-sequence space, the map ϕ_m: $X \to K_m$ is continuous, and $\phi_m(U(m))$ is relatively open in K_m. Thus the map ϕ: $X \to \Pi_{m=1}^{\infty} K_m$ is an isomorphism into if

$$
\begin{tikzcd}
X = \text{proj lim} \arrow[d, "\phi_m"] & \cdots \arrow[r] & X_{m+k+1} \arrow[r, "j_{m+k}"] \arrow[d, "R_{k+1}"'] \arrow[dr, "S_{k+1}"] & X_{m+k} \arrow[r] \arrow[d, "R_k"] & \cdots \arrow[r] & X_{m+1} \arrow[r, "j_m"] \arrow[d, "R_1"'] \arrow[dr, "S_1"] & X_m \arrow[r] \arrow[d, "i_m"] & \cdots \\
K_m = \text{proj lim} & \cdots \arrow[r] & \ell_p \arrow[r, "T_{k+1}"'] & \ell_p \arrow[r, "T_k"'] & \cdots \arrow[r, "T_2"'] & \ell_p \arrow[r, "T_1"'] & \ell_p &
\end{tikzcd}
$$

Figure 1

$\phi(x) = (\phi_1(x), \phi_2(x), \ldots)$. It follows from [8, p. 409] that this product space is a Köthe ℓ_p-sequence space and clearly it is Schwartz and Fréchet.

REMARKS 1: If X is a Banach space with a f.d.d. and $X \oplus X$ is isomorphic to X, then there is an analogous theorem between S_X and the Schwartz Fréchet sequence spaces modeled on X (see [3]).

2. Let D and D' be compact diagonal maps: $\ell_p \to \ell_p$, if e_n is the usual basis for ℓ_p, we will say D is weaker than D' if $D'e_n \neq 0 \Rightarrow De_n \neq 0$ and $\|De_n\|/\|D'e_n\| \to \infty$ as $n \to \infty$. The proof of Theorem remains valid if we replace T_k (any or all) by weaker compact diagonal maps: $\ell_p \to \ell_p$.

It is well known that there is no countable cofinal subset in this ordering. Hence there is a D: $\ell_p \to \ell_p$ which is weaker than any of the T_k's. Let $K_p(D)$ be the resulting Köthe sequence space that is obtained in the Theorem with D replacing all the T_k's. We have:

COROLLARY 1: *For each $E \in \mathscr{B}_p$, there is a diagonal compact map $D: \ell_p \to \ell_p$, so that E is isomorphic to a subspace of $K_p(D)$.*

COROLLARY 2: *Each Schwartz Fréchet space is a subspace of a Schwartz Fréchet space with an unconditional basis.*

PROOF: Schwartz-Köthe ℓ_p-sequence spaces have an unconditional basis.

COROLLARY 3: *Let $1 \leq p \leq \infty$, then S_p has a universal generator which is separable barreled and bornological.*

PROOF: Let $X \in S_p$. Since S_p is an operator defined prevariety [1] (X is $\mathscr{I}$-space in the sense of [11] with $\mathscr{I}$ = compact maps which factor through a subspace of ℓ_p) and by a Proposition 3.1 of [1], X is isomorphic to a subspace of a product of Fréchet spaces in S_p. Thus by Corollary 1, X is a subspace of a product of $U_p = \Pi\{K_p(D): D$ diagonal compact: $\ell_p \to \ell_p\}$, and U_p is a universal generator for S_p.

Each $K_p(D)$ is separable, since it is a Fréchet Montel space, which is separable by [8, p. 370]. The cardinality of $(D: \ell_p \to \ell_p$ diagonal compact$) = c =$ cardinality of the Reals, since card $(c_0) = c$. Hence by [7, p. 32], U_p is separable and is easily seen to be barreled and bornological.

The Proposition below shows that Corollary 3 cannot be improved to produce a Fréchet universal generator. By Theorem 5.4 of [3], $S_\infty \supset S_p \supset S_2$ for $1 \leq p \leq \infty$.

PROPOSITION: *Let $\mathcal{P}$ be any prevariety satisfying $S \supset \mathcal{P} \supset S_2$, then $\mathcal{P}$ has no Fréchet universal generator.*

PROOF: The proof requires the notion of approximate dimension. We refer the reader to [10, pp. 160–165] for notation and definitions.

Suppose E is a Fréchet universal generator for $\mathcal{P}$. Then $F = E^N$ is a Fréchet space in $\mathcal{P}$, which contains every Fréchet space in $\mathcal{P}$ as a subspace. Let $\{U(m)\}$ be a basis of neighborhoods of the origin of F with $U(m+1)$ precompact with respect to the seminorm $\rho_{U(m)}$. Let $\phi_n(\epsilon) = M_\epsilon(U(n+1), U(n))$, we note that $\phi_n(\epsilon)$ is defined for $\epsilon > 0$ and $\phi_n(\epsilon) \to \infty$ as $\epsilon \to 0$. There is a $\phi(\epsilon)$, defined on $(0, \infty)$, so that $\phi^{-1}(\epsilon)\phi_n(\epsilon) \to 0$ as $\epsilon \to 0$, for $n = 1, 2, \ldots$ and $\phi(\epsilon) \to \infty$ as $\epsilon \to 0$. $(\phi(\epsilon) = \epsilon^{-1} \sup\{\phi_m(\epsilon): m \leq n\}$ for $\epsilon \in (1/n+1, 1/n)$.) Hence $\phi(\epsilon) \in \Phi_U(F)$, and by Proposition 1 of [10, p. 165], $\phi(\epsilon) \in \Phi_U(X)$ for each Fréchet space $X \in \mathcal{P}$. We complete the proof by constructing a Fréchet space X in S_2 with $\phi(\epsilon) \notin \Phi_U(X)$ obtaining a contradiction.

Let $N_0 = 0$ and inductively choose $N_n > N_{n-1}$ so that $N_n > \phi(1/n^2)$. Let $\lambda_k = 1/n$ for $N_{n-1} < k \leq N_n$ and let $T: \ell_p \to \ell_p$ be the diagonal map which sends e_n to $\lambda_n e_n$. Let $\delta = 1/m$, $n \geq m$ and $U =$ unit ball of ℓ_2. It is easy to see that $M_{1/n^2}(T(\delta U), U) \geq N_n = K$ since $\delta\lambda_K \geq (1/m)(1/n) \geq 1/n^2$. Thus $\phi^{-1}(\epsilon)M_\epsilon(T(\delta U), U) \not\to 0$ as $\epsilon \to 0$. Let $\mu^k = (\mu^k_n)$ be the sequences with $\mu^k_n > 0$ and $(\mu^k_n)^{2^k} = \lambda_n$ and let $T_k: \ell_2 \to \ell_2$ be the diagonal map which sends e_n to $\mu^k_n e_n$. Let $X \in S_2$ be the projective limit:

$$\begin{tikzcd} \ell_2 \arrow[r, "\theta"] \arrow[d, "id"'] & X = \text{proj lim} \ldots \arrow[r] & \ell_2 \arrow[r, "T_2"] & \ell_2 \arrow[r, "T_1"] & \ell_2 \arrow[d, "id"] \\ \ell_2 \arrow[rrrr, "T"] & & & & \ell_2 \end{tikzcd}$$

and let $\theta : \ell_2 \xrightarrow{\theta} X$ that makes Figure 2 a commutative diagram. To see that $\phi(\epsilon) \notin \Phi_U(X)$, let U = unit ball in the ℓ_2 in Figure 2 which is the range of T, and suppose V is a neighborhood of X. There is $\delta = 1/m$ so that $\theta(\delta U) \subset V$ and thus $\phi^{-1}(\epsilon)M_\epsilon(V, U) \geq \phi^{-1}(\epsilon)M_\epsilon$ $\phi^{-1}(\epsilon)M_\epsilon(T(\delta U), U) \nrightarrow 0$ as $\epsilon \to 0$. This completes the proof.

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