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p-ADIC L-FUNCTIONS FOR ELLIPTIC CURVES WITH COMPLEX MULTIPLICATION I

Pierrette Cassou-Noguès*

1. Introduction

Let K be an imaginary quadratic field with class number one, lying inside the complex field $\mathbb{C}$, and $\mathcal{O}$ the ring of integers of K. Let E be an elliptic curve defined over K, whose ring of endomorphisms is isomorphic to $\mathcal{O}$. Since K has class number 1, we can choose a Weierstraß model for E

$$(1) \qquad y^2 = 4x^3 - g_2x - g_3$$

where g_2 and g_3 belong to $\mathcal{O}$, and where the discriminant of (1) is divisible only by the primes of K where E has a bad reduction, and possibly by the primes of K above 2 and 3. Let $\mathfrak{p}(z)$ be the associated Weierstraß function and L its period lattice. As K has class number one, we can choose $\Omega \in L$ such that $L = \Omega\mathcal{O}$. We fix, once and for all, an algebraic closure $\bar{K}$ of K, which we suppose lies inside the complex field $\mathbb{C}$.

Let S be the set of rational primes consisting of 2, 3, and all $\mathfrak{q}$ such that E has a bad reduction at at least one prime of K above $\mathfrak{q}$. For the rest of the paper, we shall assume that p is a rational prime, not in S, which splits in K, say $(p) = \mathfrak{p}\bar{\mathfrak{p}}$. We write $K_\mathfrak{p}$ for the completion of K at $\mathfrak{p}$, $\mathcal{O}_\mathfrak{p}$ the ring of integers of $K_\mathfrak{p}$, and $\mathbb{C}_\mathfrak{p}$ for the completion of an algebraic closure

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of $K_{\mathfrak{p}}$. We assume that we are given a fixed prime $\mathfrak{P}$ of $\bar{K}$ lying above $\mathfrak{p}$, or, what amounts to the same thing, an embedding τ of $\bar{K}$ into $\mathbb{C}_{\mathfrak{p}}$.

The aim of the present paper is to prove the existence of $\mathfrak{P}$-adic L-functions attached to E and certain abelian extensions of K, and to give several arithmetic applications of these. Functions of this type have already been constructed by Katz [9], [10], Manin–Vishik [15], and Lichtenbaum [12]. In fact, much of our construction has been based on an earlier version of Lichtenbaum's paper [12], and we wish to make quite clear our indebtedness to his work. We do, however, go further than [12] both in defining $\mathfrak{P}$-adic L-functions for a wider class of abelian extensions of K, and in the arithmetic applications we give. Also, we shall treat the case in which the class number of K is greater than 1 by similar methods in a later paper. The present paper should be viewed as an introduction to our later work.

Finally, I wish to thank J. Coates for helpful suggestions on this work.

1. Results used from elsewhere

In this section we summarize, without proofs, a number of results from related papers, which will be used in our construction of the $\mathfrak{P}$-adic L-functions. We use the notation in the introduction.

Let $\hat{E}$ be the formal group giving the kernel of reduction modulo $\mathfrak{p}$ on the curve E; for a detailed discussion of this, see [19], p. 42. A local parameter for $\hat{E}$ is given by $t = -2x/y$, where x and y are the coordinates of the model (1) of E. Since p splits in K, it is easy to see that $\hat{E}$ has height one. Let T be the completion of the maximal unramified extension of $K_{\mathfrak{p}}$, and $\mathcal{O}_T$ the ring of integers of T. It is shown in [13] that every formal group of height 1 defined over $\mathcal{O}_T$ is isomorphic over $\mathcal{O}_T$ to the formal multiplicative group G_m. From this fact, it is easy to deduce the following lemma. Let z be given by $t = -2\wp(z)/\wp'(z)$. Thus we can view z as the parameter of the formal additive group G_a.

LEMMA 1: *There exists* $g(X) \in \mathcal{O}_T[[X]]$, *and* $\gamma \in \mathcal{O}_T^\times$, *such that* $t = g(e^{\gamma z} - 1)$.

Here $\mathcal{O}_T[[X]]$ denotes the ring of formal power series in X with coefficients in $\mathcal{O}_T$.

If $\mathcal{L}$ is any lattice in the complex plane, we define, as usual

$$\sigma(z, \mathscr{L}) = z \prod_{\substack{\omega \in \mathscr{L} \\ \omega \neq 0}} \left(1 - \frac{z}{\omega}\right) e^{(z/\omega)+(1/2)(z/\omega)^2}$$

and put

$$\theta(z, \mathscr{L}) = \Delta(\mathscr{L}) e^{-6 s_2(\mathscr{L}) z^2} \sigma(z, \mathscr{L})^{12}$$

where $\Delta(\mathscr{L})$ is the discriminant function of $\mathscr{L}$, and

$$s_2(\mathscr{L}) = \lim_{\substack{s \to 0 \\ s > 0}} \sum_{\substack{\omega \in \mathscr{L} \\ \omega \neq 0}} \omega^{-2} |\omega|^{-2s}.$$

If $\mathfrak{a}$ is any integral ideal of K, we define

(2) $$\Theta(z, \mathfrak{a}) = \theta(z, L)^{N\mathfrak{a}} / \theta(z, \mathfrak{a}^{-1} L)$$

where $N\mathfrak{a}$ is the absolute norm of $\mathfrak{a}$. In fact, as is shown in Robert [16], $\Theta(z, \mathfrak{a})$ is an elliptic function for the lattice L.

Assume now that H is an arbitrary finite abelian extension of K. Let ψ be the Grössencharacter of E over K. We define $\mathfrak{h}$ to be the least common multiple of the conductor of ψ and the conductor of H/K. Let h be a generator of the ideal $\mathfrak{h}$ and define $\rho = \Omega/h$. Let $E_{\mathfrak{h}}$ be the group of $\mathfrak{h}$-division-points on E. By Lemma 2 of [1], $K(E_{\mathfrak{h}})$ is the ray class field of K modulo $\mathfrak{h}$. We now choose and fix a set B of integral ideals of K, which are prime to $\mathfrak{h}$, and which are such that $\{(\mathfrak{b}, K(E_{\mathfrak{h}})/K); \mathfrak{b} \in B\}$ is precisely the Galois group of $K(E_{\mathfrak{h}})/H$; here $(\mathfrak{b}, K(E_{\mathfrak{h}})/K)$ denotes the Artin symbol of $\mathfrak{b}$ for $K(E_{\mathfrak{h}})/K$. If $\mathfrak{a}$ is an integral ideal of K, we define

$$\Lambda(z, \mathfrak{a}) = \prod_{\mathfrak{b} \in B} \Theta(z + \psi(\mathfrak{b})\rho, \mathfrak{a})$$

It is shown in [1] (cf. Lemma 7) that $\Lambda(z, \mathfrak{a})$ is a rational function of $\wp(z)$ and $\wp'(z)$ with coefficients in H. If σ is an element of the Galois group of H over K, we write $\Lambda_\sigma(z, \mathfrak{a})$ for the rational function of $\wp(z)$ and $\wp'(z)$, which is obtained by letting σ act on the coefficients of $\Lambda(z, \mathfrak{a})$.

If $\mathfrak{c}$ is an integral ideal of K, prime to the conductor of H/K, we write $\sigma_{\mathfrak{c}}$ for the Artin symbol $(\mathfrak{c}, H/K)$. Let k be an integer ≥ 1. We introduce the partial Hecke L-function, for each σ in the Galois group of H over K,

$$\zeta_H(\sigma, k; s) = \sum_{\substack{(\mathfrak{a}, \mathfrak{h})=1 \\ \sigma_\mathfrak{a}=\sigma}} \frac{\bar{\psi}^k(\mathfrak{a})}{(N\mathfrak{a})^s},$$

where the summation is over all integral ideals $\mathfrak{a}$ of K, prime to $\mathfrak{h}$, such that $\sigma_\mathfrak{a} = \sigma$. It can be shown that $\zeta_H(\sigma, k; s)$ can be analytically continued over the whole complex plane, and we write $\zeta_H(\sigma, k)$ for its value at $s = k$. The following lemma is proven in [1]:

LEMMA 2: *For each $\sigma \in G(H/K)$, we have*

$$z \frac{d}{dz} \log \Lambda_\sigma(z, \mathfrak{a}) = \sum_{k=1}^{\infty} c_k(\mathfrak{a}, \sigma) z^k$$

where, for $k \geq 1$

$$c_k(\mathfrak{a}, \sigma) = 12(-1)^{k-1}\rho^{-k}(N\mathfrak{a}\,\zeta_H(\sigma, k) - \psi^k(\mathfrak{a})\zeta_H(\sigma\sigma_\mathfrak{a}, k)).$$

Here $\mathfrak{a}$ is any integral ideal of K, prime to $\mathfrak{h}$.

Finally we recall some basic facts about Leopoldt's Γ-transform (see [12]). Let M be any complete subfield of $\mathbb{C}_\mathfrak{p}$. Let Q_M be the set of power series $\Sigma_{n=0}^{\infty} a_n x^n$ in $M[[x]]$ such that $\lim_{n\to\infty} |a_n n| = 0$, where $|\ \ |$ denotes the valuation of $\mathbb{C}_\mathfrak{p}$. Let C_M be the set of continuous functions from $\mathbb{Z}_p$ to M. Then both Q_M and C_M are Banach algebras with the norms $\sup_n |n!a_n|$ and $\max_{z\in \mathbb{Z}_p} |f(z)|$, respectively. Let α be a residue class mod $(p-1)$. Following Leopoldt [11], Lichtenbaum has shown in [12] that one can define the Γ^α-transform. For the precise definition, see [12]. We simply note that Γ^α is a bounded linear map from Q_M to C_M. The following is a key lemma about Γ^α.

LEMMA 3: *Given $A(X) \in Q_M$, define*

$$\tilde{A}(X) = A(X) - \frac{1}{p} \sum_{\zeta} A(\zeta(X+1) - 1)$$

where ζ ranges over all p-th roots of unity. If k is an integer ≥ 0 with $k \equiv \alpha$ mod $p-1$, then

$$\Gamma^\alpha(A)(k) = \frac{d^k}{dz^k} \tilde{A}(e^z - 1)\Big|_{z=0}.$$

Let $\mathcal{O}_M$ be the ring of integers of M. Given a power series $f(X) \in \mathcal{O}_M[[X]]$, we can obtain a function $f^* \in C_M$ by $f^*(s) = f((1+p)^s - 1)$. We call f^* an *Iwasawa function* in C_M. Another basic result about Γ^α is the following (see [12]). If $A(X) \in \mathcal{O}_M[[X]]$, then $\Gamma^\alpha(A)(s)$ is an Iwasawa function.

II. $\mathfrak{p}$-adic L-functions

As before, let M denote a complete subfield of $\mathbb{C}_\mathfrak{p}$, and $\mathcal{O}_M$ the ring of integers of M. We suppose, for simplicity, that M contains the field T, which is the completion of the maximal unramified extension of $K_\mathfrak{p}$. By Lemma 1, there exists a power series $g(X) \in \mathcal{O}_T[[X]]$, and $\gamma \in \mathcal{O}_T^\times$, such that $t = g(e^{\gamma z} - 1)$. In fact, $g(X)$ defines an isomorphism from G_m to $\hat{E}$. Let $\hat{E}_\pi$, where $\pi = \psi(\mathfrak{p})$ be the kernel of the endomorphism $[\pi]$ of $\hat{E}$. Given $A(t) \in \mathcal{O}_M[[t]]$, we define, as before,

$$\tilde{A}(t) = A(t) - \frac{1}{p}\sum_{\zeta} A(\zeta(t+1) - 1),$$

where ζ runs over all p-th roots of unity in $\mathbb{C}_\mathfrak{p}$.

LEMMA 4: *Let $B(t) \in \mathcal{O}_M[[t]]$, and define $A(X) = B(g(X))$. Then, for each integer $k \geq 0$, we have*

$$\left(\frac{d}{dz}\right)^k \tilde{A}(e^z - 1)\,\Big|_{z=0} = \gamma^{-k}\left(\frac{d}{dz}\right)^k \left\{(B(t) - \frac{1}{p}\sum_{\eta \in \hat{E}_\pi} B(t * \eta))\right\}_{t=0}$$

here $t * \eta$ denotes the sum of t and η on $\hat{E}$.

PROOF: Since $t = g(e^{\gamma z} - 1)$ and $\eta = g(\zeta - 1)$, it follows from the fact that g is an isomorphism from G_m to $\hat{E}$ that $t * \eta = g(\zeta e^{\gamma z} - 1)$ (note that $\zeta e^{\gamma z} - 1$ is the product of $\zeta - 1$ and $e^{\gamma z} - 1$ on G_m). Hence

$$\begin{aligned}\left(\frac{d}{dz}\right)^k B(g(\zeta e^z - 1)) &= \gamma^{-k}\left(\frac{d}{dz}\right)^k B(g(\zeta e^{\gamma z} - 1)) \\ &= \gamma^{-k}\left(\frac{d}{dz}\right)^k B(t * \eta).\end{aligned}$$

It is clear that η ranges over $\hat{E}_\pi$ as ζ runs over the p-th roots of unity. Then the assertion of the lemma is clear.

As in §.1, let H be an arbitrary finite abelian extension of K and write $G = G(H/K)$. We assume now that p is prime to 2, 3 and $\mathfrak{f}$, where $\mathfrak{f}$ is the least common multiple of the conductor of H/K and the conductor of ψ. Let $\mathfrak{a}$ be an integral ideal of K, which is prime to $\mathfrak{f}$, and let $\Lambda(z, \mathfrak{a})$ be as defined in §.1. The prime $\bar{\mathfrak{p}}$ of $\bar{K}$ determines a prime $\mathfrak{P}$ of H lying above $\mathfrak{p}$.

LEMMA 5: *Let $\sigma \in G$. In terms of the parameter $t = -2\mathfrak{p}(z)/\mathfrak{p}'(z)$, the function*

$$\frac{d}{dz} \log \Lambda_\sigma(z, \mathfrak{a})$$

has an expansion whose coefficients all belong to $\mathcal{O}_\mathfrak{P}$, the ring of integers of the completion of H at $\mathfrak{P}$.

PROOF: By Lemma 11 of [1], $\Lambda_\sigma(z, \mathfrak{a})$ has a power series expansion $\Sigma_{k=0}^{\infty} h_k(\mathfrak{a}, \sigma)t^k$, where the $h_k(\mathfrak{a}, \sigma)$ belong to $\mathcal{O}_\mathfrak{P}$, and $h_0(\mathfrak{a}, \sigma)$ is a unit in $\mathcal{O}_\mathfrak{P}$. It follows that the logarithmic derivative, with respect to t, of this power series also belongs to $\mathcal{O}_\mathfrak{P}[[t]]$. Now we can write $z = \lambda(t)$, where λ is the logarithm map of $\hat{E}$. It is well known that $\lambda'(t)$ is a power series with coefficients in $\mathbf{Z}_p$ and leading coefficient 1. Thus $1/(\lambda'(t))$ also belongs to $\mathbf{Z}_p[[t]]$, and the assertion of Lemma 5 follows by the chain rule for differentiation.

LEMMA 6: *Let n be an integer ≥ 0. There exists $c \in \mathbf{C}$ such that*

$$\prod_q \Theta(z + q, \mathfrak{a}) = c\Theta(\pi^n z, \mathfrak{a}), \tag{3}$$

where the product on the left is taken over a set of representatives modulo L of the π^n-division points of L.

PROOF: Both sides of (3) are elliptic functions for the lattice L, and so it suffices to verify that the two sides have the same zeros and poles. The zeros of $\Theta(z, \mathfrak{a})$ occur precisely at the elements of L each with the multiplicity $12(N\mathfrak{a} - 1)$. Similarly, the poles of $\Theta(z, \mathfrak{a})$ are each of order 12, and occur precisely at the elements of $\mathfrak{a}^{-1}L$ which are not in L. Using these remarks, one immediately concludes that the right and left sides of (3) have the same zeros and poles, as required.

LEMMA 7: *Let $\sigma \in G$, and let n be an integer ≥ 0. There exists $C \in \mathbf{C}$ such that*

(4) $$\prod_q \Lambda_\sigma(z+q, \mathfrak{a}) = C\Lambda_{\sigma\sigma_\mathfrak{p}^n}(\pi^n z, \mathfrak{a})$$

where the product on the left is taken over a set of representatives modulo L of the π^n*-division points of L.*

PROOF: Let $\sigma = \sigma_\mathfrak{c}$, where $\mathfrak{c}$ is an integral ideal of K prime to $\mathfrak{f}$. Then it is shown in the proof of Lemma 8 of [1] that

$$\Lambda_\sigma(z, \mathfrak{a}) = \prod_{\mathfrak{b}\in B} \Theta(z + \psi(\mathfrak{b}\mathfrak{c})\rho, \mathfrak{a}).$$

On the other hand, recalling that $\pi = \psi(\mathfrak{p})$, it follows from (3) that

$$\prod_q \Theta(z + q + \psi(\mathfrak{b}\mathfrak{c})\rho, \mathfrak{a}) = \Theta(\pi^n z + \psi(\mathfrak{b}\mathfrak{p}^n\mathfrak{c})\rho, \mathfrak{a}).$$

Taking the product of both sides of this equation over the $\mathfrak{b} \in B$, and using (5) with $\mathfrak{c}$ replaced by $\mathfrak{c}\mathfrak{p}^n$, the assertion of Lemma 7 follows.

We now apply Lemma 4 with $B_\sigma(t)$ given by the expansion in t of $\frac{d}{dz}\log \Lambda_\sigma(z, \mathfrak{a})$. By Lemma 5, this expansion does, in fact, belong to $\mathcal{O}_T[[t]]$. Taking the logarithm derivative with respect to z of both sides of (4), we conclude that

$$\left(\frac{d}{dz}\right)^k \left(\sum_{\eta\in\hat{E}_\pi} B_\sigma(t*\eta)\right)_{z=0} = \left(\frac{d}{dz}\right)^{k+1} \log \Lambda_{\sigma\sigma_\mathfrak{p}}(\pi z, \mathfrak{a})\Big|_{z=0}$$

Hence, if $A_\sigma(X) = B_\sigma(g(X))$, Lemma 4 implies that

$$\left(\frac{d}{dz}\right)^k \tilde{A}_\sigma(e^z-1)_{z=0} = \gamma^{-k}\left(\frac{d}{dz}\right)^{k+1}\left\{(\log \Lambda_\sigma(z, \mathfrak{a}) - \frac{1}{p}\log \Lambda_{\sigma\sigma_\mathfrak{p}}(\pi z, \mathfrak{a}))\right\}_{z=o}.$$

Thus, in view of Lemma 2 and 3, we have established the following result. Write $\lambda_k = 12(-1)^{k-1}\rho^{-k}(k-1)!$. Let α fixed be a residue class mod $(p-1)$. We define

(6) $$\zeta_{H,\mathfrak{p}}(\sigma, k) = \zeta_H(\sigma, k) - \frac{\psi^k(\mathfrak{p})}{N\mathfrak{p}}\zeta_H(\sigma\sigma_\mathfrak{p}, k).$$

THEOREM 8: *Let* $B_\sigma(t) = B(t, \sigma, \mathfrak{a})$ *be the expansion in* t *of* $\frac{d}{dz}\log \Lambda_\sigma(z, \mathfrak{a})$. *Put* $A_\sigma(t) = B_\sigma(g(t))$. *Then for all integers* $k \geq 0$ *with* $k \equiv \alpha$ *mod* $(p-1)$, *we have*

$$\Gamma^{\alpha}(A_\sigma)(k) = \gamma^{-k}\lambda_{k+1}(N\mathfrak{a}\zeta_{H,\mathfrak{p}}(\sigma, k+1) - \psi^{k+1}(\mathfrak{a})\zeta_{H,\mathfrak{p}}(\sigma\sigma_{\mathfrak{a}}, k+1)).$$

We now use Theorem 8 to construct $\mathfrak{P}$-adic L-functions. Suppose χ is a homomorphism of G into $\bar{K}$. Replacing H by the fixed field of the kernel of χ if necessary, we can assume that the kernel of χ is trivial.

Let us denote also by χ the homomorphism of G into $\mathbb{C}_{\mathfrak{p}}^{\times}$ given by $\tau \circ \chi$. For each integer $k \geq 1$, we define the number $\Omega^{-k}L(\bar{\psi}^{-k}\chi^{-1}, k)$ in $\mathbb{C}_{\mathfrak{p}}$ by

$$\Omega^{-k}L(\bar{\psi}^{k}\chi^{-1}, k) = \sum_{\sigma \in G} \chi^{-1}(\sigma)\zeta_H(\sigma, k)\Omega^{-k} \tag{7}$$

Let $\mathcal{O}_{T,\chi}$ be the ring of integers of the field obtained by adjoining the values of χ to T, and write $\Lambda_\chi = \mathcal{O}_{T,\chi}[[X]]$.

Now take $\mathfrak{a}$ an integral ideal in K, prime to $\mathfrak{f}$ and $\mathfrak{p}$, and let $A_\sigma(t) = A_\sigma(t, \mathfrak{a})$ be the power series in t, which is defined in Theorem 8. Let α be an arbitrary residue class modulo $(p-1)$. It follows from Lemma 5 that there is a power series $r_\alpha(X; \chi, \mathfrak{a})$ in Λ_χ such that

$$r_\alpha((1+p)^s - 1; \chi, \mathfrak{a}) = \sum_{\sigma \in G} \chi^{-1}(\sigma)\Gamma^{\alpha-1}(A_\sigma)(-s) \tag{8}$$

for all s in $\mathbb{Z}_p$.

Lemma 9: *For all integers $k \geq 0$, with $k \equiv \alpha - 1$ mod $(p-1)$, we have*

$$r_\alpha((1+p)^{-k} - 1; \chi, \mathfrak{a}) = \gamma^{-k}\lambda_{k+1}(N\mathfrak{a} - \psi^{k+1}(\mathfrak{a})\chi(\mathfrak{a})) \times \left(1 - \frac{\chi(\mathfrak{p})\psi^{k+1}(\mathfrak{p})}{N\mathfrak{p}}\right) L(\chi^{-1}\psi^{-k+1}, k+1).$$

Proof: This is immediate from Theorem 8 and the definitions (7) and (8).

If x is a unit in $K_{\mathfrak{p}}$, we write as usual $x = \omega(x)\langle x \rangle$, where $\omega(x)$ is a $(p-1)$-th root of unity, and $\langle x \rangle \equiv 1$ mod $\mathfrak{p}$. Since $\psi(\mathfrak{a})$ generates the ideal $\mathfrak{a}$, and $\mathfrak{a}$ is prime to $\mathfrak{p}$ by hypothesis, the number $\psi(\mathfrak{a})$ is a unit in $K_{\mathfrak{p}}$ when viewed under the canonical inclusion of K in $K_{\mathfrak{p}}$. Define $\beta(\mathfrak{a})$ in Z_p by the equation

$$\langle \psi(\mathfrak{a}) \rangle = (1+p)^{\beta(\mathfrak{a})}$$

and $a_\alpha(X; \chi, \mathfrak{a})$ by

(9) $$a_\alpha(X;\chi,\mathfrak{a}) = N\mathfrak{a} - \psi(\mathfrak{a})\chi(\mathfrak{a})\omega(\psi(\mathfrak{a}))^{\alpha-1}(1+X)^{-\beta(\mathfrak{a})}.$$

It is clear that for all integers $k \geq 0$ with $k \equiv \alpha - 1 \bmod (p-1)$, we have

$$a_\alpha((1+p)^{-k}-1;\chi,\mathfrak{a}) = N\mathfrak{a} - \psi^{k+1}(\mathfrak{a})\chi(\mathfrak{a}).$$

Since $\mathfrak{a} \neq 1$ and $\psi(\mathfrak{a})$ generates $\mathfrak{a}$, it is easy to see that $a_\alpha(X;\chi,\mathfrak{a})$ is not identically zero.

Define

(10) $$f_\alpha(X;\chi,\mathfrak{a}) = \frac{r_\alpha(X;\chi,\mathfrak{a})}{a_\alpha(X;\chi,\mathfrak{a})}.$$

For $\lambda \in K$, let $S(\lambda)$ denote the trace, from K to $\mathbb{Q}$, of α. Let $\mathcal{D}$ be the different of K and d its discriminant. Let $\mathfrak{h}_0$ be the conductor of χ and $\mathfrak{h}_0^{-1}\mathcal{D}^{-1} = (\delta_0)$. We choose once for all δ_0 so that $\delta_0\sqrt{d}$ has exact denominator $\mathfrak{h}_0$. Put, [18], when χ is a proper character

$$T(\bar{\chi}) = \sum_{\lambda \bmod \mathfrak{h}_0} \bar{\chi}(\lambda)e^{2\pi i S(\lambda\delta_0)}$$

where λ runs through a full system of representatives of residue classes mod $\mathfrak{h}_0$. $T(\bar{\chi})$ is different from zero.

Let $w_{\mathfrak{h}}$ be the number of roots of unity in K congruent to 1 mod $\mathfrak{h}$.

Let θ be the canonical character giving the action of $G(H(E_{\mathfrak{p}})/\mathrm{K})$ on the group $E_{\mathfrak{p}}$ of $\mathfrak{p}$-division points on E. We define the $\mathfrak{p}$-adic L-functions $L_{\mathfrak{p}}(\chi\theta^\alpha, s)$ by

(11) $$L_{\mathfrak{p}}(\chi\theta^\alpha, s) = \frac{1}{T(\bar{\chi})\sqrt{d}\,w_{\mathfrak{g}}} f_\alpha((1+p)^s-1;\chi,\mathfrak{a}).$$

(Here $\mathfrak{g}$ is the least common multiple of the conductor of $\chi\theta^\alpha$ and $\mathfrak{f}$.) Now if $H = K$, $\chi = \chi_0$ is the trivial character with conductor (1). We take $T(\chi_0) = 1$ we consider as before $r_\alpha(X;\chi_0,\mathfrak{a})$, $a_\alpha(X;\chi_0,\mathfrak{a})$, $f_\alpha(X;\chi_0,\mathfrak{a})$ and we define

(12) $$L_{\mathfrak{p}}(\theta^\alpha, s) = \frac{1}{\sqrt{d}\,w_{\mathfrak{p}\mathfrak{f}}} f_\alpha((1+p)^s-1;\chi_0,\mathfrak{a}).$$

THEOREM 10: *For all integers $k \geq 0$, $k \equiv \alpha - 1 \bmod (p-1)$ we have*

$$(13)\quad L_\mathfrak{p}(\chi\theta^\alpha, -k) = \frac{\gamma^{-k}\lambda_{k+1}}{T(\bar{\chi})w_\mathfrak{q}\sqrt{d}}\left(1 - \frac{\chi(\mathfrak{p})\psi^{k+1}(\mathfrak{p})}{N\mathfrak{p}}\right) L(\bar{\chi}^1\bar{\psi}^{k+1}, k+1).$$

and

$$(14)\qquad L_\mathfrak{p}(\theta^\alpha, -k) = \frac{\gamma^{-k}\lambda_{k+1}}{w_\mathfrak{q}\sqrt{d}}\left(1 - \frac{\psi^{k+1}(\mathfrak{p})}{N\mathfrak{p}}\right) L(\bar{\psi}^{k+1}, k+1).$$

REMARKS:

1) The functions $L_\mathfrak{p}(\theta^\alpha, s)$ have been also constructed in [5].

2) The factor $\left(1 - \frac{\chi(\mathfrak{p})\psi^{k+1}(\mathfrak{p})}{N\mathfrak{p}}\right)$ is the Euler factor of $\mathfrak{p}$ in the Euler product of $L(\chi\psi^{k+1}, 1)$. In fact $L(\chi^{-1}\psi^{k+1}, k+1)$ and $L(\chi\psi^{k+1}, 1)$ are linked by the functional equation of $L(\chi^{-1}\bar{\psi}^{k+1}, s)$ [7].

3) We have chosen this normalisation of $L(\chi\theta^\alpha, s)$ because in §.III, we want to give a formula for $L_\mathfrak{p}(\chi\theta^\alpha, 1)$, which will be an analogue of the classical complex formula for $L(\chi\psi^0, 1)$ (see the above remark), arising from Kronecker's limit formula [18].

4) We can choose an $\mathfrak{a}$ such that $a_\alpha(X;\chi, \mathfrak{a})$ is a unit in Λ_χ. Let e denote a generator of the ideal $12\mathfrak{h} \cap Z$. Choose n to be a rational integer, prime to p, such that $(1 + ne\pi)$ is not divisible by $\bar{\mathfrak{p}}$ and take $\mathfrak{a} = (1 + ne\pi)$. Then $N\mathfrak{a} \not\equiv 1 \bmod p$; also $\sigma_\mathfrak{a} = 1$ because the conductor of H/K divides e, and $\psi^k(\mathfrak{a}) = (1 + ne\pi)^k$. Then $\psi^k(\mathfrak{a}) \equiv 1 \bmod \mathfrak{p}$ because the conductor of ψ divides e. Then $f_\alpha(X;\chi, \mathfrak{a})$ belongs to Λ_χ even when $\chi = \chi_0$ is trivial. Moreover as the right hand side of (13) and (14) is independent of the choice of $\mathfrak{a}$, and $f_\alpha((1+p)^s - 1;\chi, \mathfrak{a})$ *is a continuous function, it follows that* $L_\mathfrak{p}(\chi\theta^\alpha, s)$ and $L_\mathfrak{p}(\theta^\alpha, s)$ are *Iwasawa functions* independent of $\mathfrak{a}$.

III. Leopoldt's formula

Now we will compute the value $L_\mathfrak{p}(\chi\theta^\alpha, 1)$ to get an analogue of Leopoldt's formula and we will see that it is a $\mathfrak{p}$-adic analogue of the complex formula for $L(\chi\psi^0\theta^\alpha, 1)$.

An important role here is played by the elliptic units of Robert [16]. Let $\mathfrak{h}$ be an arbitrary integral ideal of K. We denote by $\mathcal{P}$ a pair $(\mathcal{A}, \mathcal{N})$ where $\mathcal{A} = \{\mathfrak{a}_j, j \in J\}$ and $\mathcal{N} = \{n_j, j \in J\}$; here J is an arbitrary finite index set and $\mathfrak{a}_j$ are integral ideals of K, prime to S and $(p)\mathfrak{h}$, and the n_j are rational integers satisfying $\Sigma_{j\in J} n_j(N\mathfrak{a}_j - 1) = 0$. Given such a pair $\mathcal{P}$, we put

$$\Theta(z, \mathcal{P}) = \prod_{j\in J} \Theta(z, \mathfrak{a}_j)^{n_j}$$

where $\Theta(z, \mathfrak{a}_j)$ is defined in the first part. Let ρ be a $\mathfrak{h}$-division point on E. Then Robert has shown that $\Theta(\rho, \mathcal{P})$ is a unit in $K(E_{\mathfrak{h}})$.

1) *Leopoldt's formula*

Recall that we have defined

$$L_{\mathfrak{p}}(\chi\theta^{\alpha}, s) = \frac{1}{T(\bar{\chi})w_{\mathfrak{q}}\sqrt{d}} f_{\alpha}((1+p)^s - 1; \chi, \mathfrak{a})$$

and

$$L_{\mathfrak{p}}(\theta^{\alpha}, s) = \frac{1}{w_{\mathfrak{q}}\sqrt{d}} f_{\alpha}((1+p)^s - 1; \chi, \mathfrak{a})$$

where $\mathfrak{g}$ is the least common multiple of the conductor of $\chi\theta^{\alpha}$ (resp. θ^{α}) and $\mathfrak{f}$.

This formula is not convenient for studying the value $L_{\mathfrak{p}}(\chi\theta^{\alpha}, 1)$. We will find another one.

Let $\mathcal{P}$ a pair as before (for the ideal $\mathfrak{h}$ least common multiple of the conductor of χ and $\mathfrak{f}$). For each $\sigma \in G(H/K)$, let:

$$\Lambda_{\sigma}(z, \mathcal{P}) = \prod_{j \in J} \Lambda_{\sigma}(z, \mathfrak{a}_j)^{n_j}.$$

In terms of the parameter $t = -2\wp(z)/\wp'(z)$ of $\hat{E}$, $\Lambda_{\sigma}(z, \mathcal{P})$ has an expansion whose coefficients all belong to $\mathcal{O}_{\mathfrak{p}}$. Moreover

$$\Lambda_{\sigma}(0, \mathcal{P}) = N_{K(E_{\mathfrak{h}})/H}\Theta(\rho, \mathcal{P}).$$

Thus $\Lambda_{\sigma}(0, \mathcal{P})$ is a unit in $\mathcal{O}_{\mathrm{P}}$. Hence $\mathrm{Log}\,\dfrac{\Lambda_{\sigma}(z, \mathcal{P})}{\Lambda_{\sigma}(0, \mathcal{P})}$ has an expansion in t, whose coefficients all belong to $H_{\mathfrak{P}}$.

LEMMA 11: *Let $B_{\sigma}(t, \mathcal{P})$ be given by the expansion in t of $\mathrm{Log}\,\dfrac{\Lambda_{\sigma}(z, \mathcal{P})}{\Lambda_{\sigma}(0, \mathcal{P})}$ and $A_{\sigma}(t, \mathcal{P}) = B_{\sigma}(g(t), \mathcal{P})$.*
Then for all integers $k \geq 1$, with $k \equiv \alpha$ mod $(p-1)$,

$$(15) \quad \Gamma^{\alpha}(A_{\sigma}(t, \mathcal{P}))(k) = \gamma^{-k}\lambda_k \sum_{j \in J} n_j(N\mathfrak{a}_j\zeta_{H,\mathfrak{p}}(\sigma, k) - \psi^k(\mathfrak{a}_j)\zeta_{H,\mathfrak{p}}(\sigma\sigma_{\mathfrak{a}_j}, k)).$$

PROOF: Let

$$B_{\sigma}(t, \mathcal{P}) = \sum_{n=0}^{\infty} a_n t^n$$

Define

$$B'_\sigma(t, \mathscr{P}) = \sum_{n=1}^{\infty} n a_n t^n$$

and

$$DB_\sigma(t, \mathscr{P}) = (1+t)\operatorname{Log}(1+t) B'_\sigma(t, \mathscr{P}).$$

It is easy to see that [12], for all $s \in \mathbb{Z}_p$

$$\Gamma^\alpha(DB_\sigma(t, \mathscr{P}))(s) = s\Gamma^\alpha(B_\sigma(t, \mathscr{P}))(s).$$

But

$$DB_\sigma(e^z - 1, \mathscr{P}) = z \frac{d}{dz} B_\sigma(e^z - 1, \mathscr{P}).$$

Thus

$$DB_\sigma(e^z - 1, \mathscr{P}) = z \frac{d}{dz} \operatorname{Log} \Lambda_\sigma(\gamma^{-1} z, \mathscr{P}).$$

As $\mathfrak{a}_j$ has been chosen prime to (p), we define $\beta(\mathfrak{a}_j)$ by

$$\langle \psi(\mathfrak{a}_j) \rangle = (1+p)^{\beta(\mathfrak{a}_j)}$$

and $a_\alpha(X; \chi, \mathscr{P})$ by

$$a_\alpha(X; \chi, \mathscr{P}) = \sum_{j \in J} n_j [N\mathfrak{a}_j - \chi(\mathfrak{a}_j)\omega(\psi(\mathfrak{a}_j)^\alpha)(1+X)^{-\beta(\mathfrak{a}_j)}]. \tag{16}$$

It is clear that for all integers $k \geq 0$, with $k \equiv \alpha \bmod p-1$ we have

$$a_\alpha((1+p)^{-k} - 1; \chi, \mathscr{P}) = \sum_{j \in J} n_j [N\mathfrak{a}_j - \chi(\mathfrak{a}_j)\psi^k(\mathfrak{a}_j)].$$

Using (15) and (16), we can prove the following Lemma.

LEMMA 12: *For all integers $k \geq 1$, $k \equiv \alpha \bmod p-1$ we have*

$$\frac{\sum_{\sigma \in G(H/K)} \chi^{-1}(\sigma)\Gamma^\alpha(A_\sigma(t, \mathscr{P}))(k)}{a_\alpha((1+p)^{-k} - 1; \chi, \mathscr{P})} = \gamma^{-k}\lambda_k\Big(1 - \frac{\chi(\mathfrak{p})\psi^k(\mathfrak{p})}{N\mathfrak{p}}\Big) L(\chi^{-1}\bar{\psi}^k, k).$$

LEMMA 13: *If either χ is non trivial or α different from 0, there exists a pair $\mathcal{P}$ such that $a_\alpha(X;\chi,\mathcal{P})$ is a unit in Λ_χ.*

PROOF: If χ is non trivial, there exists σ such that $\chi(\sigma)\neq 1$. Let e denote a generator of the ideal $12\mathfrak{f}\cap\mathbb{Z}$. Choose $\mathfrak{a}_1=(1+ne\pi)$ $n_2=-(N\mathfrak{a}_1-1)$; take $\mathfrak{a}_2$ to be an integral ideal of K, prime to S and p, such that $\sigma_{\mathfrak{a}_2}=\sigma$ and let $n_1=N\mathfrak{a}_2-1$.

Now if χ is trivial and $\alpha\neq 0$, let η be an element of $\mathcal{O}$, whose image in $\mathcal{O}/\mathfrak{p}$ is a generator of $(\mathcal{O}/\mathfrak{p})^\times$. Take $\mathfrak{a}_1=(1+ne\pi)$. Choose $\mathfrak{a}_2=(\alpha_2)$ where α_2 is an algebraic integer in K, satisfying $\alpha_2\equiv 1 \bmod e\bar{\pi}$ and $\alpha_2\equiv\eta \bmod \pi$. Let $n_1=N\mathfrak{a}_2-1$ and $n_2=-(N\mathfrak{a}_1-1)$. Then n_2 is prime to p and because the conductor of ψ divides e,

$$\omega(\psi(\mathfrak{a}_1))^\alpha\equiv\psi^\alpha(\mathfrak{a}_1)\equiv 1 \bmod \mathfrak{p}$$

and

$$\omega(\psi(\mathfrak{a}_2))^\alpha\equiv\psi^\alpha(\mathfrak{a}_2)\equiv\eta^\alpha \bmod \mathfrak{p}.$$

Such a choice is made in [1] Lemma 13.

Now $\dfrac{\sum_{\sigma\in G(H/K)}\chi^{-1}(\sigma)\Gamma^\alpha(A_\sigma(t,\mathcal{P}))(s)}{a_\alpha((1+p)^s-1;\chi,\mathcal{P})}$ is a

continuous function on $\mathbb{Z}_p$, which is such that for all integers $k\geq 1$, $k\equiv\alpha \bmod p-1$

$$L_\mathfrak{p}(\chi\theta^\alpha,1-k)=\frac{\gamma}{T(\bar{\chi})w_\mathfrak{f}\sqrt{d}}\,\frac{\sum_{\sigma\in G(H/K)}\chi^{-1}(\sigma)\Gamma^\alpha(A_\sigma(t,\mathcal{P}))(k)}{a_\alpha((1+p)^{-k}-1;\chi,\mathcal{P})}$$

if either χ is non trivial or α different from zero.

LEMMA 14: *If either χ is a non trivial character, or α a non zero residue class mod $(p-1)$, for all $s\in\mathbb{Z}_p$,*

$$L_\mathfrak{p}(\chi\theta^\alpha,1-s)=\frac{\gamma}{T(\bar{\chi})w_\mathfrak{f}\sqrt{d}}\,\frac{\sum_{\sigma\in G(H/K)}\chi^{-1}(\sigma)\Gamma^\alpha(A_\sigma(t,\mathcal{P}))(s)}{a_\alpha((1+p)^{-s}-1;\chi,\mathcal{P})}. \tag{17}$$

REMARK: If χ is trivial and α is zero

$$a_0(0;\chi_0,\mathcal{P})=\sum_{j\in J} n_j(N\mathfrak{a}_j-1)=0.$$

But:

$$\Gamma^0(A(t,\mathcal{P}))(0)=\tilde{A}(0,\mathcal{P})=B(0,\mathcal{P})-\frac{1}{p}\sum_{n\in\hat{E}_\pi} B(\eta,\mathcal{P})$$
$$B(0,\mathcal{P})=0$$

and

$$\frac{1}{p}\sum_{\eta\in\hat{E}_\pi} B(\eta,\mathcal{P})=\frac{1}{p}\sum_{\alpha}\operatorname{Log}_p\frac{\Lambda(\alpha,\mathcal{P})}{\Lambda(0,\mathcal{P})}$$

where the sum on the right is taken over a set of representatives modulo L of the π-division points of L. Then

$$\frac{1}{p}\sum_{\alpha}\operatorname{Log}_p\frac{\Lambda(\alpha,\mathcal{P})}{\Lambda(0,\mathcal{P})}=\left(\frac{1}{p}-1\right)\operatorname{Log}_p\Lambda(0,\mathcal{P}).$$

But

$$\Lambda(0,\mathcal{P})=N_{K(E_\mathfrak{f})K}\Theta(\rho,\mathcal{P})$$

where ρ is a $\mathfrak{f}$-division point of $\mathbb{C}$ mod L. This is a unit in K, then a root of unity and

$$\operatorname{Log}_p N_{K(E_\mathfrak{f})/K}\Theta(\rho,\mathcal{P})=0.$$

Even when χ is trivial and α is zero

$$\frac{\Gamma^0(A(t,\mathcal{P}))(s)}{a_0((1+p)^s-1;\chi_0,\mathcal{P})}$$

is a continuous function on $\mathbb{Z}_p$ and we have

$$L_\mathfrak{p}(\theta^0,1-s)=\frac{\gamma}{w_\mathfrak{q}\sqrt{d}}\frac{\Gamma^0(A(t,\mathcal{P}))(s)}{a_0((1+p)^s-1;\chi_0,\mathcal{P})}.$$

But this formula is not useful for computing $L_\mathfrak{p}(\theta^0,1)$.
Now we come back to the case where χ is non trivial, and $\alpha=0$. From (17) we have

$$L_{\mathfrak{p}}(\chi, 1) = \frac{\gamma}{T(\bar{\chi})w_{\mathfrak{q}}\sqrt{d}} \frac{\sum_{\sigma\in G(H/K)} \chi^{-1}(\sigma)\Gamma^0(A_\sigma(t, \mathcal{P}))(0)}{a_0(0;\chi, \mathcal{P})}$$

$$\Gamma^0(A_\sigma(t, \mathcal{P}))(0) = \tilde{A}_\sigma(0, \mathcal{P}) = B_\sigma(0, \mathcal{P}) - \frac{1}{p}\sum_{\eta\in\hat{E}_\pi} B_\sigma(\eta, \mathcal{P})$$

by lemma 4

$$B_\sigma(0, \mathcal{P}) = 0$$

and

$$\frac{1}{p}\sum_{\eta\in\hat{E}_\pi} B_\sigma(\eta, \mathcal{P}) = \frac{1}{p}\sum_\alpha \operatorname{Log}\frac{\Lambda_\sigma(\alpha, \mathcal{P})}{\Lambda_\sigma(0, \mathcal{P})}.$$

where the sum on the right is taken over a set of representatives modulo L of the π-division points of L. Now from Lemma 7, we have

$$\Gamma^0(A_\sigma(t, \mathcal{P}))(0) = \frac{1}{p}\operatorname{Log}\frac{\Lambda_{\sigma\sigma_{\mathfrak{p}}}(0, \mathcal{P})}{\Lambda_\sigma(0, \mathcal{P})^p}.$$

THEOREM 15: *If χ is not trivial*

(18) $$L_{\mathfrak{p}}(\chi, 1) = \frac{\gamma}{T(\bar{\chi})w_{\mathfrak{q}}\sqrt{d}}\left(\frac{\chi(\mathfrak{p})}{p} - 1\right) \times \frac{\sum_{\sigma\in G(H/K)} \chi^{-1}(\sigma)\operatorname{Log}_p[N_{K(E_{\mathfrak{b}})/H}\Theta(\rho, \mathcal{P})]^\sigma}{a_0(0;\chi, \mathcal{P})}$$

We now proceed to find a similar formula for $\alpha \neq 0$. As before, define

$$T(\bar{\theta}) = \sum_{\lambda \bmod \mathfrak{p}} \bar{\theta}(\lambda)e^{2\pi i S(\lambda\delta_0)}$$

where δ has been chosen once for all such that $\mathfrak{p}^{-1}\mathcal{D}^{-1} = (\delta)$ and $\delta\sqrt{d}$ has exact denominator $\mathfrak{p}$, and where λ runs through a full system of representatives of the residue classes mod $\mathfrak{p}$. Let us denote by ζ the p-th root of unity $e^{2\pi i S(\delta)}$. As p splits in K, $\mathcal{O}/\mathfrak{p}$ is isomorphic to Z/pZ. Then, we will write

$$T(\bar{\theta}) = \sum_{\lambda \bmod p} \bar{\theta}(\lambda)\zeta^\lambda.$$

LEMMA 16: *For each α, congruence class mod $(p-1)$ for each rational integer n, prime to p*

$$\sum_{\lambda \bmod p} \bar{\theta}^{\alpha}(\lambda)\zeta^{\lambda n} = \omega^{\alpha}(n)T(\theta^{\alpha}).$$

PROOF: Let $m \in \mathbb{Z}$, such that

$$m \equiv n \bmod p$$

and

$$m \equiv 1 \bmod f \text{ (where } f = \mathfrak{f} \cap \mathbb{Z})$$

$$\sum_{\lambda \bmod p} \bar{\theta}^{\alpha}(\lambda)\zeta^{\lambda n} = \sum_{\lambda \bmod p} \bar{\theta}^{\alpha}(\lambda)\zeta^{\lambda m} = \theta^{\alpha}(m) \sum_{\lambda \bmod p} \bar{\theta}^{\alpha}(\lambda)\zeta^{\lambda}.$$

By definition

$$\theta(m) = \omega(\psi(m)) = \omega(m) = \omega(n).$$

Then Lemma 16 is proved.

Let M be any complete subfield of $\mathbb{C}_p$, and $A \in Q_M$. For each α congruence class mod$(p-1)$, let

$$A_{\alpha}(u) = \frac{1}{T(\bar{\theta}^{\alpha})} \sum_{\lambda \bmod p} \bar{\theta}^{\alpha}(\lambda)A(\zeta^{\lambda}(u+1)-1).$$

A_{α} belongs to Q_M and does not depend on ζ.

LEMMA 17: *For each $s \in \mathbb{Z}_p$*

$$\Gamma^{\alpha-\beta}(A)(s) = \Gamma^{-\beta}(A_{\alpha})(s).$$

PROOF: Because of the linearity of $\Gamma^{\alpha-\beta}$ and $\Gamma^{-\beta}$ we have just to prove the equality for $A(u) = (1+u)^n$. Then

$$A_{\alpha}(u) = \frac{1}{T(\bar{\theta}^{\alpha})} \sum_{\lambda \bmod p} \bar{\theta}^{\alpha}(\lambda)\zeta^{\lambda n}(1+u)^n$$

$$A_{\alpha}(u) = \omega^{\alpha}(n)(1+u)^n.$$

By definition [12]:

$$\Gamma^{\alpha-\beta}(A)(s) = \omega^{\alpha-\beta}(n)\langle n\rangle^s \qquad \text{if } p \nmid n$$
$$= 0 \qquad \text{if } p \mid n$$

and

$$\Gamma^{-\beta}(A_\alpha)(s) = \omega^{-\beta}(n)\omega^{\alpha}(n)\langle n\rangle^s \qquad \text{if } p \nmid n$$
$$= 0 \qquad \text{if } p \mid n.$$

Now let us consider

$$A_{\sigma,\alpha}(t, \mathscr{P}) = \frac{1}{T(\bar{\theta}^\alpha)} \sum_{\lambda \bmod p} \bar{\theta}^\alpha(\lambda) A_\sigma(\zeta^\lambda(t+1) - 1, \mathscr{P}).$$

Then

$$\Gamma^\alpha(A_\sigma(t, \mathscr{P}))(0) = \Gamma^0(A_{\sigma,\alpha}(t, \mathscr{P}))(0).$$

Moreover

$$\Gamma^0(A_{\sigma,\alpha}(t, \mathscr{P}))(0) = A_{\sigma,\alpha}(0, \mathscr{P}) - \frac{1}{p} \sum_{\zeta'} A_{\sigma,\alpha}(\zeta' - 1, \mathscr{P})$$

where ζ' runs over all p-th roots of unity in $\mathbb{C}_\mathfrak{p}$. But:

$$\sum_{\zeta'} A_{\sigma,\alpha}(\zeta' - 1, \mathscr{P}) = \frac{1}{T(\bar{\theta}^\alpha)} \sum_{\zeta'} \sum_{\lambda \bmod p} \bar{\theta}^\alpha(\lambda) A_\sigma(\zeta^\lambda \zeta' - 1, \mathscr{P}).$$

Then

$$\sum_{\zeta'} A_{\sigma,\alpha}(\zeta' - 1, \mathscr{P}) = 0$$

and:

$$\Gamma^0(A_{\sigma,\alpha}(t, \mathscr{P}))(0) = \frac{1}{T(\bar{\theta}^\alpha)} \sum_{\lambda \bmod p} \bar{\theta}^\alpha(\lambda) A_\sigma(\zeta^\lambda - 1, \mathscr{P}).$$

Recall that by definition $A_\sigma(t, \mathscr{P}) = B_\sigma(g(t), \mathscr{P})$ where $B_\sigma(t, \mathscr{P})$ is given by the expansion $\operatorname{Log} \dfrac{\Lambda_\sigma(z, \mathscr{P})}{\Lambda_\sigma(0, \mathscr{P})}$.

Define:

$$\Lambda^1(z, \mathfrak{a}_j) = \prod_{\mathfrak{b} \in B} \Theta(z + \psi(\mathfrak{b})\rho + q, \mathfrak{a}_j)$$

and

$$\Lambda^1(z, \mathscr{P}) = \prod_{j \in J} \Lambda^1(z, \mathfrak{a}_j)^{n_j}$$

where q is an element of $\mathbb{C}$ such that $\xi(q)$ is the $\mathfrak{p}$-division point on E which corresponds to ζ. Then

$$A_\sigma(\zeta - 1, \mathscr{P}) = \operatorname{Log} \frac{\Lambda_\sigma(q, \mathscr{P})}{\Lambda_\sigma(0, \mathscr{P})} = \operatorname{Log} \frac{\Lambda^1_\sigma(0, \mathscr{P})}{\Lambda_\sigma(0, \mathscr{P})}.$$

By Lubin Tate theory, we know that $G(K(E_\mathfrak{p})/K)$ is naturally isomorphic to the group of units of $\mathcal{O}/\mathfrak{p}$; moreover $G(H(E_\mathfrak{p})/H$ is isomorphic to $G(K(E_\mathfrak{p})/K)$ [14]. Then to each λ mod p corresponds $\sigma_\lambda \in G(H(E_\mathfrak{p})/H)$ and

$$A_\sigma(\zeta^\lambda - 1, \mathscr{P}) = \operatorname{Log} \frac{\Lambda^1_{\sigma\sigma_\lambda}(0, \mathscr{P})}{\Lambda_\sigma(0, \mathscr{P})}$$

THEOREM 18: *If α is a non zero residue class mod $p - 1$*

$$L_\mathfrak{p}(\chi\theta^\alpha, 1) = \frac{\gamma}{T(\bar{\chi}) w_\mathfrak{q} \sqrt{d}} \frac{\sum_{\sigma \in G(K(E_{\mathfrak{h}\mathfrak{p}})/K} \chi^{-1}\theta^{-\alpha}(\sigma) \operatorname{Log}_p \Theta(p + q, \mathscr{P})^\alpha}{a_\alpha(0; \chi, \mathscr{P})}. \tag{19}$$

(2) *Analogy with complex formula*

Let H be an arbitrary finite abelian extension of K and let $\mathfrak{h}$ be the least common multiple of the conductor of ψ and $\mathfrak{h}_0$, the conductor of H/K. Let χ' be a ray class character mod $\mathfrak{h}$ such that χ, the proper ray-class character associated with χ' has conductor $\mathfrak{h}_0$.

We will see that we have complex formula for $L(\chi', 1)$ which is analogue of (17) and (18).

We take the notation of Robert [16]. Let us consider the set $A(\mathfrak{h})$ of pairs $\{t, \mathfrak{b}\}$ where $t \in \mathbb{C}$ and $\mathfrak{b}$ is an ideal of K, such that $\mathfrak{h} = \{\alpha \in \mathcal{O} \mid \alpha t \in \mathfrak{b}\}$. One says that $\{t, \mathfrak{b}\}$ is equivalent to $\{t', \mathfrak{b}'\}$ if and only if, there exists $\theta \in K^*$ such that $t'/\theta t$ is congruent to 1 mod $\mathfrak{h}$ and $\mathfrak{b}' = \theta\mathfrak{b}$. Denote by $\sim$ this equivalence. For each $\{t, \mathfrak{b}\} \in A(\mathfrak{h})$, $t\mathfrak{h}\mathfrak{b}^{-1}$ is an integral ideal, prime to $\mathfrak{h}$. Denote by $C_{\{t, \mathfrak{b}\}}$ the ideal class of $t\mathfrak{h}\mathfrak{b}^{-1}$. Robert has shown that the map $\{t, \mathfrak{b}\} \mapsto C_{\{t, \mathfrak{b}\}}$ defines an isomorphism between $A(\mathfrak{h})$ and the ray class group mod $\mathfrak{h}$, $Cl(\mathfrak{h})$. Let $[w_1, w_2]$ be a basis of $\mathfrak{b}$; we define

$$\varphi^{12}(t, \mathfrak{b}) = \theta^{12}(t; w_1, w_2) \exp(-\mathscr{H}(t, t)/16)$$

where $\mathscr{K}(t, t) = 12i\pi\bar{t}t/(w_2\bar{w}_1 - w_1\bar{w}_2)$. It can be shown that $\varphi^{12h}(t, \mathfrak{b})$ depends only on $C_{\{t, \mathfrak{b}\}}$ and we set

$$\varphi_{\mathfrak{h}}(C) = \varphi^{12h}(t, \mathfrak{b}).$$

Now if we consider the pair $\{\rho, \mathcal{O}\}$ where $\rho = \frac{\Omega}{h}$. Then $C_{\{\rho, \mathcal{O}\}} = C_0$ the identity in the ray class group mod $\mathfrak{h}$. So

$$\Theta^{12h}(\rho, \mathfrak{a}_j) = \varphi(C_0)^{N\mathfrak{a}_j}/\varphi(C_0 C_{\mathfrak{a}_j}).$$

Then:

$$\text{(20)}\quad \frac{\sum_{\sigma \in G(K(E_{\mathfrak{h}})/K)} \chi'(\sigma) \operatorname{Log}|\Theta(\rho, \mathscr{P})^\sigma|}{a_0(0; \chi, \mathscr{P})} = \frac{1}{12h} \sum_{C \in Cl(\mathfrak{h})} \chi'(C) \operatorname{Log} |\varphi_{\mathfrak{h}}(C)|.$$

Moreover it can be proved that [16]:

$$\text{(21)}\quad \frac{1}{w_{\mathfrak{h}} h} \sum_{C \in Cl(\mathfrak{h})} \chi'(C) \operatorname{Log} |\varphi_{\mathfrak{h}}(C)| = \frac{X(\chi)}{w_{\mathfrak{h}_0} h_0} \sum_{C \in Cl(\mathfrak{h}_0)} \chi(C) \operatorname{Log}|\varphi_{\mathfrak{h}_0}(C)|$$

when

$$X(\chi) = \prod_{\mathfrak{q} | \mathfrak{h}} (1 - \bar{\chi}(\mathfrak{q})).$$

Now Siegel [18] has shown that

$$\text{(22)}\qquad L(\chi, 1) = \frac{2\pi}{T(\bar{\chi})\sqrt{d}\, w_{\mathfrak{h}_0} 6 h_0} \sum_{C \in Cl(\mathfrak{h}_0)} \chi(C) \operatorname{Log}|\varphi_{\mathfrak{h}_0}(C)|.$$

So, from (20), (21), (22) we have

$$\text{(23)}\qquad X(\chi) L(\chi, 1) = \frac{\pi}{T(\bar{\chi}) w_{\mathfrak{h}} \sqrt{d}} \frac{\sum_{\sigma \in G(K(E_{\mathfrak{h}})/K)} \chi'(\sigma) \operatorname{Log}|\Theta(\rho, \mathscr{P})|^\sigma}{a_0(0; \chi, \mathscr{P})}.$$

This formula is the complex analogue of (17) and (18). We will try to explain why this holds. We have

$$L(\bar{\chi}', 0) = X(\chi) L(\bar{\chi}, 0)$$

and

$$L(\bar{\chi}', 0) = L(\psi^0\bar{\chi}', 0) = L(\psi^0\bar{\chi}, 0).$$

Moreover, from the functional equation [7], we have

$$L(\bar{\chi}, 0) = L(\chi, 1)\frac{\sqrt{d}\,T(\bar{\chi})}{2\pi}.$$

Then

$$X(\chi)L(\chi, 1) = \frac{2\pi}{\sqrt{d}\,T(\bar{\chi})}\,L(\psi^0\bar{\chi}, 0)$$

and this is to compare with Lemma 12 and 13, if we could put $k = 0$.

IV. $\mathfrak{p}$-adic residue formula

Again, we suppose throughout this section that H is an arbitrary finite abelian extension of K. As before, we write $\mathfrak{h}$ for the least common multiple of the conductor of H over K, and the conductor of the Grossencharacter ψ of E over K. Finally, p is a rational prime, with $p \neq 2, 3$ and $(p, \mathfrak{h}) = 1$, which splits in K, say $(p) = \mathfrak{p}\bar{\mathfrak{p}}$. For simplicity, we write

$$F = H(E_{\mathfrak{p}}).$$

By analogy with Leopoldt's work, in the cyclotomic case, our aim is to use the result of §.III to find the residue at $s = 1$ of a $\mathfrak{p}$-adic function that can be viewed almost as the $\mathfrak{p}$-adic zeta function of F. Such a formula has been studied independently of us by Vishik [20] and Lichtenbaum. We begin by recalling the complex analogue of this formula. By class field theory, we have

$$\zeta_F(s) = \zeta_K(s)\prod_{\chi\neq 1} L(\chi, s)$$

where the product on the right is taken over the non trivial characters χ of the Galois group of F over K, and $L(\chi, s)$ is the primitive complex L-function attached to χ. Let Δ, W, g denote respectively the discriminant of F over $\mathbf{Q}$ the number of roots of unity in F, and the degree of F over K. Let d, w denote the discriminant of K over $\mathbb{Q}$, and the number of roots of unity in K. Finally, let R_∞ denote the regulator of F, and h the class number of F. Multiplying by $s-1$ in the above formula and letting $s \to 1$ we obtain

$$\text{(24)} \qquad \frac{(2\pi)^g h R_\infty}{W\sqrt{|\Delta|}} = \frac{2\pi}{w\sqrt{|d|}} \prod_{\chi \neq 1} L(\chi, 1).$$

Let $R_\mathfrak{p}$ be the $\mathfrak{p}$-adic regulator of F over K, as defined on p. 13 of [4]. Also, we can view $\sqrt{|\Delta|}$ and $\sqrt{|d|}$ as lying inside $\mathbb{C}_\mathfrak{p}$ by taking their images under our fixed embedding $\tau: \bar{K} \to \mathbb{C}_\mathfrak{p}$ (for simplicity, we identify these elements with their images under τ).

Let $\mathcal{P}$ be the pair defined in the previous section; $\rho = \dfrac{\Omega}{hp}$, where $(h) = \mathfrak{h}$. Let for $\sigma \in G(F/K)$

$$\text{(25)} \qquad E(\sigma) = \frac{\prod_{\mathfrak{b} \in B} \Theta(\psi(\mathfrak{b})\rho, \mathcal{P})^\sigma}{\prod_{\mathfrak{b} \in B} \Theta(\psi(\mathfrak{b})\rho, \mathcal{P})}.$$

Let $\mathscr{E}_1$ be the group generated by the $E(\sigma)^{\sigma'}$, with $\sigma' \in G(F/K)$. It is a group of units in F.

Let us denote by

$$\text{(26)} \qquad A(\mathcal{P}) = \prod_{\chi \neq 1} a_0(0; \chi, \mathcal{P})$$

by

$$\text{(27)} \qquad X = \prod_{\chi \neq 1} X(\chi)$$

and

$$\text{(28)} \qquad w' = \prod_{\chi \neq 1} w_{\mathfrak{g}_\chi}$$

where $\mathfrak{g}_\chi$ is the least common multiple of the conductor of χ and ψ, where χ runs over all primitive character of $G(F/K)$.

LEMMA 19: *The index of $\mathscr{E}_1$ in the group of all units in F is given by*

$$2^{g-1}h\frac{ww'}{W}A(\mathscr{P})X.$$

PROOF: It is well known that the index of $\mathscr{E}_1$ in the group of all units in F is equal to $\frac{U}{R_\infty}$ where $U = \det(\log|E(\sigma)^{\sigma'}|)$ with $\sigma, \sigma' \in G(F/K)$. From (24) we have

$$\prod_{\chi\neq 1} L(\chi, 1) = (2\pi)^{g-1}\frac{w}{W}\frac{R_\infty\sqrt{|d|}}{\sqrt{|\Delta|}}h. \tag{29}$$

Moreover from (23)

$$\begin{aligned}\prod_{\chi\neq 1} L(\chi, 1) = \pi^{g-1}&\frac{\sqrt{|d|}}{(\sqrt{|d|})^g\prod_{\chi\neq 1}T(\bar{\chi})}\frac{1}{w'A(\mathscr{P})X}\\ &\times\prod_{\chi\neq 1}\sum_{\sigma\in G(F/K)}\chi(\sigma)\operatorname{Log}\left|\prod_{\mathfrak{b}\in B}\Theta(\psi(\mathfrak{b})\rho, \mathscr{P})^\sigma\right|.\end{aligned} \tag{30}$$

But we know [18] that

$$U = \prod_{\chi\neq 1}\sum_{\sigma\in G(F/K)}\chi(\sigma)\operatorname{Log}\left|\prod_{\mathfrak{b}\in B}\Theta(\psi(\mathfrak{b})\rho, \mathscr{P})^\sigma\right|. \tag{31}$$

Combining (29) and (30), we have the lemma, recalling that $(\sqrt{|d|})^g\,\Pi_{\chi\neq 1}\,T(\bar{\chi}) = \sqrt{|\Delta|}$.
Let us denote

$$P = \left(1-\frac{1}{p}\right)^{-1}\prod_{\mathfrak{P}}(1-N(\mathfrak{P}))^{-1}$$

where the product is taken over all primes of F above $\mathfrak{p}$.

THEOREM 20:

$$\prod_{\chi\neq 1} L_{\mathfrak{p}}(\chi, 1) = (2\gamma)^{g-1}h\frac{w}{W}\frac{R_{\mathfrak{p}}\sqrt{|d|}}{\sqrt{|\Delta|}}PX \textit{ up to } \pm 1.$$

where the product on the left is taken over all non trivial character of $G(F/K)$.

PROOF: From (17) and (18), we know that

$$L_{\mathfrak{p}}(\chi, 1) = \frac{\gamma}{\sqrt{d}\,T(\bar{\chi})w_{\mathfrak{g}_\chi}}$$

$$\frac{\sum_{\sigma \in G(F/K)} \chi(\sigma) \operatorname{Log}_p \left(\prod_{\mathfrak{b} \in B} \Theta(\psi(\mathfrak{b})\rho, \mathcal{P})^{\sigma} \right)}{a_0(0; \chi, \mathcal{P})} \left(1 - \frac{\chi(\mathfrak{p})}{p}\right).$$

Then

$$\prod_{\chi \neq 1} L_{\mathfrak{p}}(\chi, 1) = \frac{\gamma^{g-1}\sqrt{|d|}}{\sqrt{|\Delta|}w'} \frac{P}{A(\mathcal{P})} \prod_{\chi \neq 1} \sum_{\alpha \in G(F/K)} \chi(\sigma) \operatorname{Log}_p \left(\prod_{\mathfrak{b} \in B} \Theta(\psi(\mathfrak{b})\rho, \mathcal{P})^{\sigma} \right).$$

Let

$$U_{\mathfrak{p}} = \prod_{\chi \neq 1} \sum_{\sigma \in G(F/K)} \chi(\sigma) \operatorname{Log}_p \left(\prod_{\mathfrak{b} \in B} \Theta(\psi(\mathfrak{b})\rho, \mathcal{P})^{\sigma} \right).$$

Then

$$U_{\mathfrak{p}} = \det(\log_p E(\sigma)^{\sigma'})\sigma, \sigma' \in G(F/K).$$

But $U_{\mathfrak{p}}/R_{\mathfrak{p}}$ is equal to the index of $\mathscr{E}_1$ in the group of all units in F, up to ± 1. Then

$$U_{\mathfrak{p}} = R_{\mathfrak{p}} 2^{g-1} h \frac{ww'}{W} A(\mathcal{P}) X \text{ up to } \pm 1.$$

Then Theorem 20 is proved.

(2) *Kummer's criterion*

Let us recall what is known about Kummer's criterion in the elliptic case. Let $L_0(\psi^k, s)$ be the *primitive* Hecke L-function of ψ^k for each $k \geq 1$. Let $L_0^*(\psi^k, k) = w(k-1)!\, L_0(\psi^k, k)$, $k \equiv 0 \bmod w$. If p is a prime number not in the exceptional set S, which splits in K, it is shown in [4] that the numbers

$$(N) \qquad L_0^*(\psi^k, k)(1 \leq k < p - 1; k \equiv 0 \bmod w)$$

are all p-integral. Let $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ and $H_{\mathfrak{p}}$ the ray class field of K modulo $\mathfrak{p}$. It is shown in [4] the Kummer's criterion i.e.

p divides at least one of the numbers (N) if and only if there exists a Z/pZ-extension of $H_{\mathfrak{p}}$, which is unramified outside the prime of $H_{\mathfrak{p}}$ above $\mathfrak{p}$ and which is distinct from the ray class field mod $\mathfrak{p}^2$.

The proof of this theorem is divided in two parts. In the first part, the authors use class field theory to establish a Galois theoretic p-adic residue formula for F an arbitrary finite extension of K. Denote by K_∞ the unique Z_p-extension of K, which is unramified outside $\mathfrak{p}$ and $F_\infty = K_\infty F$. The notations are those of the previous section.

LEMMA 21: *Let M be the maximal abelian p-extension of F, which is unramified outside the primes of F lying above $\mathfrak{p}$. Then $G(M/F_\infty)$ is finite if and only if $R_\mathfrak{p} \neq 0$. If $R_\mathfrak{p} \neq 0$, the order of $G(M/F_\infty)$ is equal to the inverse of the p-adic valuation of*

$$\frac{p^e h}{W} \frac{R_\mathfrak{p}\sqrt{|d|}}{\sqrt{|\Delta|}} P$$

where the integer e is defined by $F \cap K_\infty = K_e$.

Then they combine this with a function theoretic p-adic residue formula due to Katz and Lichtenbaum for the p-adic zeta function of $H_\mathfrak{p}$ over K.

Let now H be an arbitrary finite abelian extension of K and $F = H(E_\mathfrak{p})$. Let us consider the numbers

$$N' \begin{cases} \lambda_k\left(1 - \dfrac{\chi(\mathfrak{p})\psi^k(\mathfrak{p})}{N\mathfrak{p}}\right) L(\bar{\chi}\bar{\psi}^k, k) (1 \leq k < p-1, k \not\equiv 0 \bmod w) \\ \lambda_k\left(1 - \dfrac{\chi(\mathfrak{p})\psi^k(\mathfrak{p})}{N\mathfrak{p}}\right) L(\bar{\chi}\bar{\psi}^k, k) \prod_{\mathfrak{q}|\mathfrak{f}} \cdot \left(1 - \bar{\chi}(\mathfrak{q}) \dfrac{\bar{\psi}^k(\mathfrak{q})}{N\mathfrak{q}^k}\right)^{-1} \\ \qquad\qquad\qquad\qquad (1 \leq k < p-1, k \equiv 0 \bmod w) \end{cases}$$

for all primitive character χ associated to the characters of the Galois group $G(F/K)$.

Let $\mathfrak{P}$ denote any prime of H above $\mathfrak{p}$.

THEOREM 22: *$\mathfrak{P}$ divides at least one of the numbers (N') if and only if there exists a $Z/p\,Z$-extension of F, which is unramified outside the primes of $H(E_\mathfrak{p})$ above $\mathfrak{p}$ and which is distinct from $H(E_{\mathfrak{p}^2})$.*

PROOF: Theorem 20 shows that

$$\left|\prod_{\chi\theta^\alpha \neq 1} L_\mathfrak{p}(\chi\theta^\alpha, 1)\right| = \left|\frac{h}{W} \frac{R_\mathfrak{p}\sqrt{|d|}}{\sqrt{|\Delta|}} XP\right|_\mathfrak{p}.$$

For all $\chi\theta^\alpha$, $L_\mathfrak{p}(\chi\theta^\alpha, s)$ is an Iwasawa function. Then, for all integers $k \geq 0$

$$L_\mathfrak{p}(\chi\theta^\alpha, 1) \equiv L_\mathfrak{p}(\chi\theta^\alpha, 1-k) \bmod \mathfrak{p}.$$

But from theorem 10, if $k \equiv \alpha \bmod (p-1)$ $k \geq 1$

$$L_\mathfrak{p}(\chi\theta^\alpha, 1-k) = \gamma^{1-k}\lambda_k\left(1 - \frac{\chi(\mathfrak{p})\psi^k(\mathfrak{p})}{N\mathfrak{p}}\right) L(\bar{\chi}\bar{\psi}^k, k).$$

This shows that if $k \equiv \alpha \bmod p-1$, $k \geq 1$

$$L_\mathfrak{p}(\chi\theta^\alpha, 1) \equiv \left(1 - \frac{\chi(\mathfrak{p})\psi^k(\mathfrak{p})}{N\mathfrak{p}}\right) \gamma^{1-k}\lambda_k L(\bar{\chi}\bar{\psi}^k, k) \bmod \mathfrak{p}.$$

Moreover, if $k \equiv \alpha \bmod p-1$

$$\prod_{\mathfrak{q}|\mathfrak{p}} (1 - \bar{\chi}\theta^{-\alpha}(\mathfrak{q})) = \prod_{\mathfrak{q}|\mathfrak{f}} (1 - \bar{\chi}(\mathfrak{q})\,\omega^{-k}(\psi(\mathfrak{q}))).$$

And if $k \equiv 0 \bmod w$

$$X(\chi\theta^\alpha) \equiv \prod_{\mathfrak{q}/\mathfrak{f}} (1 - \bar{\chi}(\mathfrak{q})\psi^{-k}(\mathfrak{q})) \bmod \mathfrak{p}$$

Or

$$X(\chi\theta^\alpha) \equiv \prod_{\mathfrak{q}/\mathfrak{f}} \left(1 - \bar{\chi}(\mathfrak{q})\frac{\bar{\psi}^k(\mathfrak{q})}{N\mathfrak{q}^k}\right) \bmod \mathfrak{p}.$$

Thus, if $\alpha \equiv k \bmod p-1$

$$X(\chi\theta^\alpha)^{-1}L_\mathfrak{p}(\chi, 1) \equiv \left(1 - \frac{\chi(\mathfrak{p})\psi^k(\mathfrak{p})}{N\mathfrak{p}}\right)\lambda_k\gamma^{1-k}L(\bar{\chi}\bar{\psi}^k, k) \bmod \mathfrak{p}$$

$$\text{if } k \not\equiv 0 \bmod p-1$$

$$X(\chi\theta^\alpha)^{-1}L_\mathfrak{p}(\chi, 1) \equiv \left(1 - \frac{\chi(\mathfrak{p})\psi^k(\mathfrak{p})}{N\mathfrak{p}}\right)\lambda_k\gamma^{1-k}L(\bar{\chi}\bar{\psi}^k, k)\prod_{\mathfrak{q}/\mathfrak{f}}\left(1 - \bar{\chi}(\mathfrak{q})\frac{\bar{\psi}^k(\mathfrak{q})}{N\mathfrak{q}^k}\right)(\mathfrak{p})$$

$$\text{if } k \equiv 0 \bmod p-1.$$

Now we have just to use Lemma 20.

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