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TOWARDS THE JANTZEN CONJECTURE III

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Abstract

A comparison technique introduced in the first paper of this series is refined through a result of Bernstein [1]. Combined with recent results of Borho, Kraft and Procesi [5, 8] it completes the proof of the Jantzen conjecture.

1. Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $U(\mathfrak{g})$ its enveloping algebra given the canonical filtration ([7], 2.3.1) and Prim $U(\mathfrak{g})$ the set of primitive ideals of $U(\mathfrak{g})$. Our aim is to determine for each $I \in$ Prim $U(\mathfrak{g})$ the zero variety $\mathcal{V}(\text{gr } I)$ of the associated graded ideal gr I. When $\mathfrak{g}$ has only type A_n factors (Cartan notation) we show that this variety is the Zariski closure of the appropriate Richardson orbit. For arbitrary $\mathfrak{g}$ a slightly weaker result is obtained for any primitive ideal which is a minimal prime ideal containing the annihilator of an induced module. The proof is based on a comparison theorem and on a technique initiated by Borho for computing these varieties. The notation is that of the first paper in this series [11], hereafter referred to as I.

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2. The Comparison Theorem

2.1 For each $k \in \mathbb{N}$, set $U^k(\mathfrak{g}) = \text{lin span}\{X^\ell : X \in \mathfrak{g}, \ell \in \{0, 1, 2, \ldots, k\}\}$. Given $y \in U^k(\mathfrak{g})$, $y \notin U^{k-1}(\mathfrak{g})$ we write deg $y = k$.

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2.2 (Notation I, 3.2). Let V be a finitely generated $\mathfrak{k}$-finite U module. In particular V is a left and a right $U(\mathfrak{g})$ module and for each subspace V^0 of V, we set $\ell(V^0) = \{a \in U(\mathfrak{g}) : aV^0 = 0\}$, $r(V^0) = \{a \in U(\mathfrak{g}) : V^0 a = 0\}$. The following refines (I, 2.4).

THEOREM: $\sqrt{\text{gr } \ell(V)} = \sqrt{\text{gr } r(V)}$.

Let V^0 be a finite dimensional generating subspace for V which we can assume $\mathfrak{k}$-stable without loss of generality. Then V^0 generates V as either a left or as a right $U(\mathfrak{g})$ module and moreover $U^k(\mathfrak{g})V^0 = V^0 U^k(\mathfrak{g})$, for all $k \in \mathbb{N}$. Thus if we set $V^k = U^k(\mathfrak{g})V^0$, $V_k = V^k/V^{k-1}$, $\text{gr } V = \bigoplus V_k$, it follows easily that gr V defined as an $S(\mathfrak{g})$ module is independent as to whether left or right action of $U(\mathfrak{g})$ is considered. Now in ([1], Prop. 1.4) it is shown that $x \in \sqrt{\text{gr } \ell(V^0)}$ if and only if there exists $y \in \ell(V^0)$ with gr $y = x$ such that for some $k \in \mathbb{N}$ one has $y^k V^\ell \subset V^{\ell + k \deg y - 1}$ for all $\ell \in \mathbb{N}$. This is equivalent to the statement that $x^k \in \text{Ann gr } V$. Hence $\sqrt{\text{gr } \ell(V^0)} = \sqrt{\text{Ann gr } V} = \sqrt{\text{gr } r(V^0)}$. Yet $V^0 U(\mathfrak{g}) = V$ and so $\ell(V^0) = \ell(V)$. Similarly $r(V^0) = r(V)$ and thus the conclusion of theorem is obtained.

2.3 (Notation I, 3.1, 3.3). Fix $\lambda, \mu \in \mathfrak{h}^*$ dominant with $\lambda - \mu \in P(R)$. Recalling ([12], 4.6) assume that $w \in W_\lambda$ is the unique maximal element, under the Bruhat ordering, of $W(\lambda)wW(\mu)$. Then from 2.2 and ([12], 4.12) taking $V = V(-w\mu, -\lambda)$ we obtain the

COROLLARY: $\mathcal{V}(\text{gr } I(w\mu)) = \mathcal{V}(\text{gr } I(w^{-1}\lambda))$.

2.4 Fix $\nu \in \mathfrak{h}^*$ regular and let $\hat{F}_\nu$ denote the intersection of $\nu + P(R)$ with the upper closure of the facette containing ν(cf. [12], 2.1).

LEMMA: *For each $\nu' \in \hat{F}_\nu$ one has*

$$\mathcal{V}(\text{gr } I(\nu)) = \mathcal{V}(\text{gr } I(\nu')).$$

We apply 2.3 twice. First with μ, λ dominant, regular and $w\mu = \nu$ (which fixes w). This gives $\mathcal{V}(\text{gr } I(\nu)) = \mathcal{V}(\text{gr } I(w^{-1}\lambda))$. Second with μ, λ dominant, λ regular and $w\mu = \nu'$ (with w as before). To apply 2.3 we must require that w be maximal in $wW(\mu)$. This is exactly (c.f. [12], 4.6) the requirement that $\nu' \in \hat{F}_\nu$ and so 2.2 applies giving $\mathcal{V}(\text{gr } I(\nu')) = \mathcal{V}(\text{gr } I(w^{-1}\lambda))$.

2.5 The above result answers positively a question of Borho and allows us from now on to restrict our attention to the determination of $\mathcal{V}(\operatorname{gr} I(\lambda))$ for λ regular.

2.6 Fix $\lambda \in \mathfrak{h}^*$ dominant, regular and define $\mathscr{X}_{\hat{\lambda}}$ as in (I, 3.4). In ([13], 5.5) we gave a decomposition of $\mathscr{X}_{\hat{\lambda}}$ into disjoint subsets $(\mathscr{X}_{\hat{\lambda}})_\sigma$ indexed by a certain subset Ω_λ of $\hat{W}_\lambda$. We remark that card $(\mathscr{X}_{\hat{\lambda}})_\sigma = \dim \sigma$, for each $\sigma \in \Omega_\lambda$ and $\Omega_\lambda = \hat{W}_\lambda$ if and only if R_λ has only simple type A_n factors, so this generalizes to arbitrary $\mathfrak{g}$ the result anticipated by Jantzen in type A_n.

LEMMA: *For each $\sigma \in \Omega_\lambda$, the variety $\mathcal{V}(\operatorname{gr} I)$ is independent of the choice of $I \in (\mathscr{X}_{\hat{\lambda}})_\sigma$.*

To each $w \in W_\lambda$ we associated in ([13], 5.2) a certain element $\mathbf{a}(w)$ of $\mathbb{C}W_\lambda$. Suppose $I(w\lambda)$, $I(w'\lambda) \in (\mathscr{X}_{\hat{\lambda}})_\sigma$. By definition of $(\mathscr{X}_{\hat{\lambda}})_\sigma$ ([13], 5.5) we have $\mathbf{a}(w') \in [\mathbf{a}(w)\mathbb{C}W_\lambda]$ (notation [13], 5.2). Then by ([13], 5.2 (iii)) we have $\mathbf{a}(w'^{-1}) \in [\mathbb{C}W_\lambda \mathbf{a}(w^{-1})]$ and by ([13], 5.3) we obtain $I(w'^{-1}\lambda) \supset I(w^{-1}\lambda)$. Taking $\lambda = \mu$ in 2.3 gives $\mathcal{V}(\operatorname{gr} I(w\lambda)) = \mathcal{V}(\operatorname{gr} I(w^{-1}\lambda)) \supset \mathcal{V}(\operatorname{gr} I(w'^{-1}\lambda)) = \mathcal{V}(\operatorname{gr} I(w'\lambda))$. Interchanging w, w' establishes the reverse inclusion and hence the assertion of the lemma.

2.7 (Notation I, 3.1, 3.3). Take $\lambda - \mu \in P(R')^{++}$ with $\lambda - \mu \in P(R)$.

LEMMA: $\mathcal{V}(\operatorname{gr} I_{B'}(\lambda)) = \mathcal{V}(\operatorname{gr} I_{B'}(\mu))$.

Take $V = L(M_{B'}(\lambda), M_{B'}(\mu))$ in 2.2. The assertion will follow if we can show that $\ell(V) = I_{B'}(\mu)$, $r(V) = I_{B'}(\lambda)$. The first assertion follows from (I, 4.8(i)) and (I, 5.5(ii)). The second from (I, 4.6) and (I, 5.5(i)).

3. The Jantzen Conjecture

Throughout this section we fix $-\lambda \in \mathfrak{h}^*$ dominant and regular.

3.1 The importance of 2.4, 2.6 derive from the fact that it is possible to compute $\mathcal{V}(\operatorname{gr} I(w\lambda))$ for certain special choices of w. The method of computation is due to Borho and in fact motivated [5, 8].

3.2 (Notation I, 3.1, 3.3). Let $B_\lambda \subset R_\lambda^+$ be a simple system of roots for R_λ. Fix $B' \subset B_\lambda$ and define $W_{B'}$, $w_{B'}$ as in (I, 3.1). A basic problem is to determine $\mathcal{V}(\operatorname{gr} I(w_{B'}\lambda))$. An important special case is when there exists $w \in W$ such that $wB' \subset B$. (This always holds for example when $\mathfrak{g}$ has only type A_n factors.) Then by ([9], 4.1) we can assume $B' \subset B$ without loss of generality. Assume this holds and (cf. I, 8.2) let $\mathcal{O}_{\mathcal{B}'}$ denote the corresponding Richardson orbit. Given any subset $S \subset \mathfrak{g}^*$ let $I(S)$ denote the ideal in $S(g)$ of polynomial functions in $\mathfrak{g}^*$ which vanish on S. Assume that $\mathfrak{g}$ has only type A_n factors.

THEOREM: (Borho–Kraft–Procesi). *Suppose* $2(\lambda, \alpha)/(\alpha, \alpha) = -1$, *for all* $\alpha \in B'$. *Then* $\operatorname{gr} I(w_{B'}\lambda) = I(\mathcal{O}_{\mathcal{B}'})$.

By ([2], 2.3a)) we have $\operatorname{gr} I(w_{\mathcal{B}'}\lambda) \subset I(\mathcal{O}_{\mathcal{B}'})$. For each $\sigma \in \hat{\mathfrak{g}}$ and each $\mathfrak{g}$ finite module M, let $[M:\sigma]$ denote the number of times the irreducible representation σ occurs in M. To prove the theorem it is enough to show that with respect to adjoint action we have

$$(*) \qquad [U(\mathfrak{g})/I(w_{B'}\lambda):\sigma] = [S(\mathfrak{g})/I(\mathcal{O}_{\mathcal{B}'}):\sigma],$$

for all $\sigma \in \hat{\mathfrak{g}}$. The left hand side of (*) has been computed by Conze–Berline and Duflo. Indeed set $\mu = w_{B'}\lambda$. Then (notation *I*, 3.2, 4.5) $\mu \in P(R')^\vee$ and by ([6], 2.12, 4.7, 5.5, 6.3) we obtain $U(\mathfrak{g})/I(\mu) = L_{B'}(\mu, \mu)$, up to isomorphism. Let E_σ be the simple $\mathfrak{g}$ module defined by σ and $\mathfrak{r}$ the reductive part of $\mathfrak{p}_{B'}$. Since $2(\mu, \alpha)/(\alpha, \alpha) = 1$, for all $\alpha \in B'$, it follows from the above and Frobenius reciprocity ([7], 5.5.7, 5.5.8) that

$$(**) \qquad [U(\mathfrak{g})/I(w_{B'}\lambda):\sigma] = \dim E_\sigma^{\mathfrak{r}}.$$

It remains to show that

$$(***) \qquad [S(\mathfrak{g})/I(\mathcal{O}_{\mathcal{B}'}):\sigma] = \dim E_\sigma^{\mathfrak{r}},$$

for all $\sigma \in \hat{\mathfrak{g}}$. (This formula was originally conjectured by Dixmier). Given $X \in \mathcal{O}_{\mathcal{B}'}$, let G_X denote its stabilizer in the adjoint group G. By ([5], 1.7, 6.3) (***) holds whenever G_X is connected and the Zariski closure of $\mathcal{O}_{\mathcal{B}'}$ is a normal variety. When $\mathfrak{g}$ has only type A_n factors the former assertion is classical and the latter has been recently established by Kraft and Procesi ([8], 3.3). This proves the theorem.

REMARK: Modulo the Kraft–Procesi normality theorem this is ([4], Thm. 2.6) the proof being along the lines of ([3], Satz 3.5).

3.3 (Notation I, 9.2). Assume $\mathfrak{g}$ has only type A_n factors. Then W_λ is a product of symmetric groups and so as in (I. 9.2 or [10], Sect. 2) we have a Robinson map $\Phi: w \mapsto (A(w), B(w))$ of W_λ into pairs of standard tableaux. Taking $B_\lambda \subset B$, the columns of $A(w)$ define a subset B' of B and hence a Richardson orbit $\xi(w)$. (For this see I, 8.3).

COROLLARY: *For all $w \in W_\lambda$, one has $\mathcal{V}(\operatorname{gr} I(w\lambda)) = \overline{\xi(w)}$ (Zariski closure).*

As in (I, 9.2) this follows from 2.4, 2.6 and 3.2.

REMARK 1: We recall (cf. [10], Thm. 1) that $I(w\lambda) = I(w'\lambda)$ if and only if $A(w) = A(w')$ and so this establishes the (obvious generalization) of the Jantzen conjecture as stated in ([4], 5.9).

REMARK 2: If $\mathfrak{g}$ has only type A_n factors then $\sqrt{\operatorname{gr} I} \in \operatorname{Spec} S(\mathfrak{g})$ for all $I \in \operatorname{Prim} U(\mathfrak{g})$.

3.4 Three main difficulties arise in attempting to compute $\sqrt{\operatorname{gr} I}$ in general. The first is that not all orbits are of Richardson type. Secondly $G_X(X$ nilpotent) need not be connected. For certain Richardson orbits this difficulty can be overcome through ([5], 7.7). Let X be a nilpotent element of $\mathfrak{g}$. We shall say that the G-orbit generated by X admits a stable polarization $\mathfrak{p}$ if there exists a parabolic subgroup P_X of G_X which satisfies $P_X = G_X$ and whose Lie algebra is $\mathfrak{p}$. Unfortunately not all Richardson orbits have this property. For example the referee remarked that it fails for the C_2 type orbit in C_3. The latter condition can always be satisfied as follows. By the Jacobson–Morozov theorem there exist $H, F \in \mathfrak{g}$ such that $\mathbb{C}X \oplus \mathbb{C}H \oplus \mathbb{C}Y$ is a simple subalgebra of $\mathfrak{g}$. Fix a Cartan subalgebra $\mathfrak{h}$ such that $H \in \mathfrak{h}$ and set

$$\mathfrak{p} = \mathfrak{h} \oplus \sum_{\alpha \in R : (\alpha, H) \geq 0} \mathbb{C}X$$

Then G_X normalizes $P([18], 4.16)$ and so $G_X = P_X$. Yet $\mathfrak{p}$ is only a polarization if $\dim \mathfrak{g}^X = \dim \mathfrak{g}^H$. The third difficulty is that $\bar{\mathcal{O}}$ need not be a normal variety. This can happen even for a Richardson orbit $\mathcal{O}_{\mathcal{B}}$. ([8], 0.5). Under the assumption $G_X = P_X$ (where P is the parabolic subgroup with Lie algebra $\mathfrak{p}_{B'}$) non-normality is equivalent to the strict inequality

$$[S(\mathfrak{g})/I(\mathcal{O}_{\mathcal{B}'}) : \sigma] < \dim E_\sigma^{\mathfrak{r}},$$

for some $\sigma \in \mathfrak{g}^{\wedge}$. In virtue of (**) we must have gr $I(w_{B'}) \subsetneq I(\mathcal{O}_{\mathcal{B}'})$, so gr $I(w_{B'}\lambda)$ can at best be primary. Yet by ([5], 7.2) both sides of (*) admit multiplicity functions (in the sense of ([3], 2.1)) which are polynomials with the same leading term (i.e. are quasi-equal in the sense of ([3], 2.2)). In the sense of ([3], 3.5d), $\bar{\mathcal{O}}_{\mathcal{B}'}$ is then the largest irreducible component of $V(\text{gr } I(w_{B'}\lambda))$.

LEMMA: *Fix $B' \subset B$, $\mu \in P(R')^{++}$. If $\mathcal{O}_{\mathcal{B}'}$ admits a stable polarization, then $V(\text{gr } I_{B'}(\mu))$ contains $\bar{\mathcal{O}}_{\mathcal{B}'}$ as its largest irreducible component. They coincide if the latter is a normal variety.*

By 2.7 it is enough to prove the assertion in the special case when $\mu = w_{B'}\lambda$ with $-\lambda \in \mathfrak{h}^*$ dominant, regular and satisfying $2(\lambda, \alpha)/(\alpha, \alpha) = -1$. Then $I_{B'}(\mu) = I(w_{B'}\lambda)$ and as noted in ([15], 5.7) $I(w_{B'}(\lambda + \nu))/I(\lambda + \nu)$ is independent of $\nu \in B'^{\perp}$ as long as $-(\lambda + \nu)$ remains dominant and regular. Thus we can further assume $\lambda \in P(R)$. Let $\mathfrak{p}$ be a stable polarization for $\mathcal{O}_{\mathcal{B}'}$. We can take $\mathfrak{p}$ to be of the form $\mathfrak{p}_{B''}: B'' \subset B$ where of course $\mathcal{O}_{\mathcal{B}''} = \mathcal{O}_{\mathcal{B}'}$. After Lusztig (cf. I, 8.6) the latter condition is equivalent to $P_{\mathcal{B}'} = P_{\mathcal{B}''}$. Now choose $-\lambda' \in P(R)$ dominant and regular such that $2(\lambda', \alpha)/(\alpha, \alpha) = -1$, for all $\alpha \in B''$. We show that $\mathcal{V}(\text{gr } I(w_{B'}\lambda)) = \mathcal{V}(\text{gr } I(w_{B''}\lambda'))$. Indeed $\lambda - \lambda' \in P(R)$ and if σ is the irreducible representation of W_λ defined by $P_{\mathcal{B}'}$, one has $I(w_{B'}\lambda) \in (\mathcal{X}_{\hat{\lambda}})_\sigma$, $I(w_{B''}\lambda') \in (\mathcal{X}_{\hat{\lambda}'})_\sigma$. Thus the assertion follows from 2.4, 2.6. Finally, $\mathcal{V}(\text{gr } I(w_{B''}\lambda'))$ satisfies the conclusion of the lemma, by the hypotheses, ([5], 7.2) and the remarks preceeding the statement of the lemma.

REMARKS: It follows that Borho's problem (I, Sect. 9) has a positive answer when $\mathfrak{g}$ has only type A_n factors. In general we have only a much weaker statement and for example as pointed out by the referee one does not even know whether or not two nilpotent orbits of the same dimension can have multiplicity functions of different degrees. If this cannot happen then the largest irreducible component coincides with the closure of the unique orbit of maximal dimension.

3.5 The main application of 3.4 derives from the following lemma.

LEMMA: *Let $I \in \text{Prim } U(\mathfrak{g})$ be a minimal prime ideal containing some induced ideal $I_{B'}(\mu): B' \subset B$, $\mu \in P(R')^{++}$. Then $\mathcal{V}(\text{gr } I) = \mathcal{V}(\text{gr } I_{B'}(\mu))$.*

Let $\{I_j\}_{j=1}^n$ denote the set of minimal prime ideals containing $I_{B'}(\mu)$ with p_i the Goldie polynomial defining $rk\, U(\mathfrak{g})/I_i$ (see [17]). Let σ be the irreducible representation of W_λ defined by the simple module $P_{\mathscr{B}'}$. By 2.6, 2.7 it is enough to show that $I_j \in (\mathscr{X}_{\hat{\mu}})_\sigma$ for all j. Define $p_{B'}$ as in (I, 6.6) and recall that $p_{B'}(\mu) = \dim V_{B'}(\mu)$ (notation I, 3.1). Choose $w \in W_\lambda$ such that $w_{B'}w^{-1}\mu$ is antidominant. By ([14], 3.9(i)) we have $d(U(\mathfrak{g})/I_j) = d(U(\mathfrak{g})/I_{B'}(\mu))$ and so by ([16], 8.1 (iii)) and ([17], 5.12) there exist $z'_j \in \mathbb{N}^+$ such that

$$\sum_{j=1}^{n} z'_j p_j = w p_{B'}. \tag{$*$}$$

(Here $z'_j = z_j(q_j/p_j)$, where z_j is defined by the conclusion of ([16], 8.1 (iii)) and (q_j/p_j) is the positive integer defined through ([17], 5.12)). Now $wp_{B'} \in P_{\mathscr{B}'}$, so by ($*$) and the linear independence ([13], Thm. 5.5) of the Goldie polynomials we obtain $p_j \in P_{\mathscr{B}'}$. Hence $I_j \in (X_{\hat{\mu}})_\sigma$, as required.

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