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REPRESENTATIONS OF THE GROUP OF FUNCTIONS TAKING VALUES IN A COMPACT LIE GROUP

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Introduction

In our paper [3] a series of important unitary representations of the group G^X of all smooth functions on Riemannian manifold X taking values in a compact semisimple Lie group G has been constructed and considered. Those representations can be obtained in the following way.

Let $\Omega(X;\mathfrak{g})$ be the space of smooth 1-forms $\omega(x)$ on X taking values in the Lie algebra $\mathfrak{g}$ of the group G, that is $\omega(x)$ is a linear operator from the tangent space T_xX into $\mathfrak{g}$. Let us introduce the norm in $\Omega(X;\mathfrak{g})$ by the formula

$$\|\omega\|^2 = \int_X \mathrm{Sp}(\omega(x)\omega^*(x))dx,$$

where $\omega^*(x):\mathfrak{g}\to T_xX$ is the operator conjugate to $\omega(x)$ (it is defined since T_xX and $\mathfrak{g}$ have natural structures of Euclidean spaces), dx is the Riemannian measure on X.

Define the unitary representation $V(\tilde{g})$ of the group G^X in the space $\Omega(X;\mathfrak{g})$ by

$$(V(\tilde{g})\omega)(x) = \mathrm{Ad}g(x)\circ\omega(x).$$

Define the Maurer–Cartan cocycle $\beta\tilde{g}$ on G^X taking values in $\Omega(X;\mathfrak{g})$ by

$$(\beta\tilde{g})(x) = (dg(x))g^{-1}(x)$$

(here $dg(x): T_xX\to T_{g(x)}G$ and $(\beta\tilde{g})(x): T_xX\to T_eG=\mathfrak{g}$).

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The representation U of the group G^X is $\mathrm{EXP}_\beta\, V$ in the sense of [1], [2]. It means that U acts in the Fock space $\mathrm{EXP}\, H = \bigoplus_{n=0}^{\infty} S^n H_{\mathrm{C}}$ where $H = \overline{\Omega(X;\mathfrak{g})}$ is the completion of $\Omega(X;\mathfrak{g})$, H_{C} is the complexification of H, $S^n H_{\mathrm{C}}$ is the symmetrized tensor product of n copies of H_{C}. The action of the operator $U(\tilde{g})$ on the vectors $\mathrm{EXP}\,\omega = \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \underbrace{\omega \otimes \cdots \otimes \omega}_{n}$, $\omega \in \Omega(X;\mathfrak{g})$ (which form a total set in $\mathrm{EXP}\, H$) is defined by

$$U(\tilde{g})\,\mathrm{EXP}\,\omega = e^{(-\|\beta\tilde{g}\|^2/2) - \langle V(\tilde{g})\omega, \beta\tilde{g}\rangle}\,\mathrm{EXP}(V(\tilde{g})\omega + \beta\tilde{g}).$$

(A more convenient realization of this representation using the Gaussian measure is given in §5).

The representation U of the group G^X draws a great interest. After the paper [1], it has been almost simultaneously discovered by several authors: [17], [7], [3], [15]. There are a lot of variants and modifications of this construction in [3]: constructing of representations by a vector field, a fibre-bundle and so on.

In this paper we continue to consider these representations and correct a mistake which took place in the proof of its irreducibility in [3] (the statement of lemma 4 §2 of [3] is false). This correction requires a new development of the treatment of measures on an infinite-dimensional space of functions on X. It is interesting that some properties of the measures depend on the dimension of X. Let us mention one of the results (lemma 10): if $\dim X \geq 4$, μ is the Gaussian measure on the space of distributions on $\mathring{X}$ with Fourier transform $\exp(-\frac{1}{2}\|\ \|^2_{\mathring{W}^1_2(X)})$, ν_1, ν_2 are two singular measures concentrated on the set of generalized functions of the form $\Sigma\, \lambda_i \delta_{x_i}$, then the measures $\mu * \nu_i$ and $\mu * \nu_2$ are singular. A proof of this result is based on the following property of the Sobolev space $\mathring{W}^1_2(X)$ (lemma 3): if X is a compact Riemannian manifold and $\dim X \geq 4$ then there exists a Hilbert–Schmidt extension of $\mathring{W}^1_2(X)$ not containing the generalized functions of the form $\Sigma\, \lambda_i \delta_{x_i}$.

The main result of this work is the proof of the irreducibility of the representations $U(\tilde{g})$ for $\dim X \geq 4$. Thus not only the question of irreducibility for $\dim X = 1$ is open, as it has been stated in [3], but also that for the cases $\dim X = 2$ and $\dim X = 3$. Irreducibility for $\dim X \geq 5$, $G = SU(2)$, X being an open set in $\mathbb{R}^m$ has been proved before in [7]. Besides, for $\dim X \geq 4$ we give a new proof of the non-equivalence of the representations corresponding to different Riemannian metrics on X.

In the main, the plane of the proof of the irreducibility is a repetition of that of [3]. The analysis of the spectral function of the Laplace–Beltrami operator is the crucial point (§1), dimension 4 being critical (see footnote to §6). This paper can be read independently of [3], though its contents doesn't include that of [3].

Let us give briefly the plan of the paper. In §1, we prove the main lemmas about Hilbert–Schmidt extensions of the space $\mathring{W}_2^1(X)$, $\dim X \geq 4$. There are auxiliary facts about representations of abelian groups in §2. In §3, we propose an example of non-singularity of the Gaussian measure with its convolution and a criterion of singularity of measures. §4 contains the main results about convolutions of the Gaussian measures generated by the Laplace–Beltrami operators on X whose measures are concentrated on the delta-functions. In §5, the spectrum of the restriction of the representation of G^X to a commutative subgroup is considered. At last in §6, we prove the theorems of irreducibility and non-equivalence of the representations of G^X for $\dim X \geq 4$ and consider the difficulties appearing in dimensions 1, 2, 3. In an appendix the formulas for the representations of the Lie algebra of G^X are given.

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§1. The extensions of the space $\mathring{W}_2^1(X)$

Let X be a compact Riemannian manifold with a boundary $\partial X \neq \phi$, $\mathring{X} = X \backslash \partial X$, dx a Riemannian measure on X, Δ the Laplace–Beltrami operator. We consider the real Sobolev space $\mathring{W}_2^1(X)$ (i.e. the completion of the space of compactly supported functions on $\mathring{X}$ with respect to the norm $\|f\| = |\int_X \Delta f \cdot f dx|^{1/2}$).

Let $\{u_k\}_{k=1}^{\infty}$ be the orthonormal basis in $\mathring{W}_2^1(X)$ of the eigenfunctions of the Dirichlet problem for the Laplace–Beltrami operator Δ, ordered accordingly non-decrease of the eigenvalues.

LEMMA 1: *If* $\dim X \geq 4$ *then there exists* $c = \{c_k\} \in l^2$, $c_k > 0$ *such that*

$$\sum_{k=1}^{\infty} c_k u_k^2(x) = \infty$$

everywhere on $\mathring{X}$.

PROOF: Let us take $c_k = \dfrac{1}{\sqrt{k+1}\ln(k+1)}$. Then $\sum_{k=1}^{\infty} c_k^2 < \infty$. Let $m = \dim X$, λ_k be the kth eigenvalue of Δ. We will prove that the following estimate is true in $\mathring{X}$:

$$\sum_{\lambda_k \le \lambda} \frac{1}{\sqrt{k+1}\ln(k+1)} u_k^2(x) \sim Cp(x)\varphi_m(\lambda), \quad \varphi_m(\lambda) = \int^{\lambda} \frac{d\lambda}{\lambda^{2-m/4}\ln\lambda}; \tag{1}$$

with $C > 0$ and $p(x) > 0$ on $\mathring{X}$. Since $\varphi_m(\lambda) = \ln\ln\lambda$ for $m = 4$ and $\varphi_m(\lambda) > \lambda^{1/4-\epsilon}$ for $m > 4$, this estimate implies the lemma.

Denote by $v_k(x)$ the eigenfunctions of the Laplace–Beltrami operator, normalized in $L^2(X, dx)$, then $u_k(x) = \dfrac{v_k(x)}{\sqrt{\lambda_k}}$. Since $\lambda_k \sim C_1 k^{2/m}$ we have $k + 1 \sim C_2 \lambda_k^{m/2}$. Therefore

$$\begin{aligned}\sum_{\lambda_k \le \lambda} \frac{1}{\sqrt{k+1}\ln(k+1)} u_k^2(x) &= \sum_{\lambda_k \le \lambda} \frac{v_k^2(x)}{\lambda_k\sqrt{k+1}\ln(k+1)} \\ &\sim C_3 \sum_{a < \lambda_k \le \lambda} \frac{v_k^2(x)}{\lambda_k^{1+m/4}\ln\lambda_k}. \quad (a > 1)\end{aligned}$$

Let us express the right-hand sum in terms of the spectral function $E(x, x; \lambda)$ of Δ. Since $E(x, x; \lambda) = \sum_{\lambda_k \le \lambda} v_k^2(x)$ we have

$$\begin{aligned}\sum_{a < \lambda_k \le \lambda} \frac{v_k^2(x)}{\lambda_k^{1+m/4}\ln\lambda_k} &= \int_a^{\lambda} \frac{dE(x, x; \lambda)}{\lambda^{1+m/4}\ln\lambda} \\ &= \frac{E(x, x; \lambda)}{\lambda^{1+m/4}\ln\lambda} - \int_a^{\lambda} E(x, x; \lambda)\, d\left(\frac{1}{\lambda^{1+m/4}\ln\lambda}\right).\end{aligned}$$

Now we use the classical asymptotic formula of Carleman for the spectral function $E(x, x; \lambda)$ [14]:

$$E(x, x; \lambda) \sim C_4 p(x) \lambda^{m/2}, \qquad p(x) > 0.$$

We get

$$\int_a^{\lambda} \frac{dE(x, x; \lambda)}{\lambda^{1+m/4}\ln\lambda} \sim C_5 p(x) \int_a^{\lambda} \frac{d\lambda}{\lambda^{2-m/4}\ln\lambda}$$

and therefore

$$\sum_{\lambda_k \le \lambda} c_k u_k^2(x) \sim C_6 p(x) \int_a^{\lambda} \frac{d\lambda}{\lambda^{2-m/4}\ln\lambda}.$$

COROLLARY. *If* $\dim X \geq 4$ *then* $\sum_k u_k^4(x) = \infty$.

PROOF: By the Cauchy inequality

$$\infty = \Big(\sum_k c_k u_k^2(x)\Big)^2 \leq \sum_k c_k^2 \cdot \sum_k u_k^4(x),$$

and the statement follows from $\sum_k c_k^2 < \infty$.

REMARK: It can be proved that the series $\sum_k u_k^4(x)$ diverges as a power one for $\dim X > 4$ and as a logarithmic one for $\dim X = 4$.

Now we prove a statement generalizing lemma 1.

LEMMA 2: *Let* $n \in \mathbb{Z}_+$ *and* $x_1, \ldots, x_n$ *be mutually disjoint points in* $\mathring{X}$. *Then for any* $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, $\sum \lambda_i^2 \neq 0$ *we have*

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}\,\ln(k+1)} (\lambda_1 u_k(x_1) + \cdots + \lambda_n u_k(x_n))^2 = \infty.$$

PROOF: Owing to (1) it is sufficient to prove that

$$\sum_{\lambda_k \leq \lambda} \frac{1}{\sqrt{k+1}\,\ln(k+1)} u_k(x) u_k(y) = o\left(\int_a^\lambda \frac{d\lambda}{\lambda^{2-m/4} \ln \lambda}\right) \tag{2}$$

on any compact subset of the complement of the diagonal in $\mathring{X} \times \mathring{X}$.

Repeating the same reasonings as in the proof of lemma 1 we get

$$\begin{aligned} \sum_{\lambda_l \leq \lambda} \frac{1}{\sqrt{k+1}\,\ln(k+1)} u_k(x) u_k(y) &\sim C \int_a^\lambda \frac{dE(x, y; \lambda)}{\lambda^{1+m/4} \ln \lambda} \\ &= C\left(\frac{E(x, y; \lambda)}{\lambda^{1+m/4} \ln \lambda} - \int_a^\lambda E(x, y; \lambda)\, d\left(\frac{1}{\lambda^{1+m/4} \ln \lambda}\right)\right). \end{aligned} \tag{3}$$

Let us use the following estimate for the spectral function which is true on any compact subset of $\mathring{X} \times \mathring{X} \backslash \mathrm{diag}$:

$$|E(x, y; \lambda)| < C(1+\lambda)^{m/2} \tag{4}$$

(see [12]). It is clear that (3) and (4) imply (2).

Let $C^\infty(\mathring{X})$ the space of all compactly supported C^∞-functions on $\mathring{X}$ and $\mathscr{F}(X) = [C^\infty(\mathring{X})]'$ the space of Schwartz's distributions. Every nonzero element $f \in \mathring{W}_2^1(X)$ defines the nonzero functional on $C^\infty(\mathring{X})$ by the formula:

$$\langle f, \varphi\rangle = \int_X f(x)\varphi(x)dx.$$

Then we can imbed $\mathring{W}_2^1(X)$ in $\mathscr{F}(X)$.

LEMMA 3: *Let X be a compact Riemannian manifold,* $\dim X \geq 4$. *There exists a Hilbert–Schmidt extension $\tilde{H}$ of $\mathring{W}_2^1(X)$, $\tilde{H} \subset \mathscr{F}(X)$ which doesn't contain the generalized functions of the form $\Sigma\ \lambda_i\delta_{x_i}$, x_i being mutually disjoint points in $\mathring{X}$ and $\Sigma\ \lambda_i^2 \neq 0$.*

PROOF: Let us consider the (generalized) decomposition of the space of distributions on $\mathring{X}$ with respect to the system $\{u_k\}_{k=1}^\infty$, i.e. the mapping

$$\mathscr{F}(X) \xrightarrow{T} \mathbb{R}^\infty : f \mapsto \{\langle f, u_k\rangle\}_{k=1}^\infty. \tag{5}$$

Its natural domain of definition in $\mathscr{F}(X)$ contains any Hilbert-Schmidt extension of $\mathring{W}_2^1(X)$. Indeed, the image of T contains any space being a Hilbert–Schmidt extension of l^2.

T maps the space $\mathring{W}_2^1(X)$ into l^2, a delta-function δ_x into the sequence $\{u_k(x)\}_{k=1}^\infty$ and any linear combination $\Sigma_i\, \lambda_i\delta_{x_i}$ into the sequence $\{\Sigma_i\ \lambda_i u_k(x_i)\}_{k=1}^\infty$. Let us take c_k from lemma 2 and consider the extension H' of the space l^2 with respect to the operator $\Gamma : \Gamma e_k = c_k e_k$. It is a Hilbert–Schmidt extension and by lemma 2 the space $H' \subset \mathbb{R}^\infty$ doesn't contain the sequence $\{\Sigma_i\ \lambda_i u_k(x_i)\}_{k=1}^\infty$, x_i being mutually disjoint points in $\mathring{X}$ and $\Sigma_i\ \lambda_i^2 \neq 0$. The space $\tilde{H} = T^{-1}H'$ is the required Hilbert–Schmidt extension of $\mathring{W}_2^1(X)$.

REMARK: As a matter of fact the statement of lemma 3 is true for any Riemann manifolds X, $\dim X \geq 4$. This result can be got from lemma 3 by partitioning X into compact manifold. However we need not obtain the result in this paper.

Now we make clear the difference between the cases $\dim X = 4$ and $\dim X > 4$. For $\dim X > 4$ the space $\tilde{H}$ from lemma 3 can be described explicitly. Namely, it is known from the Sobolev space theory [11] that if and only if $l \leq \frac{m}{2}$ then $\mathring{W}_2^l(X)$ as subset of $\mathscr{F}(X)$ doesn't contain delta-functions δ_x and their linear combinations. On the other hand, if and only if $l > 1 + \frac{m}{4}$ the implication $\mathring{W}_2^1(X) \subset \mathring{W}_2^l(X)$ is a Hilbert–Schmidt extension. Consequently if $\frac{m}{2} \geq l > 1 + \frac{m}{4}$ then $\mathring{W}_2^l(X)$ is a space required in lemma 3. This inequality can be solved for $m \geq 5$ $\left(l = \frac{m}{2}\right)$.

This reasoning shows that for $m = 4$ $\tilde{H}$ can not be described in terms of $\mathring{W}_2^l(X)$ (and any "power" terms, but only "logarithmic" terms). It is clear that a choice of an extension $\left(c_k = \frac{1}{\sqrt{k+1}\ln(k+1)}\right)$ suitable for $m = 4$ suits for $m > 4$ as well.

§2. Disjointness of spectral measures

Let X be a Borel space and $\{\mu_\alpha\}$ be a measurable family of measures on X where α runs through a space A with a measure ν. (Measurability of $\{\mu_\alpha\}$ means that the mapping $(\alpha, Y) \mapsto \mu_\alpha(Y)$, where $\alpha \in A$, $Y \in \mathfrak{A}, \mathfrak{A}$ is the algebra of the measurable sets in X, is measurable on $A \times \mathfrak{A}$.)

Definition: The family $\{\mu_\alpha\}$ is called ν-singular mod 0 if for almost all (with respect to $\nu \times \nu$) pairs (α', α'') the measures $\mu_{\alpha'}$ and $\mu_{\alpha''}$ are singular. The family $\{\mu_\alpha\}$ is called ν-disjoint (or disjoint if it is clear what measure ν is considered) if for any measurable subsets A_1 and A_2 in A of positive ν-measure, such that $\nu(A_1 \cap A_2) = 0$, the measures $\int_{A_1} \mu_\alpha d\nu(\alpha)$ and $\int_{A_2} \mu_\alpha d\nu(\alpha)$ are singular.

Remark: As easy examples show, generally speaking singularity of $\{\mu_\alpha\}$ doesn't imply disjointness.

Let G be an abelian topological group possessing a sufficient set of continuous characters $\chi : G \to S^1$. We assume that for any continuous unitary representation U of G in a complex Hilbert space $\mathscr{H}$ there exists an isomorphism T of $\mathscr{H}$ onto a direct integral of Hilbert spaces,

$$T : \mathscr{H} \to \int_{\hat{G}}^{\oplus} \mathscr{H}_\chi d\mu(\chi)$$

with μ a Borel measure on $\hat{G}$ ($\hat{G}$ is the space of the measurable characters), which transfers $U(g)$, $g \in G$ into the operators

$$(TU(g)T^{-1}f)(\chi) = \chi(g)f(\chi).$$

This assumption is true of course for any locally compact group and for some others. In particular it is true for the group $\mathfrak{a}^X$ of §5.[1] The rest of §2 relates to the groups with the assumption being true.

The measure μ on $\hat{G}$ is defined by U uniquely up to equivalence and is called the spectral measure of U. The realization of the representation of G in the space $\int_{\hat{G}}^{\oplus} \mathcal{H}_\chi d\mu(\chi)$ is called the spectral decomposition of the initial representation U.

We give here two statements about the spectral measures.

(1) Two unitary representations of G are disjoint (that is they contain no equivalent subrepresentations) if and only if their spectral measures are singular.

(2) The spectral measures of the direct sum and the tensor product of two representations of G are equivalent to correspondingly the sum and the convolution of their spectral measures.

DEFINITION: A measurable family of unitary representations U_α of G where α runs through (A, ν) is called disjoint if the family of the spectral measures μ_α of U_α is disjoint (with respect to the given Borel measure ν on A).

In other words the set of representations U_α is called disjoint if for any subsets A_1, A_2 of A with positive ν-measure and $\nu(A_1 \cap A_2) = 0$, the representations $\int_{A_1}^{\oplus} U_\alpha d\nu(\alpha)$ and $\int_{A_2}^{\oplus} U_\alpha d\nu(\alpha)$ are disjoint.

Let us assume that a representation U of G is decomposed into a direct integral of representations

$$U = \int_A^{\oplus} U_\alpha d\nu(\alpha).$$

It means that U is equivalent to the representation in a direct integral of Hilbert spaces

$$H = \int_A^{\oplus} H_\alpha d\nu(\alpha)$$

[1] It is false for example for the multiplicative group of classes of mod 0 measurable mappings $S^1 \to S^1$.

given by

$$(U(g)f)(\alpha) = U_\alpha(g)f(\alpha),$$

U_α being a representation of G in H_α.

The next lemma is a simple measure theoretic variant of Schur's lemma.

LEMMA 4: *If a family of representations of G is disjoint then the commutant of U (that is the ring of operators in H commuting with $U(g)$) consists of*

$$B = \int_A B_\alpha d\nu(\alpha)$$

where B_α is a measurable operator function taking values in the commutant of U_α.

PROOF: Let an operator B commute with the operators of the representation. Then for any $A_1 \subset A$ with $\nu(A_1) > 0$ the space $\int_{A_1}^{\oplus} H_\alpha d\nu(\alpha)$ is invariant with respect to B (owing to disjointness with its direct complement). Consequently B is decomposed into a direct integral.

COROLLARY: *Under the assumption of the lemma the W^*-algebra generated by the operators $U(g)$ contains the operators of multiplication by every bounded ν-measurable function $a(\alpha): f(\alpha) \mapsto a(\alpha)f(\alpha)$.*[2]

Indeed, every such operator commutes with the operators B i.e. it belongs to the bicommutant of U which by von-Neumann's theorem is the weak closure of the algebra generated by operators $U(g)$.

LEMMA 5: *Let a representation U of G be decomposed in a tensor product $U = U' \otimes U''$ of representations U' and U'' with the corresponding spectral measures μ' and μ''. If the family of measures $\{\mu'_\chi\}$, $\mu'_\chi = \mu'(\cdot - \chi)$ is μ''-disjoint then the weakly closed operator*

[2] In [3] this statement (§2 lemma 5) contained the condition of singularity (instead of disjointness) of the U_α. Notice that without the assumption of singularity of the U_α the statement is false.

algebra generated by the operators $U(g), g \in G$ contains all operators $E \otimes U''(g)$ (and therefore all operators $U'(g) \otimes E$).

PROOF: Let $U'' = \int_{\hat{G}}^{\oplus} U''_x d\mu''(x)$ be the spectral decomposition of the representation U''. Then

$$U = \int_{\hat{G}}^{\oplus} (U' \otimes U''_\chi) d\mu''(\chi).$$

Since the spectral measure of $U' \otimes U''_\chi$ is $\mu'_\chi = \mu'(\cdot - \chi)$ it follows that by the condition of the lemma the family of representations $U' \otimes U''_\chi$ is disjoint. It follows from corollary of lemma 4 that the W^*-algebra generated by $U(g)$ contains the operators of multiplication by functions $a_g(\chi) = \langle \chi, g \rangle$, i.e. the operators $E \otimes U''(g)$.

We emphasize that lemma 5 is true for any multiplicities of the spectra of U' and U''.

§3. A condition of singularity of the Gaussian measure together with its convolution

Let μ be the standard Gaussian measure in $\mathbb{R}^\infty$, i.e. the Gaussian measure with zero mean and Fourier transform

$$\int_{\mathbb{R}^\infty} e^{i\langle f, x\rangle} d\mu(x) = \exp(-\tfrac{1}{2}\|f\|^2_{l^2}).$$

The following proposition is widely known. The measure μ and its translation $\mu_y = \mu(\cdot - y)$ are equivalent (that is mutually absolute continuous) if and only if $y \in l^2$. It is also well known that if two Gaussian measures are not equivalent, then they are singular (see for example [13]).

Let now ν be a Borel measure in $\mathbb{R}^\infty$. Let us consider the convolution $\mu * \nu$ that is the measure which is defined on cylindrical sets A by

$$(\mu * \nu)(A) = \int_{\mathbb{R}^\infty} \mu_y(A) d\nu(y) = \int_{\mathbb{R}^\infty} \mu(A - y) d\nu(y).$$

It is clear that singularity of μ and $\mu * \nu$ implies that $\nu(l^2) = 0$. Taking in mind the statement given above one could assume that the opposite implication is true as well, that is $\nu(l^2) = 0$ implies singularity of μ and $\mu * \nu$. (It is just that what was affirmed in lemma 4 of §2 in [3]. The mistake was the unjustified passage to the limit in the expression for the density.) However this is wrong. A counterexample can be given even when ν is a Gaussian measure.

EXAMPLE: Let ν be a Gaussian measure with zero mean whose correlation operator Γ is Hilbert-Schmidt but not nuclear. The convolution $\mu * \nu$ is a Gaussian measure as well the correlation operator $C = E + \Gamma$ (E is the identity operator). Owing to Feldman's theorem [10] the measures μ and $\mu * \nu$ are equivalent. On the other hand, since Γ is not nuclear, $\nu(l^2) = 0$ by the Minlos-Sazonov theorem (see [10]).

A criterion of singularity of the standard Gaussian measure μ together with its convolution can be gotten from the following lemma generalizing Ismagilow's lemma [7].

LEMMA 6: *Let Γ be a strictly positive Hilbert-Schmidt operator in ℓ^2, $\tilde{H} \subset \mathbb{R}^\infty$ be the completion of l^2 with respect to the norm $\|y\|_\Gamma = \langle \Gamma y, y\rangle^{1/2}$. Then there exists a subset $A \subset \mathbb{R}^\infty$ such that $\mu(A) = 1$ and $\mu(A - z) = 0$ for any $z \notin \tilde{H}$.*

PROOF: Let us introduce the expression

$$\Phi_\Gamma(x) \equiv :\langle \Gamma x, x\rangle: = \lim_n (\langle \Gamma_n x, x\rangle - \mathrm{Sp}\, \Gamma_n),$$

where $\Gamma_n = P_n \Gamma P_n$, P_n is the projection onto the space spanned by the first n eigenvectors of Γ (about Wick regularisation – the sign $:\ \ :$ – see for example [9]). It is known (see [9]) that if Γ is a Hilbert–Schmidt operator then $\Phi_\Gamma(x)$ is defined and finite almost everywhere with respect to μ. (Notice that if $\mathrm{Sp}\, \Gamma = \infty$, the sequences $\langle \Gamma_n x, x\rangle$ and $\mathrm{Sp}\, \Gamma_n$ don't converge.) Let us put

$$A = \{x : \Phi_\Gamma(x) \text{ is finite}\}.$$

According to what has been said above, $\mu(A) = 1$. We prove that $\mu(A - z) = 0$ for any $z \notin \tilde{H}$.

For that purpose we use the following simple fact about the standard Gaussian measure μ in $\mathbb{R}^\infty$ (see for example [6]). The series

$\Sigma_{n=1}^{\infty} d_n y_n$ either converges almost everywhere with respect to μ (if $d \in l^2$) or has neither a finite nor infinite sum almost everywhere (if $d \notin l^2$). Thus $\mu\{y : \Sigma_{n=1}^{\infty} d_n y_n = \infty\} = 0$ for any $d \in \mathbb{R}^{\infty}$.

Since μ doesn't change by orthogonal transformations, the operator Γ in lemma can be assumed to be diagonal: $\Gamma e_n = c_n e_n$, $c_n > 0$, where $e_n = (0, \ldots, 0, 1, 0, \ldots)$, the unit is on the nth place. According to the remark above

$$\mu\{x : \lim_n \langle \Gamma_n x, z\rangle = \infty\} = \mu\left\{x : \sum_{n=1}^{\infty} c_n z_n x_n = \infty\right\} = 0$$

for any $z \in \mathbb{R}^{\infty}$. Let $z \notin \tilde{H}$. Then since $\lim_n \langle \Gamma_n z, z\rangle = \langle \Gamma z, z\rangle = \infty$, the following implication is true:

$$\{x : \lim_n (2\langle \Gamma_n x, z\rangle + \langle \Gamma_n z, z\rangle) \text{ is finite}\} \subset \{x : \lim_n \langle \Gamma_n x, z\rangle = \infty\}.$$

Consequently

$$\mu\{x : \lim_n (2\langle \Gamma_n x, z\rangle + \langle \Gamma_n z, z\rangle) \text{ is finite}\} = 0.$$

Since $\mu\{x : \lim_n (\langle \Gamma_n x, x\rangle - \operatorname{Sp} \Gamma_n)$ is finite$\} = 1$ it follows that $\mu\{x : \Phi_\Gamma(x + z)$ is finite$\} = \mu\{x : \lim_n (\langle \Gamma_n x, x\rangle - \operatorname{Sp} \Gamma_n + 2\langle \Gamma_n x, z\rangle + \langle \Gamma_n z, z\rangle)$ is finite$\} = 0$ i.e. $\mu(A - z) = 0$.

COROLLARY 1: *With the notations of lemma 6, if ν is a Borel measure on $\mathbb{R}^{\infty}$ such that $\nu(\tilde{H}) = 0$ then μ and $\mu * \nu$ are singular.*

Indeed $(\mu * \nu)(A) = \int_{\mathbb{R}^{\infty} \setminus H} \mu(A - z) d\nu(z) = 0$ because, by lemma 6, $\mu(A - z) = 0$ if $z \notin \tilde{H}$. Since $\mu(A) = 1$ the measures $\mu * \nu$ and μ are singular.

A convenient criterion for the singularity of the measures is given by the next corollary.

COROLLARY 2: *Let μ be the standard Gaussian measure in $\mathbb{R}^{\infty}$. If there exists an element $c = \{c_k\} \in l^2$, $c_k > 0$ such that $\nu\{x : \Sigma_{k=1}^{\infty} c_k x_k^2 < \infty\} = 0$ then μ and $\mu * \nu$ are singular.*

Indeed one can apply corollary 1 to the case when $\Gamma e_n = c_n e_n$, $n = 1, 2, \ldots$.

§4. Convolutions with a Gaussian measure in the space of distributions on X

Let X be a Riemannian manifold with a boundary ∂X (∂X may be empty), $\mathring{X} = X \setminus \partial X$, μ be the Gaussian measure in the space $\mathscr{F}(X)$ of distributions on $\mathring{X}$ associated with the Laplace–Beltrami operator, i.e. the measure with Fourier transform $\exp(-\frac{1}{2}\| \quad \|^2_{W_2^1})$. We introduce for convenience the following notations:

$$\Phi = \left\{\sum_{i=1}^{n} \lambda_i \delta_{x_i} : x_i \in \mathring{X}, \lambda_i \in \mathbb{Z}, n \in \mathbb{Z}_+\right\},$$

$$\Phi_{\mathrm{R}} = \left\{\sum_{i=1}^{n} \lambda_i \delta_{x_i} : x_i \in \mathring{X}, \lambda_i \in \mathbb{R}, n \in \mathbb{Z}_+\right\}.$$

LEMMA 7: *If X is a compact Riemannian manifold,* $\dim X \geq 4$ *then there exists a subset $A \subset \mathscr{F}(X)$ such that $\mu(A) = 1$ and $\mu(A - \varphi) = 0$ for any $\varphi \in \Phi_{\mathrm{R}}$, $\varphi \neq 0$.*

The statement is a straight consequence of lemmas 3 and 6.

COROLLARY: *If X is a compact Riemannian manifold,* $\dim X \geq 4$, *and ν is a measure on the space $\mathscr{F}(X)$ such that $\nu(\Phi_{\mathrm{R}} \setminus \{0\}) = 1$ then the measures μ and $\mu * \nu$ are singular.*

Now we consider a Riemannian manifold X. Let $Y \subset X$ be an open subset with compact closure $\bar{Y}$. Let us consider the functions with supports in Y and restrict to them distributions from $\mathscr{F}(X)$. We get the projection $\pi : \mathscr{F}(X) \to \mathscr{F}(Y)$ and $\pi\mu_X = \mu_Y$ where μ_X, μ_Y are the Gaussian measures associated with the corresponding Laplace–Beltrami operators on X and Y.

LEMMA 8: *Let $Y \subset X$ be the same as above,* $\dim X \geq 4$. *Then there exists a subset $A \subset \mathscr{F}(X)$ such that $\mu_X(A) = 1$, $\mu_Y(\pi A - \pi\varphi) = 0$ for any $\varphi \in \Phi$ provided $\pi\varphi \neq 0$.*

The statement follows from lemma 7.

COROLLARY: *Let X be a Riemannian manifold,* $\dim X \geq 4, \{Y_n\}_{n=1}^{\infty}$ *be a countable base of open sets in X with compact closure, π_n be the projection $\mathscr{F}(X) \to \mathscr{F}(Y)$. There exists a subset $A \subset \mathscr{F}(X)$ such that $\mu_X(A) = 1$ and $\mu_{Y_n}(\pi_n A - \pi_n\varphi) = 0$ for any n and $\varphi \in \Phi$ provided $\pi_n\varphi \neq 0$.*

Let us put for any n $A_n = \pi_n^{-1}(\pi_n A)$ and introduce for any $\varphi \in \Phi$ the set

$$A^{\varphi} = A \backslash \Big(\bigcup_{n,\psi} (A_n - \psi) \Big),$$

where $\psi \in \Phi$, $\operatorname{supp} \psi \subset \operatorname{supp} \varphi$ and $\pi_n \psi \neq 0$ (similar sets were considered in [7]).

LEMMA 9: *Let A be a set as in the corollary of lemma* 8 ($\dim X \geq 4$). *Then $\varphi_1, \varphi_2 \in \Phi$ and $\varphi_1 \neq \varphi_2$ imply $(A^{\varphi_1} + \varphi_1) \cap (A^{\varphi_2} + \varphi_2) = \phi$.*

PROOF: Let us assume on the contrary that there exist $\varphi_1 \neq \varphi_2$ such that $(A^{\varphi_1} + \varphi_1) \cap (A^{\varphi_2} + \varphi_2) \neq \phi$, i.e. there exist $a_i \in A^{\varphi_i}, i = 1, 2$, such that $a_1 + \varphi_1 = a_2 + \varphi_2$. Since $\varphi_1 \neq \varphi_2$ there exists a point x_1 belonging to the support of φ_1 with a coefficient λ and to that of φ_2 with a coefficient $\lambda' \neq \lambda$. Let for definiteness $\lambda \neq 0$, λ' can be equal to 0. Let Y_n be a basis of neighbourhood of x_1 and let Y_n contain no other points of the supports of φ_1 and φ_2. Then from the equality $\pi_n(a_1 + \varphi_1) = \pi_n(a_2 + \varphi_2)$ it follows that $\pi_n a_1 = \pi_n a_2 + k\delta_{x_1}$, where $k \neq 0$. But this is impossible since on the one hand $\pi_n a_1 \in \pi_n A \backslash (\pi_n A + k\delta_{x_1})$ and on the other $\pi_n a_2 + k\delta_{x_1} \in \pi_n A + k\delta_{x_1}$.

LEMMA 10: *Let X be a Riemannian manifold,* $\dim X \geq 4$, *μ be the Gaussian measure in the space of distributions on $\mathring{X}$ with Fourier transform* $\exp(-\frac{1}{2}\| \ \|^2_{W_2^1})$, *$\nu_1$ and ν_2 be singular measures on $\mathcal{F}(X)$,* $\nu_1(\Phi) = \nu_2(\Phi) = 1$. ***Then the measures $\mu * \nu_1$ and $\mu * \nu_2$ are singular. In particular if $\nu_1(\{0\}) = 0$ then the measures $\mu * \nu_1$ and μ are singular.***

PROOF: We take A in the same way as above and put $B^{\varphi} = A^{\varphi} + \varphi (\varphi \in \Phi)$. The mapping $\varphi \mapsto B^{\varphi}$ is measurable as a mapping of Φ into the family $\mathcal{B}(\mathcal{F}(X))$ of the Borel subsets of $\mathcal{F}(X)$ (both spaces are provided with the natural Borel structure). Therefore if $Q \subset B$ is a Borel set then $B^Q = \bigcup_{\varphi \in Q} B^{\varphi}$ is measurable with respect to every Borel measure. Let $Q_1 \cap Q_2 = \phi$ and $\nu_1(Q_1) = \nu_2(Q_2) = 1$. By lemma 9 $B^{Q_1} \cap B^{Q_2} = \bigcup_{\substack{\varphi_1 \in Q_1 \\ \varphi_2 \in Q_2}} (B^{\varphi_1} \cap B^{\varphi_2}) = \phi$. On the other hand

$$(\mu * \nu_1)(B^{Q_1}) = \int_{Q_1} \mu \Big(\bigcup_{\varphi \in Q_1} B^{\varphi_1} - \psi \Big) d\nu_1(\psi) =$$

$$= \int_{Q_1} \mu \Big(\bigcup_{\varphi \in Q_1} (A^{\varphi} + \varphi - \psi) \Big) d\nu_1(\psi) \geq \int_{Q_1} \mu(A^{\psi}) d\nu_1(\psi) = 1$$

because $\mu(A^{\psi}) = \mu(A) = 1$. By analogy $(\mu * \nu_2)(B^{Q_2}) = 1$ and the lemma is proved.

Using the disjointness of a measure family (see §2) one can re-formulate the statement of lemma 10 in the following way.

LEMMA 10′: *Let X and μ be the same as in lemma* 10 *and let ν be a measure in the space of distributions on X such that $\nu(\Phi) = 1$. Then the family of measures $\mu_\varphi = \mu(\cdot - \varphi)$ is ν-disjoint.*

REMARK: All lemmas of this section will remain true if the set $\mathbb{Z}$ in the definition of Φ is substituted by any countable subset of $\mathbb{R}$.

LEMMA 11: *Let two Riemannian structures τ_1, τ_2 be given on a compact manifold X, $\dim X \geq 4$, and μ_1, μ_2 be Gaussian measures generated by these structures in the space of distributions $\mathcal{F}(X)$. Then there exists a subset $A \subset \mathcal{F}(X)$ which satisfies one of the following conditions*:

(i) $\mu_2(A) = 1$ *and* $\mu_1(A - \varphi) = 0$ *for any* $\varphi \in \Phi_{\mathbb{R}} \setminus \{0\}$,
(ii) $\mu_1(A) = 1$ *and* $\mu_2(A - \varphi) = 0$ *for any* $\varphi \in \Phi_{\mathbb{R}} \setminus \{0\}$.

PROOF: The Riemannian structure τ_i generates an inner product in the function space on $\mathring{X}$:

$$(f, g)_i = -\int_X \Delta_i f \cdot g d_i x$$

Δ_i being the Laplace-Beltrami operator on the Riemannian manifold (X, τ_i), $i = 1, 2$. It is clear that these inner products are equivalent, i.e. $(f, g)_2 = (Bf, g)_1$ where B is a positive bounded invertible operator. Let, as above, $\mathring{W}_2^1(X)$ be a completion of the space of compactly supported functions on $\mathring{X}$, $\{u_k\}_{k=1}^\infty$ be the orthonormal basis in $\mathring{W}_2^1(X)$ of the eigenfunctions of the Dirichlet problem for Δ_1. Let us consider the mapping (5) from the space of distributions on X into $\mathbb{R}^\infty$. It maps $\mathring{W}_2^1(X)$ into l^2, functions $\varphi = \Sigma_i \lambda_i \delta_{x_i}$ into sequences $\{\Sigma_i \lambda_i u_k(x_i)\}_{k=1}^\infty$, the measures μ_1, μ_2 correspondingly into the standard Gaussian measure μ_1' and the Gaussian measure μ_2' with the correlation operator $B = \|b_{ij}\|$.

Let us consider the following sets in $\mathbb{R}^\infty$:

$$A_1' = \left\{x : \sum_k c_k(x_k^2 - 1) < \infty\right\}, \qquad A_2' = \left\{x : \sum_k c_k(x_k^2 - b_{kk}) < \infty\right\},$$

where $c_k = \dfrac{1}{\sqrt{k+1}\ln(k+1)}$. Since $\Gamma = \|c_k\delta_{ik}\|$ is a Hilbert–Schmidt operator with respect to the inner products $(,)_1$ and $(,)_2$, the expressions $\Sigma_k\, c_k(x_k^2 - 1)$ and $\Sigma_k\, c_k(x_k^2 - b_{kk})$ give almost everywhere (correspondingly with respect to μ_1' and μ_2') finite quadratic functionals with zero mean. Therefore $\mu_i'(A_i') = 1, i = 1, 2$.

We assume now that $\Sigma_k\, c_k(1 - b_{kk}) \neq -\infty$ and show that it implies $\mu_1'(A_2' - a) = 0$ for any $a = \{a_k\}$, $a_k = \Sigma_i\, \lambda_i u_k(x_i)$. Indeed

$$A_2' - a = \left\{x : \sum_k c_k[(x_k + a_k)^2 - b_{kk}] < \infty\right\} =$$

$$= \left\{x : \sum_k c_k[(x_k^2 - 1) + 2a_k x_k + (1 - b_{kk}) + a_k^2] < \infty\right\}.$$

Since $\Sigma_k\, c_k a_k^2 = \infty$ by lemma 2 and since $\Sigma_k\, c_k(1 - b_{kk}) \neq -\infty$, the series $\Sigma_k\, c_k(1 - b_{kk} + a_k^2)$ diverges. On the other hand $\Sigma_k\, c_k(x_k^2 - 1) < \infty$ almost everywhere with respect to μ_1' and since $\Sigma_k\, c_k^2 a_k^2 < \infty$ it follows that $\Sigma_k\, c_k a_k x_k < \infty$ almost everywhere with respect to μ_1'. Consequently $\mu_1'(A_2' - a) = 0$.

Let A_2 be the inverse image of A_2' in $\mathscr{F}(X)$. Then $\mu_2(A_2) = 1$ and the statement proved above implies that $\mu_1(A_2 - \varphi) = 0$ for any $\varphi = \Sigma_i\, \lambda_i \delta_{x_i}$ $(\varphi \neq 0)$.

If $\Sigma_k\, c_k(1 - b_{kk}) = -\infty$ then similar reasonings prove that the inverse image A_1 of A_1' satisfies $\mu_2(A_1 - \varphi) = 0$ for any $\varphi = \Sigma_i\, \lambda_i \delta_{x_i}$ $(\varphi \neq 0)$.

COROLLARY 1: *Under the conditions of lemma* 11, *if* ν *is a measure on* $\mathscr{F}(X)$ *concentrated on* $\Phi_{\mathbf{R}} \backslash \{0\}$ *then either* $\mu_1 \perp \mu_2 * \nu$ *or* $\mu_2 \perp \mu_1 * \nu$.

COROLLARY 2: *Under the same conditions, if* $\mu_1 \perp \mu_2$ *and* ν *is a measure on* $\mathscr{F}(X)$ *concentrated on* $\Phi_{\mathbf{R}}$ *and satisfying* $\nu(\{0\}) > 0$ *then* $\mu_1 * \nu$ *and* $\mu_2 * \nu$ *are not equivalent.*

Indeed, by the corollary 1 and the singularity of the measures μ_1, μ_2 either $\mu_1 \perp \mu_2 * \nu$ or $\mu_2 \perp \mu_1 * \nu$. On the other hand $\nu(\{0\}) > 0$ implies that $\mu_1 < \mu_1 * \nu$, $\mu_2 < \mu_2 * \nu$. Consequently $\mu_1 * \nu \not\sim \mu_2 * \nu$.

REMARK: Notice that convolutions of two singular Gaussian measures μ_1, μ_2 with a measure ν can turn out equivalent.

EXAMPLE: Let μ_1 be the standard Gaussian measure on $\mathbb{R}^\infty$, μ_2 and ν be the Gaussian measures with correlation functionals $\Sigma_n\,(1-n^{-1/2})x_n^2$ and $\Sigma_n\,nx^2$. Then $\mu_1 \perp \mu_2$ and at the same time $\mu_1 * \nu \sim \mu_2 * \nu$.

§5. On the spectrum of a representation of the abelian group $\mathfrak{a}^X$

We consider the group G^X of smooth mappings of a Riemannian manifold X into a compact semisimple Lie group G. Let U be the unitary representation of G^X being defined in the Introduction. We will use here another model of this representation (see [3]).

Let $\Omega(X;\mathfrak{g})$ be the pre-Hilbert space of $\mathfrak{g}$-values 1-forms on X with the unitary representation V of G^X acting there and $\beta: G^X \to \Omega(X;\mathfrak{g})$ be the Maurer–Cartan cocycle (see Introduction). Let further $\mathcal{F}$ be the conjugate space to $\Omega(X;\mathfrak{g})$ and μ be the Gaussian measure on $\mathcal{F}$ with Fourier transform $\chi(\omega) = \exp(-\frac{1}{2}\|\omega\|^2)$. U acts in the Hilbert space $L^2_\mu(\mathcal{F})$ by the following formula

$$(U(\tilde{g})\Phi)(F) = e^{i\langle F, \beta\tilde{g}\rangle}\Phi(V^{-1}(\tilde{g})F).$$

(see [3] about equivalence of the representations).

Let now $\mathfrak{a} \subset \mathfrak{g}$ be a Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of G, $\mathfrak{a}^X$ be the abelian group of compactly supported C^∞-mappings $X \to \mathfrak{a}$. $\mathfrak{a}^X$ is said in §2 to satisfy the condition formulated there and therefore for any unitary representation of $\mathfrak{a}^X$ the spectral measure is defined. Let us define the unitary representation of $\mathfrak{a}^X$ in $L^2_\mu(\mathcal{F})$ by the formula

$$W(a) = U(\exp a), \quad a \in \mathfrak{a}^X.$$

Here we give a summary of properties of W and its spectral measure (see [3] §4 lemmas 1–4).

(1) Let $\mathfrak{m}$ denote the orthogonal complement in $\mathfrak{g}$ to $\mathfrak{a}$, $\mathcal{F}_\mathfrak{a}$, $\mathcal{F}_\mathfrak{m}$ be the subspaces of, correspondingly, $\mathfrak{a}$-valued and $\mathfrak{m}$-valued generalized 1-forms on X, $\mu_\mathfrak{a}, \mu_\mathfrak{m}$ be the standard Gaussian measures on $\mathcal{F}_\mathfrak{a}$ and $\mathcal{F}_\mathfrak{m}$. Then

$$W = W_\mathfrak{a} \otimes W_\mathfrak{m},$$

where $W_\mathfrak{a}, W_\mathfrak{m}$ are the representations in the corresponding spaces

$L^2_{\mu_{\mathfrak{a}}}(\mathscr{F}_{\mathfrak{a}})$ and $L^2_{\mu_{\mathfrak{m}}}(\mathscr{F}_{\mathfrak{m}})$ which are defined by

$$(W_{\mathfrak{a}}(a)\Phi)(F) = e^{i\langle F, da\rangle}\Phi(F),$$
$$(W_{\mathfrak{m}}(a)\Phi)(F) = \Phi(\mathrm{Ad}^{-1}(\exp a)F).$$

Thus the spectral measure of W is equal to the convolution of the spectral measures of $W_{\mathfrak{a}}$ and $W_{\mathfrak{m}}$.

(2) $$W_{\mathfrak{m}} \cong \bigoplus_{n=0}^{\infty} S^n V_{\mathfrak{m}},$$

where $V_{\mathfrak{m}}$ is the representation in the space $\overline{\Omega(X;\mathfrak{m})_C}$ given by

$$(V_{\mathfrak{m}}(a)\omega)(x) = \mathrm{Ad}(\exp a(x)) \cdot \omega(x),$$

$S^n V_{\mathfrak{m}}$ $(n > 0)$ is the symmetrized tensor product of n copies of $V_{\mathfrak{m}}$; $S^0 V_{\mathfrak{m}}$ is the unity representation.

(3) The spectral measure μ of $W_{\mathfrak{a}}$ is equivalent to the Gaussian measure on $(\mathfrak{a}^X)'$ with zero mean and Fourier transform $\chi(a) = \exp(-\frac{1}{2}\langle da, da\rangle)$, $a \in \mathfrak{a}^X$.

(4) By the decomposition $W_{\mathfrak{m}} \cong \bigoplus_{n=0}^{\infty} S^n V_{\mathfrak{m}}$, the spectral measure ν of $W_{\mathfrak{m}}$ is equivalent to the sum of the spectral measure ν_0 of the unity representation (i.e. the measure concentrated on the point 0) and the spectral measures ν_n of $S^n V_{\mathfrak{m}}$ $(n = 1, 2, \ldots)$.

Let Σ denote the set of the roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Let $\nu_n^{\alpha_1,\ldots,\alpha_n}$, $\alpha_i \in \Sigma$, $n = 1, 2, \ldots$, denote the measure in $(\mathfrak{a}^X)'$ concentrated on the set of distributions of the form $\varphi_{x_1}^{\alpha_1} + \cdots + \varphi_{x_n}^{\alpha_n}$ where $\langle \varphi_x^{\alpha}, a\rangle = \alpha(a(x))$ and equivalent on this set to the measure $dx_1 \ldots dx_n$. Then ν_n is equivalent to the sum of $\nu_n^{\alpha_1,\ldots,\alpha_n}$ $(n = 1, 2, \ldots)$.

We denote now by Ψ the set of distributions of the form $\varphi_{x_1}^{\alpha_1} + \cdots + \varphi_{x_n}^{\alpha_n}$ where $\alpha_i \in \Sigma$, $x_i \in \mathring{X}$, $n = 0, 1, \ldots$ and note that $\nu(\Psi) = 1$.

Lemma 12: *If* $\dim X \geq 4$ *then the family of measures* $\mu_\varphi = \mu(\cdot - \varphi)$, $\varphi \in \Psi$ *is* ν*-disjoint* (*see* §2).

Proof: Let us fix a unit vector $e \in \mathfrak{a}$ such that the numbers $\alpha(e)$ are mutually distinct as α runs through the non-zero elements of Σ. Let $\mathbb{R}_1 \subset \mathfrak{a}$ be the one-dimensional space spanned by e, $\mathbb{R}_1^X \subset \mathfrak{a}^X$ be the space of compactly supported C^∞-smooth real functions on X, let $(\mathbb{R}_1^X)'$ be the space conjugate to $\mathbb{R}_1^X$. We consider the projection

$$\tau : (\mathfrak{a}^X)' \to (\mathbb{R}_1^X)'$$

and denote by μ', ν' the projections of μ and ν on $(\mathbb{R}_1^X)'$. Owing to the properties (3), (4) of the measures μ and ν, μ' is the Gaussian measure on $(\mathbb{R}_1^X)'$ with zero mean and Fourier transform $\chi(f) = \exp(-\frac{1}{2}\|f\|^2_{\mathring{W}_2^1})$, ν' is concentrated on the set $\tau\Psi = \{\alpha_1(e)\delta_{x_1} + \cdots + \alpha_n(e)\delta_{x_n} : x_i \in \mathring{X}, \alpha_i \in \Sigma, n = 0, 1, 2, \ldots\}$.

Consequently, by lemma 10′ the family of measures $\mu'_{\varphi'} = \mu'(\cdot - \varphi'), \varphi' \in \tau\Psi$ is ν'-disjoint. Then the family of measures $\mu''_\varphi = \mu'_{\tau\varphi}$ is ν-disjoint since, by the choice of e, the mapping $\Psi \to \tau\Psi$ is a bijection and $\tau\nu = \nu'$.

It follows now from this disjointness that the family of measures $\mu_\varphi = \mu(\cdot - \varphi), \varphi \in \Psi$ is ν-disjoint on $\mathfrak{a}^X$, i.e. for any $A_1, A_2 \subset \Psi$ of positive ν-measures with $\nu(A_1 \cap A_2) = 0$ the measures $\int_{A_i} \mu_\varphi d\nu(\varphi), i = 1, 2$ are mutually singular. Indeed, since the family of measures μ''_φ is ν-disjoint on Ψ then there exists $B_1, B_2 \subset (\mathbb{R}_1^X)'$ such that $B_1 \cap B_2 = \phi$ and $\int_{A_i} \mu''_\varphi(B_i) d\nu(\varphi) = \nu(A_i)$. Let $\bar{B}_i$ denote the inverse image of B_i in $(\mathfrak{a}^X)'$. Then for any $\varphi \in \Psi$ the set $\bar{B}_i + \varphi$ is the inverse image of $B_i + \tau\varphi$ and therefore $\mu_\varphi(\bar{B}_i) = \mu''_\varphi(B_i)$. Consequently $\int_{A_i} \mu_\varphi(\bar{B}_i) d\nu(\varphi) = \nu(A_i)$, $i = 1, 2$ and since $\bar{B}_1 \cap \bar{B}_2 = \phi$ the measures $\int_{A_i} \mu_\varphi d\nu(\varphi)$, $i = 1, 2$ are mutually singular.

Corollary 1: *The spectral measure of $W_\mathfrak{a}$ and those of $W_\mathfrak{a} \otimes S^n V_\mathfrak{m}$, $n = 1, 2, \ldots$ are mutually singular.*

Corollary 2: *The weakly closed operator algebra generated by the operators of W contains all operators of multiplication by $e^{i\langle \cdot, da \rangle}$.*

That follows from lemmas 12 and 5.

Lemma 13: *Let X be a smooth manifold, $\dim X \geq 4$, τ_1, τ_2 be two Riemannian structures on X, W^1, W^2 be the representations of $\mathfrak{a}^X$ corresponding to these structures. If $\tau_1 \neq \tau_2$ then W^1 and W^2 are not equivalent.*

Proof: Without a loss of generality we can assume X to be a compact manifold. Indeed, in the opposite case one can take an arbitrary neighbourhood $Y \subset X$ with a compact closure on which $\tau_1 \neq \tau_2$, and consider instead of W^i their restrictions to $\mathfrak{a}^Y$.

We note that $\mathfrak{a}^X = \mathbb{R}_1^X \oplus \cdots \oplus \mathbb{R}_r^X$, $\mathbb{R}_k = \mathbb{R}$ $(r = \dim \mathfrak{a})$. Therefore it is sufficient to prove that the restrictions $\overline{W^i}$ of W^i to $\mathbb{R}_1^X$ are not equivalent. For this purpose we will find the spectral measures of $\overline{W^i}$.

We have $W^i = W^i_{\mathfrak{a}} \otimes W^i_{\mathfrak{m}}$ and $W^i_{\mathfrak{a}}$ is a tensor product of representations of the groups $\mathbb{R}^X_k$. Let us denote by $\overline{W^i_{\mathfrak{a}}}$ and $\overline{W^i_{\mathfrak{m}}}$ the restrictions of $W^i_{\mathfrak{a}}$ and $W^i_{\mathfrak{m}}$ to $\mathbb{R}^X_1$. Then $\overline{W^i} = \overline{W^i_{\mathfrak{a}}} \otimes \overline{W^i_{\mathfrak{m}}}$.

By definition, the spectral measure of $\overline{W^i_{\mathfrak{a}}}$ is the standard Gaussian measure $\bar{\mu}_i$ in the space of distributions $\mathscr{F}(X)$ generated by the Riemannian structure τ_i. The inequality $\tau_1 \neq \tau_2$ implies $\bar{\mu}_1 \perp \bar{\mu}_2$ (for a proof, see [3], p. 326). The spectral measures $\bar{\nu}_i$ of $\overline{W^i_{\mathfrak{m}}}$ are equivalent to each other ($\bar{\nu}_1 \sim \bar{\nu}_2 \sim \bar{\nu}$) and are concentrated on the set $\Phi = \left\{\sum_{i=1}^{n} \lambda_i \delta_{x_i} : x_i \in \mathring{X},\ \lambda_i \in Z,\ n \in Z_+\right\}$ with $\bar{\nu}(\{0\}) > 0$. Therefore by corollary 2 of lemma 11 $\bar{\mu}_1 * \bar{\nu}_1$ and $\bar{\mu}_2 * \bar{\nu}_2$ are not equivalent. Those are the spectral measures of $\overline{W^i}$ and consequently the representations $\overline{W^1}$ and $\overline{W^2}$ are not equivalent.

§6. Irreducibility and non-equivalence of representations of G^X

THEOREM 1: *If* $\dim X \geq 4$ *then the representation* U *of* G^X *is irreducible.*

A proof reduces to a check-up of the following two statements:

(1) the cyclic subspace generated by the vacuum vector $\Phi \equiv 1$ is irreducible,

(2) the vacuum vector is cyclic in the space of U.

PROOF OF STATEMENT 1: Let $\mathfrak{a} \subset \mathfrak{g}$ be an arbitrary Cartan subalgebra. We denote by $H_{\mathfrak{a}}$ the space of functionals $\Phi \in L^2_\mu(\mathscr{F})$ such that $\Phi(\cdot + F) = \Phi(\cdot)$ for any $F \in \mathscr{F}_{\mathfrak{m}}$ (see §5). By definition W, $H_{\mathfrak{a}}$ is invariant under the action of W and the representations of $\mathfrak{a}^X$ in $H_{\mathfrak{a}}$ and in its orthogonal complement are equivalent, correspondingly, to $W_{\mathfrak{a}}$ and $\bigoplus_{n=1}^{\infty}(W_{\mathfrak{a}} \otimes S^n V_{\mathfrak{m}})$. It follows from corollary 1 of lemma 12 that the representations of $\mathfrak{a}^X$ in $H_{\mathfrak{a}}$ and in its orthogonal complement are disjoint.

Let us verify now that if $\mathfrak{a}_1, \mathfrak{a}_2$ are two Cartan subalgebra with $\mathfrak{a}_1 \cap \mathfrak{a}_2 = 0$ then $H_{\mathfrak{a}_1} \cap H_{\mathfrak{a}_2} = \{c\Omega\}$, Ω being the vacuum vector. Indeed, let $\mathfrak{m}_1, \mathfrak{m}_2$ be the orthogonal complements in $\mathfrak{g}$, correspondingly, of $\mathfrak{a}_1$ and $\mathfrak{a}_2$. Since $\mathfrak{a}_1 \cap \mathfrak{a}_2 = 0$ we have $\mathfrak{m}_1 + \mathfrak{m}_2 = \mathfrak{g}$ and therefore $\mathscr{F}_{\mathfrak{m}_1} + \mathscr{F}_{\mathfrak{m}_2} = \mathscr{F}$. Hence if $\Phi \in H_{\mathfrak{a}_1} \cap H_{\mathfrak{a}_2}$ that is $\Phi(\cdot + F_1) = \Phi(\cdot + F_2) = \Phi(\cdot)$ for any $F_1 \in \mathscr{F}_{\mathfrak{m}_1}$ and $F_2 \in \mathscr{F}_{\mathfrak{m}_2}$, then $\Phi(\cdot + F) = \Phi(\cdot)$ for any $F \in \mathscr{F}$ that is $\Phi(\cdot) = c\Omega$.

Let A be an arbitrary operator in $L^2_\mu(\mathscr{F})$ which commutes with operators of U. Then A commutes with operators of W. Since the representations of $\mathfrak{a}^X$ in $H_\mathfrak{a}$ and in its orthogonal complements are disjoint, the operator A leaves every subspace $H_\mathfrak{a}$ invariant and therefore $A\Omega = c\Omega$. Statement 1 follows.[3]

PROOF OF STATEMENT 2: Follows from corollary 2 of lemma 12. Namely, the corollary implies that the weak closed operator algebra generated by operators $U(\tilde{g})$, $\tilde{g} \in G^X$ in $L^2_\mu(\mathscr{F})$ contains the operators of multiplication by functionals of the form

$$(6) \quad \exp\left(i\left\langle F, \sum_{k=1}^{n} V(\tilde{g}_k)du_k\right\rangle\right), \quad \tilde{g}_k \in G^X, \quad u_k \in \mathfrak{g}^X, \quad n = 1, 2, \ldots.$$

Then one can prove that the set of vectors of the form $\Sigma_{k=1}^n V(\tilde{g}_k)du_k$ is dense in $\Omega(X; \mathfrak{g})$ which implies that the functionals (6) form a total set in $L^2_\mu(\mathscr{F})$ and statement 2 follows (see [3] §5 lemmas 2, 3).

THEOREM 2: *Let G be a compact semisimple Lie group, X be a manifold,* $\dim X \geq 4$. *If τ_1, τ_2 are two different Riemannian structures on X then the corresponding representations U of G^X are not equivalent.*

Indeed, by lemma 13 even the restrictions of these representations to an abelian subgroup $\mathfrak{a}^X$ are not equivalent.

REMARK: It can be proved by quite other reasonings that theorem 2 stands for $\dim X = 1$. In this case the spectral measure of $W = W_\mathfrak{a} \otimes W_\mathfrak{m}$ is equivalent to that of $W_\mathfrak{a}$. At the same time the spectral measures of the representations $W_\mathfrak{a}$ corresponding to different Riemannian structures on X are singular. Therefore the spectral measures of the corresponding representations W are singular as well. (The spectral measures of W and $W_\mathfrak{a}$ are likely to be equivalent for $\dim X = 2$ as well, see the remark below.)

Another proof of nonequivalence of the representations of G^X for

[3] As a matter of fact we repeated here the proof of lemma 1 of §5 in [3] with the difference that the reference to corollary 2 of lemma 5 of §4 in [3] was changed by the reference to corollary 2 of lemma 12 of this paper.

$G = SU(2)$ and X a manifold of an arbitrary dimension can be found in [7].

In conclusion we give some remarks on the cases $\dim X \leq 3$.

1. Let X be a compact Riemannian manifold, μ be standard Gaussian measure in the space of distributions on X, ν be a measure on the set of delta-functions δ_x equivalent to dx on this set.

Our proof of irreducibility is based on the statement that for $\dim X \geq 4$ the measures μ and $\mu * \nu^n$ $(n = 1, 2, \ldots)$ are mutually singular. Let us show that this statement fails for $\dim X \leq 2$ and therefore the methods of this work can not be used for the dimensions 1 and 2.

PROPOSITION: *If* $\dim X = 1$ *then* μ *and* $\mu * \nu^n$ $(n = 1, 2, \ldots)$ *are equivalent.*

PROOF: If $\dim X = 1$ then $\mathring{W}_2^1(X) \subset C$ (see [11]). Therefore $\delta_x \in (\mathring{W}_2^1(X))'$ for any $x \in \mathring{X}$ and consequently the measure ν^n is concentrated on $(\mathring{W}_2^1(X))'$. It is known that the Gaussian measure μ is quasi-invariant under the translations by vectors from $(\mathring{W}_2^1(X))'$. Hence the convolution of every measure, concentrated on $(\mathring{W}_2^1(X))'$, with μ is equivalent to μ. In particular $\mu * \nu^n \sim \mu$.

The case $\dim X = 2$ is more subtle: since in this case $\delta_x \not\in (\mathring{W}_2^1(X))'$ for all $x \in X$, any translation of μ by a linear combination of δ-functions is singular with μ. However it has been shown by Hoegh-Krohn that for X being an Euclidean disk the measures $\mu * \nu$ and μ are equivalent[4] and $\dfrac{d(\mu * \nu)}{d\mu} \in L_\mu^2$. We give here a sketch of a proof.

Let $\{u_k\}_{k=1}^\infty$ be the orthonormal basis in $\mathring{W}_2^1(X)$ of eigenfunctions of Dirichlet problem for the Laplace operator which are ordered accordingly non-decrease of the eigenvalues, μ_n, ν_n be the projections of μ and ν on the finite-dimensional space generated by $u_1, \ldots, u_n$.

We put $p = \dfrac{d(\mu * \nu)}{d\mu}$, $p_n = \dfrac{d(\mu_n * \nu_n)}{d\mu_n}$ and calculate the density p. It is known (see [6], the theorem about convergence of martingales) that p does exist almost everywhere with respect to μ and $p = \lim p_n$. Generally speaking $\int p d\mu \leq 1$ and $\int p d\mu = 1$ if and only if $\mu * \nu$ and μ are equivalent.

[4] A more simple example of the appearance is given in §3 where ν is a Gaussian measure.

To prove the equivalence of μ and $\mu * \nu$ it is sufficient to prove that p_n converges in L^2_μ. Indeed if $p_n \to \tilde{p}$ in L^2_μ then $\tilde{p} = p$ and therefore $\int p d\mu = \int \tilde{p} d\mu = \lim_n \int p_n d\mu = 1$ (since $\int p_n d\mu = 1$ for all n). Let us calculate $\|p_n - p_m\|_{L^2_\mu}$.

Recall that if μ_n is the standard Gaussian measure in $\mathbb{R}^n$ and $\mu_n(\cdot + u)$ is its translation by a vector u, then $\dfrac{d\mu_n(\cdot + u)}{d\mu_n}(y) = e^{-\langle y,u\rangle - (1/2)\|u\|^2_n}$. Therefore, since $\delta_x \sim \Sigma_k u_k(x)$ we get

$$p_n(y) = \frac{d(\mu_n * \nu_n)}{d\mu_n}(y) = \int_X \exp\Big(-\sum_1^n y_k u_k(x) - \tfrac{1}{2}\sum_1^n u_k^2(x)\Big)dx.$$

It implies

$$\langle p_n, p_m\rangle = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m}\int_X\int_X \exp\Big(-\sum_1^n y_k u_k(x) - \tfrac{1}{2}\sum_1^n u_k^2(x) - \sum_1^m y_k u_k(x') - \tfrac{1}{2}\sum_1^m u_k^2(x') - \tfrac{1}{2}\sum_1^m y_k^2\Big)dx\,dx'\,dy. \quad (m \ge n).$$

Integrating by y we get

$$\langle p_n, p_m\rangle = \int_X\int_X \exp\Big(\sum_1^n u_k(x)u_k(x')\Big)dx\,dx' \quad (m \ge n).$$

In particular $\langle p_n, p_m\rangle = \langle p_n, p_n\rangle$ for $m \ge n$. Hence

$$\begin{aligned}\|p_n - p_m\|^2 &= \langle p_m, p_m\rangle - \langle p_n, p_m\rangle \\ &= \int_X\int_X \exp\Big(\sum_1^n u_k(x)u_k(x')\Big) \\ &\quad\times\Big[\exp\Big(\sum_{n+1}^m u_k(x)u_k(x')\Big) - 1\Big]dx\,dx' \quad (m \ge n).\end{aligned}$$

To prove that p_n converges in L^2_μ, i.e. $\|p_n - p_m\| \to 0$ for $m, n \to \infty$, it is sufficient to verify that $\lim_n \int_X \int_X \exp(\Sigma_1^n u_k(x)u_k(x'))dx\,dx'$ does exist. For that it is sufficient now to verify that

$$\int_X\int_X \exp G(x,x')dx\,dx' < \infty,$$

where $G(x,x') = \lim_n \Sigma_1^n u_k(x)u_k(x')$ is the Green function for the Laplace operator. It is known that when X is an Euclidean disk the

Green function has a logarithmic singularity on the diagonal:

$$G(x, x') \sim \frac{1}{2\pi} \ln \frac{1}{\rho(x, x')},$$

ρ being a Euclidean metric on the plane (see for example [8]). Consequently

$$\int_X \int_X \exp G(x, x')dxdx' \sim \int_{|x|\leq 1} \int_{|x'|\leq 1} [\rho(x, x')]^{-1/2\pi} dxdx',$$

and it is clear that the last integral does converge.

2. For $\dim X = 3$ equivalence of μ and $\mu * \nu$ can not be proved by the same means as for $\dim X = 2$ since $\int_X \int_X \exp G(x, x')dxdx'$ diverges. On the other hand nonequivalence of those can not be deduced from lemma 6 since for $\dim X = 3$ $\Sigma_k c_k u_k^2(x)$ converges almost everywhere for any $\{c_k\} \in l^2$.

REMARK: In [3] a projective representation of the group $G^X \cdot \Omega(X; \mathfrak{g})$ (Sugawara's group) was constructed. This representation is irreducible for $\dim X \geq 4$ since its restriction to G^X is irreducible due to theorem 1. Yet this representation is irreducible for a manifold X of an arbitrary dimension including the case $\dim X = 1$. This statement easily follows from an analysis of the decomposition of the representation with respect to the abelian normal subgroup $\Omega(X; \mathfrak{g})$ and the lemmas about the spectrum of the representation of $\mathfrak{a}^X \subset G^X$.

Appendix. The Lie algebra $\mathfrak{g}^X$ and its representation

Let as above X be a Riemannian manifold, $\mathfrak{g}$ be a Lie algebra of a compact semisimple Lie group G. We consider the space $C^\infty(X; \mathfrak{g})$ of all compactly supported C^∞-mapping $X \to \mathfrak{g}$ provided with the usual topology. Let us define in $C^\infty(X; \mathfrak{g})$ the Lie algebra structure by $[a_1, a_2](x) = [a_1(x), a_2(x)]$. This Lie algebra is called the Lie algebra of G^X and denoted by $\mathfrak{g}^X$.

Using the exponential mapping

$$\exp : \mathfrak{g}^X \to G^X$$

we may assign to any unitary representation U of G^X the representation L of its Lie algebra in the same space, defined by

$$L(a)\xi = \lim_{t\to 0}\left(\frac{1}{t}(U(\exp ta) - E)\xi\right).$$

(The operators $L(a)$ are defined on the set of such vectors in the space of U that this limit does exist.)

Let $U = \mathrm{EXP}_\beta\, V$ be the representation of G^X which is considered in the paper. Let us give the formulas for the operators L of the corresponding representation of $\mathfrak{g}^X$.

Let U be realized in the Fock space

$$\mathrm{EXP}\, H = \bigoplus_{n=0}^{\infty} S^n H_{\mathrm{C}},$$

where $H = \overline{\Omega(X;\mathfrak{g})}$, H_{C} is the complexification of H, $S^n H_{\mathrm{C}}$ is the symmetrized tensor product of n copies of H_{C}. The formulas for operators of U in this realization is given in the Introduction. From the formulas one can easily get an expression for operators of the representation of $\mathfrak{g}^X$.

Let us put

$$s^n(\omega_1 \otimes \cdots \otimes \omega_n) = \frac{1}{n!}\sum \omega_{i_1} \otimes \cdots \otimes \omega_{i_n},$$

where an ordered set $(i_1, \ldots, i_n)$ in the sum runs through the family of all permutations of $\{1, \ldots, n\}$. If $\omega_i \in H_{\mathrm{C}}$ then $s^n(\omega_1 \otimes \cdots \otimes \omega_n) \in S^n H_{\mathrm{C}}$.

PROPOSITION: *The operators $L(a)$ of the representation of $\mathfrak{g}^X$ induced by U are defined on all vectors $s^n(\omega_1 \otimes \cdots \otimes \omega_n)$, $\omega_i \in H_{\mathrm{C}}$ and are given by the formula*

$$\begin{aligned} &L(a)(s^n(\omega_1 \otimes \cdots \otimes \omega_n)) \\ &\quad = -\sqrt{n}\sum_{i=1}^{n} \langle \omega_i, da\rangle s^{n-1}(\omega_1 \otimes \cdots \otimes \omega_{i-1} \otimes \omega_{i+1} \otimes \cdots \otimes \omega_n) \\ &\qquad + n \sum_{i=1}^{n} s^n(\omega_1 \otimes \cdots \otimes \omega_{i-1} \otimes [a, \omega_i] \otimes \omega_{i+1} \otimes \cdots \otimes \omega_n) \\ &\qquad + \sqrt{n+1}\sum_{i=1}^{n+1} s^{n+1}(da \otimes \omega_1 \otimes \cdots \otimes \omega_n). \end{aligned} \tag{7}$$

In particular

$$L(a)(\underbrace{\omega \otimes \cdots \otimes \omega}_{n}) = -\sqrt{n}\langle \omega, da\rangle \underbrace{\omega \otimes \cdots \otimes \omega}_{n-1}$$
$$+\sum_{i=1}^{n} \underbrace{\omega \otimes \cdots \otimes \omega}_{i-1} \otimes [a, \omega] \otimes \underbrace{\omega \otimes \cdots \otimes \omega}_{n-i}$$
$$+\frac{1}{\sqrt{n+1}} \sum_{i=0}^{n} \underbrace{\omega \otimes \cdots \otimes \omega}_{i} \otimes da \otimes \underbrace{\omega \otimes \cdots \otimes \omega}_{n-i}.$$

REMARK: The formulas (7) give not only the representation of $\mathfrak{g}^X$ but also the representation (which is not Hermitian now) of its complexification $\mathfrak{g}_{\mathbb{C}}^X$ and therefore $\mathfrak{g}_\tau^X$ where $\mathfrak{g}_\tau$ is an arbitrary semisimple Lie algebra.

Added in proof

At the last time S. Albeverio, R. Hoegh-Krohn and D. Testard proved irreducibility of $U(\tilde{g})$ in the case dim $X \geq 3$ and reducibility in the case dim $X = 1$ (preprint 1980, Bochum).

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