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# HIGHER DE RHAM–WITT COMPLEXES OF SUPERSINGULAR K3 SURFACES

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## Contents



## Introduction

Let $X/k$ be a proper smooth variety of dimension $r$ over a perfect field $k$ characteristic $p > 0$. We define the higher de Rham–Witt complexes by

$$W\Omega^{\cdot}_X(n) = 0 \longrightarrow W\mathcal{O}_X \xrightarrow{F^n d} W\Omega^1_X \xrightarrow{d} \cdots \longrightarrow W\Omega^r_X \longrightarrow 0$$

in particular $W\Omega^{\cdot}_X(0)$ is the ordinary de Rham–Witt complex. These complexes were used in [6] and [9] to study the torsion in crystalline cohomology.

In this paper we study the higher de Rham–Witt complexes in the case where $X$ is a supersingular K3 surface defined over an algebraically closed field $k$ of characteristic $p > 2$. We take the term supersingular to mean that $\hat{B}r_X = \hat{\mathbb{G}}_a$, but do not assume that rank $NS(X) = 22 = b_2$.

It turns out that for sufficiently small $n$, the hypercohomology $\mathsf{H}^2(X, W\Omega^{\cdot}_X(n))$ is related to the cupproduct pairing on $H^2_{fl}(X, \mathbb{Z}_p(1))$, in fact for $n$ sufficiently small $\mathsf{H}^2(X, W\Omega^{\cdot}_X(n))$ injects into the dual of $H^2_{fl}(X, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} W(k)$, thus the spaces involved in Ogus'

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classification of supersingular K3 crystals [11] can be defined in terms of the higher de Rham–Witt complexes.

Ogus shows that $H^2_{fl}(X, \mathbb{Z}_p(1))$ splits as an orthogonal sum under the cupproduct pairing

$$(H^2_{fl}(X, \mathbb{Z}_p(1)), (\ ,\ )) \simeq (T_0, p\langle\ ,\ \rangle) \oplus (T_1, (\ ,\ ))$$

where the pairings $\langle\ ,\ \rangle$ and $(\ ,\ )$ are perfect. Our main result is that there is a natural isomorphism

$$T_0 \otimes k \simeq \ker F^{\sigma_0} d \subset H^2(X, W\mathcal{O}_X)$$

where $\sigma_0$ is the Artin invariant of $X$, i.e., is determined by rank $T_0 = 2\sigma_0$. This isomorphism endows $\ker F^{\sigma_0} d$ with a natural pairing namely $\langle\ ,\ \rangle \bmod p$ on $T_0 \otimes k$. Thanks to the results of [10] we have a natural basis of $\ker F^{\sigma_0} d$, we compute the matrices of the pairing $\langle\ ,\ \rangle$ and of the frobenius $f = \mathrm{id}_{T_0} \otimes \mathrm{Frob}_k$ on $T_0 \otimes k$ in this basis, and give a proof along the same lines as Ogus that $H^2_{fl}(X, \mathbb{Z}_p(1))$ has Hasse invariant $e_p = -1$.

In Section 2 we give a couple of applications of this theory. We consider the action of an endomorphism $g: X \to X$ on the global 2-forms $H^0(X, \Omega^2_X)$. Such an endomorphism induces an endomorphism on $\ker F^{\sigma_0} d$ which is either zero or orthogonal with respect to the pairing $\langle\ ,\ \rangle$, this proves that the action of $g$ on $H^0(X, \Omega^2_X)$ is either 0 or given by multiplication by a $p^{\sigma_0}+1$'st root of unity. (More precise results can be obtained in terms of the coordinates of the corresponding point in the period space of supersingular K3 crystals.) A corollary of this result is that any automorphism of $X$ has finite order on $H^2_{DR}(X/k)$.

It is conjectured that $\sigma_0 \leq 10$ (it follows from linear algebra that $\sigma_0 \leq 11$), this holds if rank $NS(X) = b_2$, hence by Artin [1] if $X$ is elliptic. We consider the case where $X$ has an involution $\theta$ which preserves the global 2-form (e.g. the Fermat surface $X_1^4 + X_2^4 + X_3^4 + X_4^4 = 0$ $p \equiv 3 \bmod 4$ or more generally a supersingular reduction of a singular K3 surface over a numberfield [15] ($p \equiv 3 \bmod 4$)), using the theory in Section 1 combined with the Woods Hole fixed point formula we show that $X$ has $\sigma_0 \leq 10$.

## 0. Review of the slope spectral sequence of a supersingular K3 surface

We state without proof the following result, which is Corollary 3.6 of [10].

(0.1) THEOREM: *Let $X/k$ be a supersingular $K3$ surface over an algebraically closed field of characteristic $p>0$, then*

(i) $H^0(X, \Omega^1_X) = 0$.

(ii) $H^2_{\mathrm{crys}}(X/W)$ *is torsion free.*

(iii) *The differential in the slope spectral sequence*

$$d : H^2(X, W(\mathcal{O}_X)) \longrightarrow H^2(X, W\Omega^1_X)$$

*is surjective.*

(iv) $H^2(X, W\Omega^1_X) = k_\sigma[[x]]$ *where the action of $F$ is given by $F(x^i) = x^{i-1}$ for $i>1$ and $F(1)=0$, $V(x^i)=0$ for all $i$.*

(v) *We have the following description of the $E_1$ term of the slope spectral sequence:*

$$\begin{array}{ccccc} k_\sigma[[V]] & \xrightarrow{\quad d \quad} & k_\sigma[[x]] & \xrightarrow{\quad 0 \quad} & W(k) \\ 0 & & W(k)\otimes H^2_{fl}(X, \mathbb{Z}_p(1)) = H^1(X, W\Omega^1_X) & & 0 \\ W(k) & & 0 & & 0 \end{array}$$

*where $d$ is given by* $d(V^i) = \begin{cases} 0 & i<\sigma_0 \\ x^{i-\sigma_0} & i\geq \sigma_0 \end{cases}$

*with $\sigma_0$ determined by $2\sigma_0 = \mathrm{ord}_p(\mathrm{discr.}\ H^2_{fl}(X, \mathbb{Z}_p))$ with cupproduct).*

(vi) *$F$ is a $p$-linear automorphism of $H^1(X, W\Omega^1_X)$.*

## 1. The cohomology of $W\Omega^{\cdot}_X(n)$

In this section we compute the hypercohomology groups $\mathrm{H}^2(X, W\Omega^{\cdot}_X(n))$ for $X$ a supersingular K3 surface. It turns out that they are torsion free for $n\leq\sigma_0$, and that there are natural inclusions

$$H^2_{\mathrm{crys}}(X/W) \longrightarrow \mathrm{H}^2(W\Omega^{\cdot}_X(1)) \longrightarrow \cdots \longrightarrow \mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0)).$$

Moreover $\mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0))$ is isomorphic to the dual of $H^1(W\Omega^1_X)$ which gives the isomorphism

$$T_0\otimes k \simeq \ker F^{\sigma_0} d$$

alluded to in the introduction.

In order to compute $\mathrm{H}^2(W\Omega^{\cdot}_X(n))$ we need a duality theorem involving the higher cycles and boundaries in the de Rham complex.

We prove this result in the general case since it might be of use in later studies of the multiplicative structure of the de Rham–Witt complex.

(1.1) DEFINITION: Let $X/k$ be a scheme over a perfect field $k$ of characteristic $p>0$. Define inductively $X^{(p^n)}$ as the pull-back of $X^{(p^{n-1})}$ along the frobenius of Spec $k$. It follows that for all $n, m$ with $n \geq m$ we have a commutative diagram:

$$\begin{tikzcd}
X^{(p^m)} \arrow[dr, "F^{n-m}"'] \arrow[drr, "F^{n-m}_{abs}"] \arrow[ddr] & & \\
& X^{(p^n)} \arrow[r] \arrow[d] & X^{(p^m)} \arrow[d] \\
& \operatorname{Spec} k \arrow[r, "F^{n-m}_{abs}"] & \operatorname{Spec} k
\end{tikzcd}$$

where the interior square is cartesian and $F^{n-m}$ (the relative frobenius) is defined by the commutativity of the exterior diagram.

Assume from now on that $X$ is smooth. Recall that the Cartier operator is a $p^{-1}$-linear surjective map

$$C: Z_1\Omega^i_X \longrightarrow \Omega^i_X$$

where $Z_1\Omega^i_X = \ker d: \Omega^i_X \to \Omega^{i+1}_X$. The kernel of $C$ is $B_1\Omega^i_X = \operatorname{Im} d: \Omega^{i-1}_X \to \Omega^i_X$, so $C$ induces an isomorphism

$$C: Z_1\Omega^i_X/B_1\Omega^i_X \longrightarrow \Omega^i_X.$$

We also have the inverse of the Cartier operator, which is a $p$-linear map

$$C^{-1}: \Omega^i_X \longrightarrow \Omega^i_X/B_1\Omega^i_X.$$

(1.2) DEFINITION: We define inductively abelian sheaves $Z_n\Omega^i_X$ and $B_n\Omega^i_X$ by

$$Z_n\Omega^i_X = \ker dC^{n-1}: Z_{n-1}\Omega^i_X \longrightarrow \Omega^{i+1}_X$$

and $B_n\Omega^i_X$ by requiring that

$$C^{-1}: B_{n-1}\Omega^i_X \longrightarrow B_n\Omega^i_X/B_1\Omega^i_X$$

is an isomorphism.

It is immediate that these sheaves are the same as those defined by Illusie in [6] 0(2.2.2) and [5] 2.3 so they are locally free $\mathcal{O}_{X^{(p^n)}}$ submodules of $F^n_*\Omega^i_X$. As $\mathcal{O}_X$ modules their structure is given by $f \cdot \omega = f^{p^n}\omega$.

(1.3) LEMMA: *There are exact sequences of $\mathcal{O}_{X^{(p^n)}}$ modules:*

(i) $0 \longrightarrow Z_n\Omega^i_X \longrightarrow F_*Z_{n-1}\Omega^i_X \xrightarrow{dC^{n-1}} F_*\Omega^{i+1}_{X^{(p^{n-1})}}$.

(ii) $F_*\Omega^{i-1}_{X^{(p^{n-1})}} \xrightarrow{C^{-(n-1)}d} F_*(F^{n-1}_*\Omega^i_X/B_{n-1}\Omega^i_X) \longrightarrow F^n_*\Omega^i_X/B_n\Omega^i_X \longrightarrow 0$.

*If $n-1 \geq m$ then there are exact sequences of $\mathcal{O}_{X^{(p^n)}}$ modules*

(iii) $0 \longrightarrow Z_n\Omega^i_X/F^{n-m}_*B_m\Omega^i_X \longrightarrow F_*(Z_{n-1}\Omega^i_X/F^{n-1-m}_*B_m\Omega^i_X) \xrightarrow{dC^{n-1}}$

$F_*\Omega^{i+1}_{X^{(p^{n-1})}}$.

(iv) $F_*\Omega^{i-1}_{X^{(p^{n-1})}} \xrightarrow{C^{-(n-1)}d} F_*(F^{n-1-m}_*Z_m\Omega^i_X/B_{n-1}\Omega^i_X) \longrightarrow$

$F^{n-m}_*Z_m\Omega^i_X/B_n\Omega^i_X \longrightarrow 0$.

PROOF: (i) and (ii) follow immediately from the definitions plus the fact that $F$ is finite and flat ($X$ being smooth), (iii) and (iv) follow from (i) and (ii) in an obvious manner.

(1.4) THEOREM: *Let $X/k$ be a proper and smooth variety of dimension $r$. Assume $n \geq m$ then for all $i, j \leq r$ there is a perfect pairing of $k$ vectorspaces*

$$H^j(X^{(p^n)}, Z_n\Omega^i_X/F^{n-m}_*B_m\Omega^i_X) \times H^{r-j}(X^{(p^n)}, F^{n-m}_*Z_m\Omega^{r-i}_X/B_n\Omega^{r-i}_X) \longrightarrow H^r(X^{(p^n)}, \Omega^r_{X^{(p^n)}}).$$

PROOF: Recall first the following lemma due to Milne [8]. Let $M, N$ be locally free $\mathcal{O}_X$ modules with a perfect pairing

$$M \times N \xrightarrow{(\,,\,)} \Omega^r_X$$

then the pairing of $\mathcal{O}_{X^{(p)}}$ modules

$$F_*M \times F_*N \xrightarrow{F_*(\,,\,)} F_*\Omega^r_X \xrightarrow{C} \Omega^r_{X^{(p)}}$$

is perfect.

We use induction to construct perfect pairings of $\mathcal{O}_{X^{(p^n)}}$ modules

$$Z_n\Omega^i_X/F_*^{n-m}B_m\Omega^i_X \times F_*^{n-m}Z_m\Omega^{r-i}_X/B_n\Omega^{r-i}_X \longrightarrow \Omega^r_{X^{(p^n)}}.$$

Assume first $m=0$.

$n=0$: The pairing

$$\Omega^i_X \times \Omega^{r-i}_X \longrightarrow \Omega^r_X$$

given by the wedge product is perfect.

$n=1$: (See also Milne [8]). Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & & 0 & & \\
Z_1\Omega^i_X \arrow[d] & \times & F_*\Omega^{r-i}_X/B_1\Omega^{r-i}_X \arrow[u] & & \\
F_*\Omega^i_X \arrow[dd, "(-1)^{i+1}d"'] & \times & F_*\Omega^{r-i}_X \arrow[u] \arrow[dr] & & \\
 & & & F_*\Omega^r_X \arrow[r, "C"] & \Omega^r_{X^{(p)}} \\
F_*\Omega^{i+1}_X & \times & F_*\Omega^r_X \arrow[uu, "d"'] \arrow[ur] & &
\end{tikzcd}$$

the diagram commutes by the following computation: Let $\omega \in F_*\Omega^i_X$ and $\tau \in F_*\Omega^{r+1-i}_X$ then

$$\begin{aligned} & C(\omega \wedge d\tau) - (-1)^{i+1}C(d\omega \wedge \tau) \\ &= C(\omega \wedge d\tau - (-1)^{i+1}d\omega \wedge \tau) = C(d(\omega \wedge \tau)) = 0. \end{aligned}$$

By Milne's lemma the two lower pairings are perfect hence we have induced a perfect pairing between the kernel and the cokernel (since $d$ is $\mathcal{O}_{X^{(p)}}$-linear).

$n=2$: By (i) and (ii) of (1.3) we have a diagram with exact rows

$$\begin{tikzcd}
0 \arrow[d] & 0 & & \\
Z_2\Omega^i_X \times \arrow[d] & F_*^2\Omega^{r-i}_X/B_2\Omega^{r-i}_X \arrow[u] & & \\
F_*Z_1\Omega^i_X \times \arrow[d, "(-1)^{i+1}dC"'] & F_*(F_*\Omega^{r-i}_X/B_1\Omega^i_X) \arrow[u] \arrow[r, "F_*(\ ,\ )"] & F_*\Omega^r_{X^{(p)}} \arrow[r, "C"] & \Omega^r_{X^{(p^2)}} \\
F_*\Omega^{i+1}_{X^{(p)}} \times & F_*\Omega^{r-(i+1)}_{X^{(p)}} \arrow[u, "C^{-1}d"'] \arrow[ur, "F_*(\wedge)"'] & &
\end{tikzcd}$$

The diagram commutes by the following computation:

$$\begin{aligned}
&\omega \in F_*Z_1\Omega^i_X, \quad \tau \in F_*\Omega^{r-(i+1)}_{X^{(p)}}\\
&C(F_*(\omega, C^{-1}d\tau)) - C((-1)^{i+1}dC\omega \wedge \tau))\\
&= C(C(\omega \wedge C^{-1}d\tau)) - (-1)^{i+1}C(dC\omega \wedge \tau)\\
&= C(C\omega \wedge d\tau - (-1)^{i+}(dC\omega \wedge \tau))\\
&= C(d(C\omega \wedge \tau)) = 0.
\end{aligned}$$

By the preceding case and by Milne's lemma the two lower pairings are perfect and since $dC$ and $C^{-1}d$ are $\mathcal{O}_{X^{(p^2)}}$ linear we have induced a perfect pairing

$$Z_2\Omega^i_X \times F_*\Omega^{r-i}_X/B_r\Omega^{r-i}_X \xrightarrow{(\ ,\ )} \Omega^r_{X^{(p^2)}}$$

where $(\ ,\ ) = C^2(\Lambda)$.

Assume now that the pairing

$$Z_{n-1}\Omega^i_X \times F^{n-1}_*\Omega^{r-i}_X/B_{n-1}\Omega^{r-i}_X \longrightarrow \Omega^r_{X^{(p^{n-1})}}$$

given by $C^{n-1}(\Lambda)$ is perfect, then the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & & 0 & & \\
Z_n\Omega^i_X \arrow[d] & \times & F^n_*\Omega^{r-i}_X/B_n\Omega^{r-i}_X \arrow[u] & & \\
F_*Z_{n-1}\Omega^i_X \arrow[d, "(-1)^{i+1}dC^{n-1}"'] & \times & F_*(F^{n-1}_*\Omega^{r-i}_X/B_{n-1}\Omega^{r-i}_X) \arrow[u] \arrow[r, "F_*(\ ,\ )"] & F_*\Omega^r_{X^{(p^{n-1})}} \arrow[r, "C"] & \Omega^r_{X^{(p^n)}} \\
F_*\Omega^{i+1}_{X^{(p^{n-1})}} & \times & F_*\Omega^{r-(i+1)}_{X^{(p^{n-1})}} \arrow[u, "C^{-(n-1)}d"'] \arrow[ur, "F_*(\Lambda)"'] & &
\end{tikzcd}$$

shows that the induced pairing

$$Z_n\Omega^i_X \times F^n_*\Omega^{r-i}_X/B_n\Omega^{r-i}_X \xrightarrow{C^n(\Lambda)} \Omega^r_{X^{(p^n)}}$$

is perfect.

For $m \neq 0$ we use the exact sequences (iii) and (iv) of (1.3) and get a commutative diagram of $\mathcal{O}_{X^{(p^n)}}$-modules:

$$
\begin{tikzcd}
0 \arrow[d] & 0 & & \\
Z_n\Omega^i_X/F_*^{n-m}B_m\Omega^i_X \arrow[d] & \times F_*^{n-m}Z_m\Omega^{r-i}_X/B_n\Omega^{r-i}_X \arrow[u] & & \\
F_*(Z_{n-1}\Omega^i_X/F_*^{n-1-m}B_m\Omega^i_X) \arrow[dd, "(-1)^{i+1}dC^{n-1}"'] & \times F_*(F_*^{n-1-m}Z_m\Omega^{r-i}_X/B_{n-1}\Omega^{r-i}_X) \arrow[u] \arrow[dr] & & \\
& & F_*\Omega^r_{X^{(p^{n-1})}} \arrow[r, "C"] & \Omega^r_{X^{(p^n)}} \\
F_*\Omega^{i+1}_{X^{(p^{n-1})}} & \times F_*\Omega^{r-(i+1)}_{X^{(p^{n-1})}} \arrow[uu, "C^{-(n-1)}d"'] \arrow[ur, "F_*(\Lambda)"'] & &
\end{tikzcd}
$$

it follows using induction that we have a perfect pairing of $\mathcal{O}_{X^{(p^n)}}$ modules

$$Z_n\Omega^i_X/F_*^{n-m}B_m\Omega^i_X \times F_*^{n-m}Z_m\Omega^{r-i}_X/B_n\Omega^{r-i}_X \longrightarrow \Omega^r_{X^{(p^n)}}.$$

Now use Grothendieck duality on the smooth proper variety $X^{(p^n)}$ to deduce the statement of the theorem.

(1.5) Definition: Let $W\Omega^{\cdot}_X(n)$ denote the complex

$$0 \longrightarrow W\mathcal{O}_X \xrightarrow{F^n d} W\Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} W\Omega^r_X \longrightarrow 0,$$

where the $W(k)$-module structure on $W\mathcal{O}_X$ is defined through $\sigma^{-n}$, the $n$'th power inverse frobenius map, such that all the differentials in the complex becomes $W(k)$-linear.

Let now $X/k$ denote a supersingular K3 surface over an algebraically closed field, let $\mathrm{ord}_p(H^2_{fl}(X, \mathbf{Z}_p(1))) = 2\sigma_0$.

(1.6) Theorem: *The $W(k)$-module $\mathrm{H}^2(X, W\Omega^{\cdot}_X(n))$ is torsion free for $n \leq \sigma_0$.*

Proof: Consider the exact sequence of (pro-) complexes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W\mathcal{O}_X & \xrightarrow{p} & W\mathcal{O}_X & \longrightarrow & W\mathcal{O}_X/p & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F^n d} & & \downarrow{\scriptstyle F^n d} & & \downarrow{\scriptstyle F^n d} & & \\
0 & \longrightarrow & W\Omega^1_X & \xrightarrow{p} & W\Omega^1_X & \longrightarrow & W\Omega^1_X/p & & \\
 & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\
0 & \longrightarrow & W\Omega^2_X & \xrightarrow{p} & W\Omega^2_X & \longrightarrow & W\Omega^2_X/p & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

we get an exact squence of hypercohomology groups

$$\mathrm{H}^1(W\Omega^\cdot_X(n)/p) \longrightarrow \mathrm{H}^2(W\Omega^\cdot_X(n)) \xrightarrow{p} \mathrm{H}^2(W\Omega^\cdot_X(n))$$

so it is enough to show that $\mathrm{H}^1(W\Omega^\cdot_X(n)/p) = 0$ for $n \leq \sigma_0$.

We show first that $W\Omega^\cdot_X(n)/p$ is quasi-isomorphic to the complex

$$0 \longrightarrow W_{n+1}\mathcal{O}_X/p \xrightarrow{F^n d} \Omega^1_X \xrightarrow{d} \Omega^2_X \longrightarrow 0.$$

We follow the same method as Illusie [6] I 3.15. Let $m \geq n$ and denote by $W_m\Omega^\cdot_X(n)$ the complex

$$0 \longrightarrow W_m\mathcal{O}_X \xrightarrow{F^n d} W_{m-n}\Omega^1_X \xrightarrow{d} W_{m-n}\Omega^2_X \longrightarrow 0$$

we want to show that the natural projection

$$W_{m+1}\Omega^\cdot_X(n)/p \longrightarrow W_m\Omega^\cdot_X(n)/p$$

is a quasi-isomorphism for all $m \geq n$.

Illusie shows [6] I 3.15 that there is an exact sequence

$$0 \longrightarrow gr^{m-1} W\Omega_X^i \xrightarrow{\underline{p}} gr^m W\Omega_X^i \longrightarrow W_{m+1}\Omega_X^i/p \longrightarrow W_m\Omega_X^i/p \longrightarrow 0$$

where $gr$ is the graded object associated to the filtration defined by

$$\mathrm{Fil}^n\, W\Omega_X^i = \ker(W\Omega_X^i \longrightarrow W_n\Omega_X^i)$$

and $\underline{p}$ is defined in [6] I. 3.4.

We have $gr^m W\Omega_X^i = V^m\Omega_X^i + dV^m\Omega_X^{i-1}$ ([6] I 3.2) so we get a four term exact sequence of complexes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & & V^{m-1}\mathcal{O}_X & \xrightarrow{\underline{p}} & V^m\mathcal{O}_X & \longrightarrow & W_{m+1}\mathcal{O}_X/p & \longrightarrow & W_m\mathcal{O}_X/p \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & & V^{m-1-n}\Omega_X^1 + dV^{m-1-n}\mathcal{O}_X & \xrightarrow{\underline{p}} & V^{m-n}\Omega_X^1 + dV^{m-n}\mathcal{O}_X & \longrightarrow & W_{m+1-n}\Omega_X^1/p & \longrightarrow & W_{m-n}\Omega_X^1/p \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & & V^{m-1-n}\Omega_X^2 + dV^{m-1-n}\Omega_X^1 & \xrightarrow{\underline{p}} & V^{m-n}\Omega_X^2 + dV^{m-n}\Omega_X^1 & \longrightarrow & W_{m+1-n}\Omega_X^2/p & \longrightarrow & W_{m-n}\Omega_X^2/p \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0 & & 0
\end{array}$$

so it is enough to show that the map between the two first complexes is a quasi-isomorphism or equivalently that the complex

$$\begin{aligned}
0 \longrightarrow\ & V^m\mathcal{O}_X/\underline{p}V^{m-1}\mathcal{O}_X \\
\xrightarrow{F^n d}\ & V^{m-n}\Omega_X^1 + dV^{m-n}\mathcal{O}_X/\underline{p}(V^{m-1-n}\Omega_X^1 + dV^{m-1-n}\mathcal{O}_X) \\
\xrightarrow{d}\ & V^{m-n}\Omega_X^2 + dV^{m-n}\Omega_X^1/\underline{p}(V^{m-1-n}\Omega_X^2 + dV^{m-1-n}\Omega_X^1) \longrightarrow 0
\end{aligned}$$

is acyclic.

Now look at the following commutative diagram

$$\begin{tikzcd}[column sep=small]
0 \arrow[d] & 0 \arrow[d] \\
V^{m-n}\mathcal{O}_X/\underline{p}V^{m-1-n}\mathcal{O}_X \arrow[r, "V^n"] \arrow[d, "d"] & V^m\mathcal{O}_X/\underline{p}V^{m-1}\mathcal{O}_X \arrow[d, "F^n d"] \\
V^{m-n}\Omega^1_X + dV^{m-n}\mathcal{O}_X/\underline{p}(V^{m-1-n}\Omega^1_X + dV^{m-1-n}\mathcal{O}_X) \arrow[r, equal] \arrow[d, "d"] & V^{m-n}\Omega^1_X + dV^{m-n}\mathcal{O}_X/\underline{p}(V^{m-1-n}\Omega^1_X + dV^{m-1-n}\mathcal{O}_X) \arrow[d, "d"] \\
V^{m-n}\Omega^2_X + dV^{m-n}\Omega^1_X/\underline{P}(V^{m-1-n}\Omega^2_X + dV^{m-1-n}\Omega^1_X) \arrow[r, equal] \arrow[d] & V^{m-n}\Omega^2_X + dV^{m-n}\Omega^1_X/\underline{p}(V^{m-1-n}\Omega^2_X + dV^{m-1-n}\Omega^1_X) \arrow[d] \\
0 & 0
\end{tikzcd}$$

the commutativity of the top square follows by the relation $FdV = d$.

It is clear that $V^n$ is bijective and since the left hand side complex is acyclic ([6] I 3.13) the right hand side is as well and so

$W_m\Omega^{\cdot}_X(n)/p \longrightarrow W_{m-1}\Omega^{\cdot}_X(n)/p$ is a quasi-isomorphism.

This reduces the theorem to proving that $\mathrm{H}^1$ of the complex

$$0 \longrightarrow W_{n+1}\mathcal{O}_X/p \xrightarrow{F^n d} \Omega^1_X \longrightarrow \Omega^2_X \longrightarrow 0$$

vanishes.

Consider again an exact sequence of complexes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & W_n\mathcal{O}_X/F \arrow[r, "V"] \arrow[d, "F^{n-1}d"] & W_{n+1}\mathcal{O}_X/p \arrow[r] \arrow[d, "F^n d"] & \mathcal{O}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^1_X \arrow[r, equal] \arrow[d] & \Omega^1_X \arrow[r] \arrow[d, "d"] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^2_X \arrow[r, equal] \arrow[d] & \Omega^2_X \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since $H^1(\mathcal{O}_X) = 0$, $\mathrm{H}^1$ of the left hand complex maps onto $\mathrm{H}^1$ of the middle complex and so it is enough to show that

$$\mathrm{H}^1(W_n\mathcal{O}_X/F \xrightarrow{F^{n-1}d} \Omega^1_X \xrightarrow{d} \Omega^2_X) = 0 \quad n \leq \sigma_0.$$

Using the isomorphism

$$W_n\mathcal{O}_X/F \xrightarrow{F^{n-1}d} B_n\Omega^1_X \quad ([6] \text{ I } 3.12\text{a})$$

one computes this hypercohomology to be $H^0(Z_1\Omega^1_X/B_n\Omega^1_X)$. By 1.4 $H^0(Z_1\Omega^1_X/B_n\Omega^1_X)$ is dual to $H^2(Z_n\Omega^1_X/B_1\Omega^1_X)$. In [10] (3.6) we have shown (see 0.1) that $dV^n : H^2(W\mathcal{O}_X) \to H^2(W\Omega^1_X)$ is surjective for $n \le \sigma_0$; using the commutative diagram

$$\begin{tikzcd}
H^2(W\mathcal{O}_X) \arrow[r, "dV^n"] \arrow[d] & H^2(W\Omega^1_X) \arrow[d] \\
H^2(\mathcal{O}_X) \arrow[r, "dV^n"] \arrow[d] & H^2(W_{n+1}\Omega^1_X) \arrow[d] \\
0 & 0
\end{tikzcd}$$

it follows that

$$dV^n : H^2(\mathcal{O}_X) \longrightarrow H^2(W_{n+1}\Omega^1_X)$$

is surjective for $n \le \sigma_0$.

By [6] 3.11.3 $F^n : W_{n+1}\Omega^1_X \to \Omega^1_X$ induces an isomorphism

$$F^n : W_{n+1}\Omega^1_X/VW_n\Omega^1_X \longrightarrow Z_n\Omega^1_X.$$

Now look at

$$\begin{tikzcd}
H^2(\mathcal{O}_X) \arrow[r, two heads, "d"] \arrow[d, "dV^n"'] & H^2(B_1\Omega^1_X) \arrow[d] \\
H^2(W_{n+1}\Omega^1_X) \arrow[r, two heads, "F_n"] & H^2(Z_n\Omega^1_X) \arrow[d] \\
& H^2(Z_n\Omega^1_X/B_1\Omega^1_X)
\end{tikzcd}$$

we conclude that $H^2(B_1\Omega^1_X) \to H^2(Z_n\Omega^1_X)$ is surjective hence $H^2(Z_n\Omega^1_X/B_1\Omega^1_X) = 0$ for $n \le \sigma_0$ and the theorem is proved.

(1.7) Definition: Define maps of complexes

$$\tilde{V}: W\Omega^{\cdot}_X(n) \longrightarrow W\Omega^{\cdot}_X(n+1) \text{ and}$$
$$\tilde{F}: W\Omega^{\cdot}_X(n) \longrightarrow W\Omega^{\cdot}_X(n+1) \text{ by}$$

$$\begin{tikzcd}
0 \arrow[r] & W\mathcal{O}_X \arrow[r, "F^n d"] \arrow[d, "V"] & W\Omega^1_X \arrow[r, "d"] \arrow[d, equal] & W\Omega^2_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W\mathcal{O}_X \arrow[r, "F^{n+1} d"] & W\Omega^1_X \arrow[r, "d"] & W\Omega^2_X \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & W\mathcal{O}_X \arrow[r, "F^n d"] \arrow[d, equal] & W\Omega^1_X \arrow[r, "d"] \arrow[d, "F"] & W\Omega^2_X \arrow[r] \arrow[d, "pF"] & 0 \\
0 \arrow[r] & W\mathcal{O}_X \arrow[r, "F^{n+1} d"] & W\Omega^1_X \arrow[r, "d"] & W\Omega^2_X \arrow[r] & 0.
\end{tikzcd}$$

It is clear from the definition of the $W(k)$-module structure on $W\mathcal{O}_X$ that $\tilde{V}$ is linear and $\tilde{F}$ is $p$-linear.

(1.8) Proposition: *The maps*

$$\tilde{V}: \mathrm{H}^2(W\Omega^{\cdot}_X(n)) \longrightarrow \mathrm{H}^2(W\Omega^{\cdot}_X(n+1))$$
$$\tilde{F}: \mathrm{H}^2(W\Omega^{\cdot}_X(n)) \longrightarrow \mathrm{H}^2(W\Omega^{\cdot}_X(n+1))$$

*are injective for $n+1 \leq \sigma_0$.*

Proof: By the definition of $W\Omega^{\cdot}_X(n)$ we have an exact sequence

$$0 \longrightarrow H^1(W\Omega^1_X) \longrightarrow \mathrm{H}^2(W\Omega^{\cdot}_X(n)) \longrightarrow \ker F^n d \longrightarrow 0$$

(we use $H^0(W\Omega^2_X) = 0$) with $\ker F^n d \subset H^2(W\mathcal{O}_X)$. This fits into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(W\Omega^1_X) \arrow[r] \arrow[d, equal] & \mathrm{H}^2(W\Omega^{\cdot}_X(n)) \arrow[r] \arrow[d, "\tilde{V}"] & \ker F^n d \arrow[r] \arrow[d, "V"] & 0 \\
0 \arrow[r] & H^1(W\Omega^1_X) \arrow[r] & \mathrm{H}^2(W\Omega^{\cdot}_X(n+1) \arrow[r] & \ker F^{n+1} d \arrow[r] & 0.
\end{tikzcd}$$

Since $V: \ker F^n d \to \ker F^{n+1} d$ is injective this shows the injectivity of $\tilde{V}$. The injectivity of $\tilde{F}$ is shown similarly

(1.9) COROLLARY: *For $n+1 \leq \sigma_0$ we have*

$$\mathbb{H}^2(W\Omega^{\cdot}_X(n+1)) = \tilde{V}\mathbb{H}^2(W\Omega^{\cdot}_X(n)) + \tilde{F}\mathbb{H}^2(W\Omega^{\cdot}_X(n)).$$

PROOF: This follows from the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{H}^2(W\Omega^{\cdot}_X(n)) \arrow[r, "\tilde{V}"] & \mathbb{H}^2(W\Omega^{\cdot}_X(n+1)) \arrow[r] & H^2(\mathcal{O}_X) \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{H}^2(W\Omega^{\cdot}_X(n-1)) \arrow[r, "\tilde{V}"] \arrow[u, "\tilde{F}"] & \mathbb{H}^2(W\Omega^{\cdot}_X(n)) \arrow[r] \arrow[u, "\tilde{F}"] & H^2(\mathcal{O}_X) \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

From now on we will suppress $\tilde{V}$ and identify $\mathbb{H}^2(W\Omega^{\cdot}_X(n))$ with its image in $\mathbb{H}^2(W\Omega^{\cdot}_X(n+1))$, so we have an ascending chain

$$H^2_{\text{crys}}(X/W) \subset \mathbb{H}^2(W\Omega^{\cdot}_X(1)) \subset \cdots \subset \mathbb{H}^2(W\Omega^{\cdot}_X(\sigma_0))$$

and by (1.9) we have for $n \leq \sigma_0$

$$\mathbb{H}^2(W\Omega^{\cdot}_X(n)) = \sum_{i=0}^{n} \tilde{F}^i H^2_{\text{crys}}(X/W).$$

Now we define a map

$$\phi_n : \mathbb{H}^2(W\Omega^{\cdot}_X(n)) \longrightarrow \operatorname{Hom}(H^1(W\Omega^1_X), W(k)) = H^1(W\Omega^1_X)^V$$

as follows:

Since $H^1(W\Omega^1_X) = H^2_{fl}(X, \mathbb{Z}_p(1)) \otimes W(k)$ we have a nondegenerate pairing on $H^1(W\Omega^1_X)$ induced by the cupproduct pairing on $H^2_{fl}(X, \mathbb{Z}_p(1))$.

Let $x \in \mathbb{H}^2(W\Omega^{\cdot}_X(n))$ then $px \in H^1(W\Omega^1_X)$ and hence we get an element in $H^1(W\Omega^1_X)^V$ by the assignment $z \to (px, z)$. By (1.9) we can find $y_0, y_1, \ldots, y_n \in H^2_{\text{crys}}(X/W)$ such that $x = \sum_{i=0}^{n} \tilde{F}^i y_i$.

Let $z \in H^1(W\Omega^1_X)$ then

$$(px, z) = \sum_{i=0}^{n} (p\tilde{F}^i y_i, z) = \sum_{i=0}^{n} (\tilde{F}^i p y_i, z) = \sum_{i=0}^{n} (F^i p y_i, z)$$

since the restriction of $\tilde{F}$ to $H^1(W\Omega^1_X)$ is the automorphism $F$. Now write $z = F^i z_i$, $z_i \in H^1(W\Omega^1_X)$, we get

$$(px, z) = \sum_{i=0}^{n} (F^i py_i, z) = \sum_{i=0}^{n} (F^i py_i, F^i z_i) = \sum_{i=0}^{n} (py_i, z_i),$$

the last equality holds since $F$ is orthogonal with respect to the pairing on $H^1(W\Omega^1_X)$. We have

$$(py_i, z_i) = (py_i, z_i)_{H^2_{\mathrm{crys}}} = p(y_i, z_i)_{H^2_{\mathrm{crys}}}$$

and so $(px, z) = p\ \Sigma_{i=1}^{n} (y_i, z_i)_{H^2_{\mathrm{crys}}}$, i.e., $(px, -) \in pH^1(W\Omega^1)^V$; now define

$$\phi_n(x) = \frac{1}{p}(px, -) \in H^1(W\Omega^1)^V.$$

(1.10) PROPOSITION: *$\phi_n$ defines an injection*

$$\mathrm{H}^2(W\Omega^{\cdot}_X(n)) \longrightarrow H^1(W\Omega^1_X)^V \quad \text{for } n \leq \sigma_0.$$

PROOF: This follows from (1.5) and from the fact that the pairing on $H^1(W\Omega^1_X)$ is non-degenerate.

(1.11) LEMMA: $\ker F^n d \subset H^2(W\mathcal{O}_X) = k_\sigma[[V]]$ *has* rank $n + \sigma_0$ *with basis* $\{1, V, \ldots, V^{\sigma_0+n-1}\}$.

PROOF: Since $\operatorname{Im} F^n d \supset \operatorname{Im} d$ for all $n \geq 0$ and since $d$ is surjective it follows that $F^n d$ is surjective, so $\operatorname{Im} F^n d = \operatorname{Im} F^{n+1} d = H^1(W\Omega^2_X)$. Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \ker F^n d \arrow[r] \arrow[d, "V"] & H^2(W\mathcal{O}_X) \arrow[r] \arrow[d, "V"] & \operatorname{Im} F^n d \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker F^{n+1} d \arrow[r] \arrow[d] & H^2(W\mathcal{O}_X) \arrow[r] \arrow[d] & \operatorname{Im} F^{n+1} d \arrow[r] & 0 \\
 & \mathrm{Coker} \arrow[r] & H^2(\mathcal{O}_X) = k & &
\end{tikzcd}$$

it follows that rank $F^{n+1}d = \text{rank } F^n d + 1$, and since $\ker d$ has rank $\sigma_0$ with $\{1, V, \ldots, V^{\sigma_0 - 1}\}$ as basis the lemma follows.

In the following we shall use the quadratic residue symbol, hence we assume $p \neq 2$.

(1.12) THEOREM: *Let* $(T_0, p\langle\ ,\ \rangle) \oplus (T_1, (\ ,\ ))$ *be an orthogonal splitting of* $H^2_{fl}(X, Z_p))$ *with* $\langle\ ,\ \rangle$ *and* $(\ ,\ )$ *perfect on* $T_0$ *and* $T_1$ *resp. (Such a splitting exists by Ogus* [11] 3.13.3) *then there is a canonical isomorphism*

$$\ker F^{\sigma_0} d \simeq T_0 \otimes k.$$

PROOF: We have $H^2_{fl}(X, Z_p(1))^V = (1/p)T_0 \oplus T_1$ so $H^1(W\Omega^1_X)^V = (1/p)T_0 \otimes W(k) \oplus T_1 \otimes W(k)$ and so $H^1(W\Omega^1_X)^V / H^1(W\Omega^1_X) = (1/p)T_0 \otimes W(k)/T_0 \otimes W(k) \simeq T_0 \otimes k$, hence there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(W\Omega^1_X) \arrow[r] \arrow[d, equal] & \mathrm{H}^2(W\Omega_X(\sigma_0)) \arrow[r] \arrow[d, "\phi_{\sigma_0}"] & \ker F^{\sigma_0} d \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1(W\Omega^1_X) \arrow[r] & H^1(W\Omega^1_X)^V \arrow[r] & T_0 \otimes k \arrow[r] & 0.
\end{tikzcd}$$

Since $\phi_{\sigma_0}$ is injective the induced map

$$\ker F^{\sigma_0} d \longrightarrow T_0 \otimes k$$

is injective and since both spaces have dimension $2\sigma_0$ it is an isomorphism.

$T_0 \otimes k$ comes equipped with a perfect pairing namely $\langle\ ,\ \rangle \bmod p$, and a frobenius $f = \mathrm{id}_{T_0} \otimes \mathrm{Frob}_k$, the above isomorphism transports these structures to $\ker F^{\sigma_0} d$. On this space we have the natural basis $\{1, V, \ldots, V^{2\sigma_0 - 1}\}$, we are going to compute the matrices of $\langle\ ,\ \rangle$ and $f$ in this basis.

(1.13) PROPOSITION: *There is a* $p$*-linear automorphism* $\underline{F}: \mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0)) \to \mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0))$ *extending the automorphism* $F$ *on* $H^1(W\Omega^1_X)$.

PROOF: Assume first that $\mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0 + 1))$ is torsion free. Then the argument of (1.10) shows that

$$\phi_{\sigma_0+1}: \mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0 + 1)) \longrightarrow H^1(W\Omega^1_X)^V$$

is injective, and it follows from the commutative diagram below and (1.12)

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0)) & \xrightarrow{\tilde{V}} & \mathrm{H}^2(w\Omega^{\cdot}_X(\sigma_0+1)) & \longrightarrow & H^2(\mathcal{O}_X) & \longrightarrow 0 \\ & \downarrow{\scriptstyle \phi_{\sigma_0}} & & \downarrow{\scriptstyle \phi_{\sigma_0+1}} & & & \\ & H^1(W\Omega^1)^V & = & H^1(W\Omega^1)^V & & & \end{array}$$

that $\phi_{\sigma_0+1}$ is an isomorphism, hence $\tilde{V}$ is surjective which is a contradiction. Let $T$ denote the torsion submodule of $\mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0+1))$ so $T \neq 0$; consider the commutative diagram

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & & \mathrm{T} \arrow[d] \arrow[drr, "\alpha"] & & \\ 0 \arrow[r] & \mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0)) \arrow[r, "\tilde{V}"] \arrow[dr, "\beta"'] & \mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0+1)) \arrow[r] \arrow[d] & H^2(\mathcal{O}_X) \arrow[r] & 0 \\ & & M \arrow[d] & & \\ & & 0 & & \end{tikzcd}$$

Since $\tilde{V}$ is additive it is clear that $T \cap \tilde{V}\mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0)) = 0$ so $T$ injects into $H^2(\mathcal{O}_X) = k$ hence $\alpha$ is an isomorphism, it follows that $\beta$ is also an isomorphism. Now we have

$$pH^2(W\Omega^{\cdot}_X(\sigma_0+1)) = pM = p\tilde{V}H^2(W\Omega^{\cdot}_X(\sigma_0)) = p\mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0)).$$

Let $x \in \mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0))$, then $\tilde{F}x \in \mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0+1))$ hence there is a $y \in \mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0))$ such that $p\tilde{F}x = py$, this $y$ is uniquely determined since $\mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0))$ is torsion free, now put $\underline{F}x = y$. Let $z$ be an element of $H^1(X\Omega^1_X)$, then $p\underline{F}z = p\tilde{F}z = \tilde{F}pz = Fpz = pFz$ so $p(\underline{F}z - Fz) = 0$, i.e., $\underline{F}z = Fz$ so $\underline{F}$ extends $F$, and the $p$-linearity follows easily.

To prove that $\underline{F}$ is an automorphism we will show that the diagram below is commutative

$$\begin{tikzcd}
0 \arrow[r] & H^1(W\Omega^1) \arrow[r] \arrow[d, "F"] & \mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0)) \arrow[r] \arrow[d, "\underline{F}"] & \ker F^{\sigma_0}d \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & H^1(W\Omega^1) \arrow[r] & \mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0)) \arrow[r] & \ker F^{\sigma_0}d \arrow[r] & 0.
\end{tikzcd}$$

The map $\mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0)) \to T_0 \otimes k$ can be described as follows: Take $x \in \mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0))$ then $px \in H^1(W\Omega^1_X)$ and $(1/p)(px, -) \in H^1(W\Omega^1_X)^V = ((1/p)T_0 \oplus T_1) \otimes W(k)$, take $(1/p)(px, -) \bmod H^1(W\Omega^1_X)$ in $H^1(W\Omega^1)^V/H^1(W\Omega^1) = (1/p)T_0T_0 \otimes k$, then multiply by $p$ to get $\overline{px} \in T_0 \otimes k$.

The frobenius $f$ on $T_0 \otimes k$ is the reduction $\bmod p$ of the restriction of $F$ on $H^1(W\Omega^1_X)$ to $T_0 \otimes W(k)$, so $f(\overline{px}) = \overline{F(px)} = \overline{\underline{F}(px)} = \overline{p\underline{F}(x)}$ which shows the commutativity of the diagram.

(1.14) PROPOSITION: *Consider the image of the composite map*

$$H^2_{\mathrm{crys}}(X/W) \longrightarrow \mathrm{H}^2(W\Omega_X(\sigma_0)) \longrightarrow \ker F^{\sigma_0}d.$$

*This is a maximal isotropic subspace with basis* $\{V^{\sigma_0}, \ldots, V^{2\sigma_0 - 1}\}$.

PROOF: Let $\mathscr{K}$ denote this image, then the diagram below shows that $\mathscr{K}$ is generated by $\{V^{\sigma_0}, \ldots, V^{2\sigma_0-1}\}$.

$$\begin{tikzcd}
0 \arrow[r] & H^1(W\Omega^1) \arrow[r] \arrow[d, dash] & H^2_{\mathrm{crys}}(X/W) \arrow[r] \arrow[d] & \ker d \arrow[r] \arrow[d, "V^{\sigma_0}"] & 0 \\
0 \arrow[r] & H^1(W\Omega^1) \arrow[r] & \mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0)) \arrow[r] & \ker F^{\sigma_0}d \arrow[r] & 0.
\end{tikzcd}$$

The following computation shows that $\mathscr{K}$ is isotropic: Let $x, y \in H^2_{\mathrm{crys}}(X/W)$, then we have $\langle \overline{px}, \overline{py} \rangle = (1/p)(px, py) \bmod p = (1/p)(px, py)_{H^2_{\mathrm{crys}}} \bmod p = p(x, y)_{H^2_{\mathrm{crys}}} \bmod p = 0$. Since $\dim \ker F^{\sigma_0}d = 2\sigma_0$, $\dim \mathscr{K} = \sigma_0$ and the pairing $\langle\ ,\ \rangle$ is non-degenerate $\mathscr{K}$ must be maximal isotropic.

(1.15) PROPOSITION: *Let* $i: \ker F^{\sigma_0-1}d \to \ker F^{\sigma_0}d$ *denote the inclusion then the following diagram is commutative*

$$\begin{tikzcd} \ker F^{\sigma_0}d^{-1} \arrow[rr, "V"] \arrow[dr, "i"'] & & \ker F^{\sigma_0}d \arrow[dl, "f"] \\ & \ker F^{\sigma_0}d & \end{tikzcd}$$

PROOF: It suffices to show that

$$\begin{tikzcd} \mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0-1)) \arrow[rr, "\tilde V"] \arrow[dr, "\tilde F"'] & & \mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0)) \arrow[dl, "F"] \\ & \mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0)) & \end{tikzcd}$$

commutes. Let $x \in \mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_0-1))$ then

$$p\underline{F}\tilde{V}x = p\tilde{F}\tilde{V}x = p\tilde{V}\tilde{F}x = \tilde{V}p\tilde{F}x = p\tilde{F}x$$

since $\mathrm{H}^2(W\Omega^{\cdot}_X(\sigma_p))$ is torsion free it follows that $\underline{F}\tilde{V}x = \tilde{F}x$. (Here we have used that $\tilde{F}\tilde{V} = \tilde{V}\tilde{F}$, which is evident from the definition.)

(1.16) COROLLARY: $\dim \mathcal{K} \cap f(\mathcal{K}) = \sigma_0 - 1$ *and* $\Sigma_{i=0}^{\sigma_0-1} f^i(\mathcal{K}) = \ker F^{\sigma_0}d$.

PROOF: This is clear from (1.14) and (1.15).

(1.17) COROLLARY: *The matrix of $f$ in the basis $\{V^{2\sigma_0-1}, V^{2\sigma_0-2}, \ldots, V, 1\}$ is given by*

$$\begin{Bmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 1 & 0 & \ddots & \vdots & a_2 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & a_{2\sigma_0} \end{Bmatrix}$$

PROOF: It follows from (1.15) that $f(V^i) = V^{i-1}$ for $i \geq 1$.

Since $\mathcal{K}$ is maximal isotropic we must have $\langle V^{2\sigma_0-1}, V^{\sigma_0-1}\rangle \neq 0$, indeed $\langle V^{2\sigma_0-n}, V^{\sigma_0-1}\rangle = \langle f^{n-1}V^{2\sigma_0-1}, f^{n-1}V^{\sigma_0+n-2}\rangle = \langle V^{2\sigma_0-1}, V^{\sigma_0+n-2}\rangle = 0$ $n > 1$ so if $\langle V^{2\sigma_0-1}, V^{\sigma_0-1}\rangle = 0$ $\mathcal{K} + kV^{\sigma_0-1}$ would be an isotropic subspace strictly containing $\mathscr{K}$. Dividing by a suitable element of $k$ we can then assume that we have a basis $\{x, f(x), f^2(x), \ldots, f^{2\sigma_0-1}(x)\}$ with $\langle x, f^{\sigma_0}(x)\rangle = 1$. Put $b_i = \langle x, f^{\sigma_0+1}(x)\rangle$ $i = 1, \ldots, \sigma_0 - 1$.

(1.18) PROPOSITION: *The matrix of the pairing $\langle\ ,\ \rangle$ in the basis $\{x, f(x), f^i(x), \ldots, f^{2\sigma_0-1}(x)\}$ is given by*

$$
\begin{array}{c}
\begin{array}{cc} \qquad\qquad \sigma_0 \qquad\qquad & \qquad\qquad \sigma_0 \qquad\qquad \end{array} \\
\left\{
\begin{array}{c|c}
\begin{array}{ccc} & & \\ & & \\ & & 0 \end{array} &
\begin{array}{ccccc}
1 & b_1 & b_2 & \cdots & b_{\sigma_0-1} \\
 & 1 & b_1^p & \cdots & b_{\sigma_0-2}^p \\
0 & & 1 & & \vdots \\
0 & & & 1 & b_1^{p^{\sigma_0-2}} \\
 & & & & 1
\end{array} \\
\hline
\begin{array}{ccccc}
1 & & & & 0 \\
b_1^{p^{\sigma_0-2}} & 1 & & & \\
\vdots & & \ddots & & \\
\vdots & & & \ddots & \\
b_{\sigma_0-1} & b_{\sigma_0-2} & \cdots & b_1 & 1
\end{array} & 0
\end{array}
\right\}
\end{array}
$$

PROOF: The $i$'th row is given by

$$(\langle f^{i-1}(x), x\rangle, \langle f^{i-1}(x), f(x)\rangle, \ldots, \langle f^{i-1}(x), f^{2\sigma_0-1}(x)\rangle).$$

Assume $i \le \sigma_0$ then $f^{i-1}(x) \in \mathscr{K}$ and

$$\langle f^{i-1}(x), x\rangle = \cdots = \langle f^{i-1}(x), f^{\sigma_0-1}(x)\rangle = 0$$

$$\langle f^{i-1}(x), f^{\sigma_0+n}(x)\rangle = \langle x, f^{\sigma_0+n-i+1}(x)\rangle = \begin{cases} 0 & n < i-1 \\ 1 & n = i-1 \\ b_{n-i+1} & n \ge 1 \end{cases}$$

(1.19) LEMMA: *The matrix of $f$ in the basis $\{x, f(x), f^2(x), \ldots, f^{2\sigma_0-1}(x)\}$ is given by*

$$\left\{\begin{array}{cccc}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & a_2 \\
0 & 1 & \vdots & \vdots \\
\vdots & \ddots & 0 & \vdots \\
0 & \cdots & 1 & a_{2\sigma_0}
\end{array}\right\}.$$

PROOF: It suffices to show that

$$f^{2\sigma_0}(x) = 1 \cdot x + a_2 f(x) + a_3 f^2(x) + \cdots + a_{2\sigma_0} f^{2\sigma_0 - 1}(x).$$

Write

$$f^{2\sigma_0}(x) = a_1 x + a_2 f(x) + a_3 f^2(x) + \cdots + a_{2\sigma_0} f^{2\sigma_0 - 1}(x)$$

we see from (1.18) that $\langle f^{\sigma_0}(x), f^n(x)\rangle = 0$ for all $n \le 2\sigma_0 - 1$ except $n = 0$, so we get

$$\langle f^{\sigma_0}(x), f^{2\sigma_0}(x)\rangle = a_1 \langle f^{\sigma_0}(x), x\rangle = a_1$$

since $f$ is orthogonal we have

$$\langle f^{\sigma_0}(x), f^{2\sigma_0}(x)\rangle = \langle x, f^{\sigma_0}(x)\rangle = 1 \text{ so } a_1 = 1.$$

(1.20) THEOREM: *Let $d$ denote the discriminant of the pairing $\langle\ ,\ \rangle$ on $T_0$, then the Legendre symbol $\left(\frac{d}{p}\right)$ satisfies*

$$\left(\frac{d}{p}\right) = -\left(\frac{-1}{p}\right)^{\sigma_0}.$$

*The Hasse invariant of $H^2_{fl}(X, \mathbb{Z}_p(1))$ is equal to $-1$.*

PROOF: Let $e_1, \ldots, e_{2\sigma_0}$ be an orthogonal basis of $T_0/pT_0$ then $d \equiv \langle e_1, e_1\rangle \ldots \langle e_{2\sigma_0}, e_{2\sigma_0}\rangle \bmod p$.

In $T_0 \otimes k$ we have

$$x = \sum_{i=1}^{2\sigma_0} \lambda_i e_i$$

and $f^n(x) = \sum_{i=1}^{2\sigma_0} \lambda_i^{p^n} e_i$, so the coordinate transformation between the bases $\{e_1, \ldots, e_{2\sigma_0}\}$ and $\{x, f(x), \ldots, f^{2\sigma_0 - 1}(x)\}$ is given by the matrix

$$\Lambda = \begin{Bmatrix} \lambda_1 & \lambda_1^p \ldots\ldots & \lambda_{1_{2\sigma_0-1}}^{p^{2\sigma_0-1}} \\ \lambda_2 & \lambda_2^p & \lambda_2^p \\ \lambda_{2\sigma_0} & \lambda_{2\sigma_0}^p & \lambda_{2\sigma_0}^{p^{2\sigma_0-1}} \end{Bmatrix}$$

and so we have

$$\Lambda^t \begin{Bmatrix} \langle e_1, e_1\rangle & & 0 \\ & \ddots & \\ 0 & & \langle e_{2\sigma_0}, e_{2\sigma_0}\rangle \end{Bmatrix} \Lambda = \left\{ \begin{array}{c|c} 0 & \begin{matrix} 1 & \cdot & b \\ 0 & & 1 \end{matrix} \\ \hline \begin{matrix} 1 & & 0 \\ b & \cdot & 1 \end{matrix} & 0 \end{array} \right\}$$

and taking determinants we get

$$(\det \Lambda)^2\bar{d} = \det \left\{ \begin{array}{cc|cc} \multicolumn{2}{c|}{0} & 1 & b \\ & & 0 & 1 \\ \hline 1 & 0 & \multicolumn{2}{c}{0} \\ b & 1 & & \end{array} \right\} = (-1)^{\sigma_0}$$

so $\left(\frac{d}{p}\right) = \left(\frac{(\det \Lambda)^{-2}}{p}\right)\left(\frac{-1}{p}\right)^{\sigma_0}$.

Let $\Lambda^F$ denote the matrix obtained by raising all the entries in $\Lambda$ to the $p$'th power then $\Lambda^F$ is the matrix of the coordinate transformation $\{e_1, \ldots, e_{2\sigma_0}\} \to \{f(x), f^2(x), \ldots, f^{2\sigma_0}(x)\}$, so

$$\Lambda^F = f \cdot \Lambda \text{ and hence}$$

$$\det \Lambda^F = (\det \Lambda)^p = \det f \cdot \det \Lambda,$$

by (1.19) we get $\det f = -1$. This shows that $\det \Lambda \notin \mathsf{F}_p$, hence $(\det \Lambda)^{-2}$ is not a square in $\mathsf{F}_p$ so $\left(\frac{(\det \Lambda)^{-2}}{p}\right) = -1$ and $\left(\frac{d}{p}\right) = -\left(-\frac{1}{p}\right)^{\sigma_0}$.

Let $d_1 = \text{discr.}(T_1, (\ ,\ ))$ then we have the following formula for the Hasse invariant $e_p$:

$$e_p(H^2_{fl}(X, \mathbb{Z}_p(1)) = e_p(T_1, (\ ,\ ))e_p(T_0, p\langle\ ,\ \rangle)(p^{2\sigma_0}d, d_1)$$

where $(p^{2\sigma_0}d, d_1)$ is the Hilbert symbol.
Since $T_1, (\ ,\ )$ is perfect $e_p(T_1) = 1$, and also $(p^{2\sigma_0}d, d_1) = 1$ so

$$e_p(H^2_{fl}(X, \mathbb{Z}_p(1)) = e_p(T_0, p\langle\ ,\ \rangle).$$

Now

$$e_p(T_0\, p\langle\ ,\ \rangle) = (p, (-1)^{\sigma_0}d) = \left(\frac{-1}{p}\right)^{\sigma_0}\left(\frac{d}{p}\right) = \left(\frac{-1}{p}\right)^{\sigma_0}\left(-\left(\frac{-1}{p}\right)\right)^{\sigma_0} = -1.$$

## 2. Applications to endomorphisms of $X$

In this section we consider an endomorphism $g: X \to X$. Such an endomorphism induces an endomorphism of $\ker F^{\sigma_0}d$ which is either zero or orthogonal with respect to the pairing $\langle\ ,\ \rangle$, this puts rather strict restrictions on the way $g$ can act on $H^2(\mathcal{O}_X)$ and $H^0(\Omega^2_X)$.

(2.1) THEOREM: *Let $g: X \to X$ be an endomorphism then the endomorphism $g_*: H^2(\mathcal{O}_X) \to H^2(\mathcal{O}_X)$ is either zero or multiplication by a $\lambda \in \mu_{p^{\sigma_0}+1}$ (the $p^{\sigma_0}+1$'st roots of unity in $k$).*

PROOF: The endomorphism $g$ induces an endomorphism of $\hat{Br}_X = \hat{\mathbb{G}}_a$, hence an element $\tilde{g}$ of $\mathrm{End}(\hat{\mathbb{G}}_a) = k_\sigma[[F]]$. Let $\tilde{g} = \lambda_0 + \lambda_1 F + \lambda_2 F^2 + \cdots$ then the action on the covariant Dieudonné module of $\hat{\mathbb{G}}_a$ which is $k_\sigma[[V]]$ is given by right multiplication by the power series $\tilde{g} = \lambda_0 + \lambda_1 V + \lambda_2 V^2 + \cdots$. We have $k_\sigma[[V]] = H^2(W\mathcal{O}_X)$ and since the de Rham–Witt complex is functorial $g$ must stabilize $\ker d$. But $\ker d$ is finite dimensional with basis $(1, V, \ldots, V^{\sigma_0-1})$ so in order to stabilize this subspace we must have $\lambda_1 = \lambda_2 = \cdots = 0$ and so the action of $g$ on $H^2(W\mathcal{O}_X)$ is given by

$$
\begin{aligned}
b_0 + b_1 V + b_2 V^2 + \cdots + b_n V^n + \cdots \longmapsto\ & b_0\lambda_0 + b_1 \lambda_0^{1/p} V \\
& + b_2 \lambda_0^{1/p^2} V^2 + \cdots + b_n \lambda_0^{1/p^n} V^n + \cdots.
\end{aligned}
$$

Assume now that $\lambda_0 \neq 0$, the isomorphism $T_0 \otimes k \simeq \ker F^{\sigma_0} d$ is functorial, and the endomorphism induced by $g$ on $T_0 \otimes k$ is necessarily orthogonal with respect to the pairing, $\langle\ ,\ \rangle$. We have $\langle V^{2\sigma_0-1}, V^{\sigma_0-1}\rangle \neq 0$, and so we get

$$
\begin{aligned}
\langle V^{2\sigma_0-1}, V^{\sigma_0-1}\rangle &= \langle \lambda_0^{1/p^{2\sigma_0-1}} V^{2\sigma_0-1}, \lambda_0^{1/p^{\sigma_0-1}} V^{\sigma_0-1}\rangle \\
&= \lambda_0^{1/p^{2\sigma_0-1}} \lambda_0^{1/p^{\sigma_0-1}} \langle V^{2\sigma_0-1}, V^{\sigma_0}\rangle
\end{aligned}
$$

and so $\lambda_0^{1/p^{2\sigma_0-1}} \lambda_0^{1/p^{\sigma_0-1}} = 1$ which is the same as $\lambda_0^{p^{\sigma_0}+1} = 1$. Now the exact sequence

$$
0 \longrightarrow H^2(W\mathcal{O}_X) \xrightarrow{V} H^2(W\mathcal{O}_X) \longrightarrow H^2(\mathcal{O}_X) \longrightarrow 0
$$

shows that the action on $H^2(\mathcal{O}_X)$ is given by multiplication by $\lambda_0$, by Serre duality the action on $H^0(\Omega_X^2)$ is multiplication by $\lambda_0^{-1}$ hence the theorem.

(2.2) REMARK: We get even stricter restrictions on $\lambda_0$ by considering the coordinates of the point in the period space of super singular K3 crystals, $\mathbb{A}^{\sigma_0-1}/\mu_{p^{\sigma_0}+1}$ [11]. Let the point corresponding to $H^2_{\mathrm{crys}}(X/W)$ have coordinates $(b_1, \ldots, b_{\sigma_0-1})$, i.e. in the notation of (1.18) $b_i = \langle x, f^{\sigma_0+i}(x)\rangle$. Assume for instance that $b_1 \neq 0$ then we have

$$
\begin{aligned}
b_1 = \langle x, f^{\sigma_0+1}(x)\rangle &= \langle \lambda_0^{1/p^{2\sigma_0-1}} x, \lambda_0^{1/p^{\sigma_0-2}} f^{\sigma_0+1}(x)\rangle \\
&= \lambda_0^{1/p^{2\sigma_0-1}} \lambda_0^{1/p^{\sigma_0-2}} b_1, \text{ i.e. } \lambda_0^{1/p^{2\sigma_0-1}} \lambda_0^{1/p^{\sigma_0-2}} = 1
\end{aligned}
$$

which gives $\lambda_0 \cdot \lambda_0^{p^{\sigma_0+1}} = 1$, so $\lambda_0^{p^{\sigma_0+1}} = \lambda_0^{-1}$. By (2.1) we also have $\lambda_0^{p^{\sigma_0}} = \lambda_0^{-1}$ hence $(\lambda_0^{p^{\sigma_0}})^p = \lambda_0^{p^{\sigma_0}}$ so $\lambda_0^{p^{\sigma_0}} \in \mathbb{F}_p$ and so also $\lambda_0 \in \mathbb{F}_p$, it follows that $\lambda_0^2 = 1$.

EXAMPLE: Let $Y$ be a singular K3 surface over $\mathbb{C}$, i.e. the Neron–Severi group of $Y$ has maximum rank $= 20$. Shioda and Inose [15] have shown that $Y$ is defined over a number field $K$. Moreover $Y$ can be constructed as a certain double covering of a Kummer surface $X = \mathrm{Km}(C \times C)$ where $C$ is the elliptic curve $H^2(Y, \mathcal{O}_Y)/pr_*H^2(Y, Z)$.

Let $\mathcal{P}$ be a prime of $k$ with $N\mathcal{P} \equiv 3 \bmod 4$ and assume that $C$ has good supersingular reduction at $\mathcal{P}$. Let the subscript $_0$ denote reduction mod $\mathcal{P}$; it follows from the results of Shioda and Inose on the $\zeta$ function of $Y$, that $Y_0$ and $X_0$ are supersingular K3 surfaces. Since $X_0$ is the Kummer surface associated to a product of supersingular elliptic curves it has $\sigma_0(X_0) = 1$ and hence by Ogus' Torelli theorem is uniquely determined; in fact it is isomorphic to the Fermat hypersurface $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 1$, but this surface has an automorphism of order 4 which multiplies the global 2-form by i, ($\mathrm{i}^2 = -1$). Since $Y$ is constructed as a double covering of $X$ it follows that $Y_0$ has an automorphism with the properties above, in particular we must have

$$p^{\sigma_0(Y_0)} \equiv -1 \bmod 4$$

which implies that $\sigma_0(Y_0)$ is *odd*. If $\sigma_0(Y_0) = 1$ then $Y_0$ itself is a Kummer surface hence isomorphic to $X_0$, if $\sigma_0(Y_0) = 3$ we can consider the corresponding point in the period space $(b_1, b_2) \in \mathbb{A}^2/\mu_{p^3+1}$, but by the remark following the proof of 2.1, we see that $b_1 = 0$, and so there is at most a 1-dimensional family of supersingular K3's with $\sigma_0 = 3$[1] which are reduction of a singular one.

Does this family exist? or do all singular K3 surfaces reduce to Kummer surfaces?

It can be noted that for a given singular K3 surface $Y$ there is only a finite number of supersingular primes in $K$ where the reduction can be non-Kummer, indeed if one considers the lattice of transcendental cycles $T_Y$, and if $\mathcal{P}$ is such that $p = N\mathcal{P} \nmid \det T_Y$, then $N\mathcal{P} \nmid \det NS(Y)$ hence $NS(Y) \otimes Z_p$ is unimodular. We have an injective specialization map

$$NS(Y) \otimes Z_p \longrightarrow NS(Y_0) \otimes Z_p$$

[1] Assuming the conjectural Torelli theorem of Ogus [11].

which splits since $NS(Y)\otimes Z_p$ is unimodular, it follows that if rank $NS(Y_0)=22$, then the $p$-adic valuation of the discriminant is 2 hence $\sigma_0(Y_0)=1$ so $Y_0$ is Kummer.

(2.3) Corollary: *Let $g:X\to X$ be an automorphism then the induced automorphism $g_*:H^2_{\mathrm{DR}}(X/k)\to H^2_{\mathrm{DR}}(X/k)$ has finite order.*

Proof: Consider the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & \longrightarrow & E_2^{0,2} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^2_{fl}(X,Z_p(1))\otimes W(k) & \longrightarrow & H^2_{\mathrm{crys}}(X/W) & \longrightarrow & E_2^{0,2} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & H^2_{fl}(X,Z_p(1))\otimes W(k) & \longrightarrow & H^2_{\mathrm{crys}}(X/W) & \longrightarrow & E_2^{0,2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H^2_{fl}(X,Z_p(1))\otimes k & \longrightarrow & H^2_{DR}(X/k) & \longrightarrow & E_2^{0,2} & \longrightarrow & 0
\end{array}$$

where $E_2^{0,2}$ is the $E_2$ term in the slope spectral sequence, it follows that we have an exact sequence

$$0\longrightarrow E_2^{0,2}\longrightarrow H^2_{fl}(X,Z_p(1))\otimes k\longrightarrow H^2_{DR}(X/k)\longrightarrow E_2^{0,2}\longrightarrow 0.$$

We have $H^2_{fl}(X,Z_p(1))\otimes k=H^2_{fl}(X,Z_p)(1))/p\otimes k$ and since $H^2_{fl}(X,Z_p(1))/p$ is a finite group it follows that the automorphism induced on $H^2_{fl}(X,Z_p(1))\otimes k$ has finite order.

By (2.1) $g_*$ has finite order on $E_2^{0,2}\subset\ker F^{\sigma_0}d$ so the exact sequence shows that there is a power $n$ of $g_*$ such that $(g_*^n-id)^2=0$ on $H^2_{DR}(X/k)$, it follows then that $(g_*^n-id)^p=0$ or $g_*^{np}=id$.

As a final application of our theory we show how we in a special case can conclude $\sigma_0\le 10$.

(2.4) Theorem: *Assume $X$ has an involution $\theta$ such that $\theta_*=id$ on $H^0(\Omega^2_X)$ then $\sigma_0\le 10$.*

Remark: It is a consequence of Artin's conjecture that $\sigma_0\le 10$ for all supersingular K3's.

Proof of (2.4): Consider the exact sequence

$$0 \longrightarrow E_2^{0,2} \longrightarrow H^2_{fl}(X, \mathbb{Z}_p(1)) \otimes k \longrightarrow H^2_{DR}(X/k) \longrightarrow E_2^{0,2} \longrightarrow 0.$$

If $\sigma_0 = 11$, then $T_0 \otimes k = H^2_{fl}(X, \mathbb{Z}_p(1)) \otimes k$, and the condition $\theta_* = id$ implies that $\theta$ acts trivially on $H^2(W\mathcal{O}_X)$ hence on $T_0 \otimes k$ and on $E_2^{0,2}$. The exact sequence above then shows that

$$\mathrm{Tr}\, \theta_* : H^2_{DR}(X/k) = 22,$$

and hence that the de Rham Lefshetz number

$$L(\theta, X) = \sum_{i=0}^{4} (-1)^i \,\mathrm{Tr}\, H^i_{DR}(X/k) \equiv 24 \bmod p.$$

Since $\theta$ has order 2, the fixed point scheme is smooth. Let $x \in X^\theta$, and consider the action $d\theta$ of $\theta$ on $T_{X,x}$. The condition $\theta_* = id$ on $H^0(\Omega_X^2)$ implies that $\det d\theta = 1$. Assume now that $X^\theta$ has a 1-dimensional component passing through $x$; this would give a vector $v \in T_{X,x}$ such that $d\theta(v) = v$, and since $d\theta$ can be diagonalized this would imply $d\theta = id$ on $T_{X,x}$ which is a contradiction to the smoothness of $X^\theta$. It follows that $X^\theta$ is discrete and hence finite.

The Lefshetz fixed point formula in de Rham cohomology then gives

$$\# X^\theta \equiv 24 \bmod p.$$

Let us compute the local factor $\det(1 - d\theta)$. We can diagonalize $d\theta$, let $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ be the matrix in diagonal form; then $ab = \det d\theta = 1$, and $a^2 = b^2 = 1$ it follows that $a = b = -1$, so $\det(1 - d\theta) = 4$. Since $\theta_*$ is trivial on $H^0(\Omega_X^2)$ it is also trivial on $H^2(\mathcal{O}_X)$, so the Wood's Hole fixed point formula gives

$$2 \equiv \sum_{x \in X^\theta} \frac{1}{\det(1 - d\theta)} = \frac{\# X^\theta}{4} \bmod p$$

hence $\# X^\theta \equiv 8 \bmod p$, so we get $8 \equiv 24 \bmod p \Rightarrow p = 2$ contradiction.

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