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Tamagawa number of reductive algebraic groups


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0. Introduction

The purpose of this paper is to give a formula for the Tamagawa number of a reductive quasi-split algebraic group $G$ defined over an algebraic number field in terms of the Tamagawa number of a maximal torus of $G$ (cf. Theorem 7.1).

The Tamagawa numbers of classical groups were determined by Weil [23]. In [15] Langlands determined the Tamagawa number of all split semisimple groups. We extend the result of Langlands to quasi-split groups.

I am most grateful to R.P. Langlands for explaining his methods to me. I would like to thank M. Rapoport for sending me his paper [18] and J. Arthur for useful suggestions.

Notations:

- $F$ = number field
- $F_v$ = completion of $F$ at the place $v$
- $\bar{F}$ = algebraic closure of $F$
- $v|\infty = v$ is an infinite place
- $v < \infty = v$ is a finite place
- $0_v = \mathcal{O}_v = \text{ring of integers of } F_v$ ($v < \infty$)
- $q = \text{order of residue field of } F_v$
- $\omega_v = \text{uniformizing element of } 0_v$ ($v < \infty$)
- $\mathbb{A} = \text{adeles of } F$, $\mathbb{A}_{\mathcal{P}} = \text{adeles trivial outside } \mathcal{P}$
- $|\cdot|_v = \text{normalised absolute value at } v$ ($v < \infty$): $|\omega_v|_v = q^{-1}$
- $|| = \text{adelic absolute value}$.

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For an algebraic group $H$ defined over $F$, we write

$$H_v = H(F_v)$$

$$H_f = \{ (h_v) \in H(A) \mid h_v = 1 \text{ if } v \mid \infty \}$$

$$H_\infty = \prod_{v \mid \infty} H_v$$

$$H_{\infty'} = \{ (h_v) \in H(A) \mid h_v = 1 \text{ if } v \not\in \mathcal{S} \}$$

$$H'_{\infty'} = \{ (h_v) \in H(A) \mid h_v \in H(0_v) \text{ if } v \not\in \mathcal{S} \}.$$

For a complex valued function $f(x)$, write $\bar{f}(x)$ for the complex conjugate of $f(x)$.

1. Quasi-split algebraic groups

1.1. Let $G$ be a connected reductive algebraic group defined over $F$. We say that $G$ is quasi-split if one of the following equivalent conditions is satisfied

(I) $G$ has a Borel subgroup $B$ defined over $F$,

(II) the centralizer in $G$ of a maximal $F$-split torus is a maximal torus of $G$,

(III) $G$ has no anisotropic roots.

In the following $G$ denotes a connected reductive quasi-split group.

1.2. Let $A$ be a maximal torus of $G$ lying in $B$ and defined over $F$, $L$ the group of characters of $A$, $\hat{\mathbb{L}} = \text{Hom}(L, \mathbb{Z})$, $\Sigma(\hat{\mathbb{L}})$ the set of roots (coroots) of $G$ with respect to $A$, $\Delta$ basis of $\Sigma$ with respect to $B$ and $\hat{\Delta}$ the elements of $\hat{\mathbb{L}}$ corresponding to $\Delta$. There is a bijection between $\hat{\mathbb{L}}$-isomorphism classes of triple $(G, B, A)$ and isomorphism classes of based root system $\psi(G) = (L, \Delta, \hat{L}, \hat{\Delta})$. This bijection yields a connected reductive $C$-group $\hat{G}^0$ with based root system $\psi_0(\hat{G}^0) = (\mathbb{L}, \Delta, L, \Delta)$. Let $\hat{A}^0$ (resp. $\hat{B}^0$) be the maximal torus (resp. Borel subgroup) defined by $\psi_0(\hat{G}^0)$.

Let $E$ be a Galois extension of $F$ such that $G$ splits over $E$. If $\sigma \in \text{Gal}(E/F)$, $\lambda \in L$, we denote the action of $\sigma$ on $\lambda$ by $\sigma\lambda$ where $\sigma\lambda(a) = \sigma(\lambda(\sigma^{-1}a))$ for $a \in A$. As $G$ is quasi-split, $\sigma\Delta = \Delta$. We can define a homomorphism $\mu : \text{Gal}(E/F) \to \text{Aut} \psi_0(G)$. Since we have canonical $\text{Aut} \psi_0(G) = \text{Aut} \psi_0(\hat{G}^0)$, we may view $\mu$ as a homomorphism of $\text{Gal}(E/F)$ into $\text{Aut} \psi_0(\hat{G}^0)$. Moreover there is a split exact sequence
(1) \( (1) \to \text{Int} \hat{G}^0 \to \text{Aut} \hat{G}^0 \to \text{Aut} \psi_0(\hat{G}^0) \to (1) \)

and a splitting yields a monomorphism

\[
\text{Aut} \psi_0(\hat{G}^0) \to \text{Aut}(\hat{G}^0, \hat{B}^0, \hat{A}^0).
\]

Together with the \( \mu \) above we get a homomorphism

\[
\mu' : \text{Gal}(E/F) \to \text{Aut}(\hat{G}^0, \hat{B}^0, \hat{A}^0)
\]

The associated group to, or \( L \)-group of, \( G \) is then by definition the semidirect product

\[
\hat{G} = \hat{G}^0 \rtimes \text{Gal}(E/F).
\]

(See Borel [3]).

1.3. Let \( Z \) be the identity component of the centre of \( G \) and \( G' \) be the derived group of \( G \). Then \( G = ZG' \) and \( A = ZA' \) where \( A' = A \cap G' \). Let \( {}^0L^+ \) be the group of rational characters of \( Z \) and \( {}^0L^- \) be the elements of \( {}^0L^+ \) which are 1 on \( Z \cap A' \). Let \( {}^1L^- \) be the lattice of roots of \( A' \). (Note that there is a bijection between the roots of \( (G, A) \) and \( (G', A') \) and the corresponding Weyl groups can be identified. We shall not use a separate notation.) We denote the Weyl group of the root system by \( W \). There exists a non-degenerate \( W \)-invariant bilinear form \( (.,.) \) on \( {}^1L^- \otimes_\mathbb{Z} \mathbb{C} \) such that its restriction to \( {}^1L^- \otimes_\mathbb{R} \mathbb{R} \) is positive definite. Let \( {}^1L \) be the lattice of rational characters of \( A' \) and

\[
{}^1L^+ = \left\{ \lambda \in {}^1L^- \otimes_\mathbb{Z} \mathbb{C} \mid \frac{2(\lambda, \alpha)}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \text{ for all roots } \alpha \right\}.
\]

Set \( L^- = {}^0L^- \oplus {}^1L^- \) and \( L^+ = {}^0L^+ \oplus {}^1L^+ \). We define dual lattices by

\[
\hat{L}^+ = \text{Hom}(L^-, Z) = \text{Hom}(0L^-, Z) \oplus \text{Hom}(1L^-, Z) = {}^0\hat{L}^+ \oplus 1\hat{L}^+,
\]

\[
\hat{L} = \text{Hom}(L, Z)
\]

\[
\hat{L}^- = \text{Hom}(L^+, Z) = \text{Hom}(0L^+, Z) \oplus \text{Hom}(1L^+, Z) = {}^0\hat{L}^- \oplus 1\hat{L}^-.
\]

We then have \( L^- \subset L \subset L^+ \subset L \otimes_\mathbb{Z} \mathbb{C} \) and \( \hat{L}^- \subset \hat{L} \subset \hat{L}^+ \subset \hat{L} \otimes_\mathbb{Z} \mathbb{C} \).

For the pairing \( L \times \hat{L} \to \mathbb{C} \), we use the notation \( \langle \lambda, \hat{\lambda} \rangle = \hat{\lambda}(\lambda) \) where \( \lambda \in L, \hat{\lambda} \in \hat{L} \) and we extend it meaningfully to the other lattices. The
form on $\mathcal{L}^+ \otimes \mathbb{C}$ adjoint to the one given above on $\mathcal{L}^- \otimes \mathbb{C}$ will also be denoted by $(\cdot, \cdot)$, i.e. if $\mu, \nu \in \mathcal{L}^- \otimes \mathbb{C}$, and if the elements $\hat{\mu}, \hat{\nu}$ of $\mathcal{L}^+ \otimes \mathbb{C}$ satisfy the equations

$$\langle \lambda, \hat{\mu} \rangle = (\lambda, \mu) \quad \text{and} \quad \langle \lambda, \hat{\nu} \rangle = (\lambda, \nu)$$

for all $\lambda \in \mathcal{L}^- \otimes \mathbb{C}$, then $(\mu, \nu) = (\hat{\mu}, \hat{\nu})$.

Suppose $v$ is a finite place of $F$. We define a map $\nu : \mathcal{A}(F_v) \to \hat{\mathcal{L}} \otimes \mathbb{Q}$ by the condition

$$|\lambda(a)|_v = |\tilde{\omega}_v|^{(\lambda, \nu(a))}$$

for all $\lambda \in \mathcal{A}$ and $a \in \mathcal{A}(F_v)$, where $\tilde{\omega}_v$ is the uniformizing element of $F_v$ and $|.|_v$ is the normalized valuation of $F_v$. For $\mu \in \mathcal{L} \otimes \mathbb{C}$, define $\hat{t}_\mu \in \mathcal{A}^0 = \text{Hom}(\hat{\mathcal{L}}, \mathbb{C}^*)$ by

$$\hat{t}_\mu(\tilde{\lambda}) = |\tilde{\omega}_v|^{(\mu, \tilde{\lambda})}$$

for all $\tilde{\lambda} \in \hat{\mathcal{L}}$. We sometimes write $\hat{t}$ for $\hat{t}_\mu$.

We write $L_F$ for the lattice of $F$-rational characters of $\mathcal{A}$. Similar notation will be used for the lattices $\mathcal{A}^0 \mathcal{L}^+$ etc.

1.4. Next we write down explicitly the Galois action on the derived group $\hat{G}'$ of $\hat{G}^0$. Put $\hat{\mathcal{A'}} = \hat{\mathcal{A}}^0 \cap \hat{G}'$. Let $\hat{\mathfrak{a}}$ be the Lie algebra of $\hat{\mathcal{A'}}$. Choose $H_1, \ldots, H_r \in \hat{\mathfrak{a}}$ so that

$$\lambda(H_i) = \langle \alpha_i, \lambda \rangle$$

where $\lambda \in \hat{\mathcal{L}}^+$ and $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ are the simple roots. Choose vectors $X_{\pm \hat{a}_i}$ belong to the $\pm \hat{a}_i$ respectively such that

$$[X_{\hat{a}_i}, X_{-\hat{a}_i}] = H_i.$$ 

For $\sigma \in \text{Gal}(E/F)$, $\hat{\sigma} \alpha = \sigma \alpha$ for $\alpha \in \Delta$. If we put $\sigma(\hat{a}_i) = \hat{a}_{\sigma(i)}$, then the Galois action on the Lie algebra $\mathfrak{g}'$ of $\hat{G}'$ is the unique isomorphism satisfying

$$\sigma(H_i) = H_{\sigma(i)}, \quad \sigma X_{\pm \hat{a}_i} = X_{\pm \hat{a}_{\sigma(i)}}$$


1.5. Let $\Sigma_F$ denote the set of $F$-roots of $G$ with respect to $A_d$, the
maximal $F$-split torus in $A$. As $G$ is quasi-split, each element of $\Sigma$ has a nontrivial restriction to $A_d$, and $\Sigma_F$ is equal to the set of restriction to $A_d$ of elements of $\Sigma$. In fact, if $G$ splits over a Galois extension $E$ of $F$, the Galois group $\text{Gal}(E/F)$ acts on $\Sigma$ and each orbit restricts to an element of $\Sigma_F$. In each orbit choose a representative $\alpha$ and denote the corresponding orbit by $O_\alpha$ and the element in $\Sigma_F$ to which the elements in $O_\alpha$ restrict, is denoted by $\alpha_F$, i.e. $\alpha_F = |A_d|.$

The Weyl group $W$ of $\Sigma$ is given by $N(A)/Z(A)$ while the rational Weyl group $W_F$ of $\Sigma_F$ is $N(A_d)/Z(A_d)$. We can identify $W_F$ as a subgroup of $W$.

Let $\Sigma_F$ be the reduced $F$-root system consisting of the indivisible $F$-roots of $\Sigma_F$, i.e. $\Sigma_F = \{\alpha_F \in \Sigma | \frac{1}{2} \alpha_F \not\in \Sigma_F\}$. $\Sigma_F = \Sigma_F \cap \Sigma_F^\vee$.

Next we define the elementary subgroup $G_{\alpha_F}$ of $G$ for $\alpha_F \in \Sigma_F$. Let $A_{\alpha_F} = (\ker \alpha_F)^0$. Then $G_{\alpha_F} = Z_G A_{\alpha_F}$ i.e. we take the centralizer in $G$ of $A_{\alpha_F}$.

It can be easily proved that $G_{\alpha_F}$ is connected reductive quasi-split group of semi simple $F$-rank 1.

1.6. There is a non-empty finite set $\mathcal{V}$ of places of $F$, containing all the infinite places such that the $F$-group $G$ can be regarded as defined above $\text{Spec}(\mathcal{O}_v)$, where $\mathcal{O}_v$ is the ring of the elements of $F$ which are integral outside $\mathcal{V}$. Thus $G(\mathcal{O}_v)$ is defined for those $v$ not in $\mathcal{V}$.

For $v \nmid \infty$, let $K_v$ be a maximal compact subgroup of $G_v$ such that $G_v = B_v \cdot K_v$ is an Iwasawa decomposition. For $v < \infty$, let $K_v$ be a special open maximal compact subgroup of $G_v$ in the sense of Bruhat-Tits [4]. In particular, for almost all $v$, $K_v$ can be taken to be $G(\mathcal{O}_v)$. Similar considerations can be given to $G_{\alpha_F}$. Therefore, when we consider the finite set $\{G, G_{\alpha_F}\}_{\alpha_F \in \Sigma_F}$ of groups taken together, except for a finite number of places, we have simultaneously

$$G_v = B_v G(\mathcal{O}_v)$$

(4)

$$G_{\alpha_F}(F_v) = B_{\alpha_F}(F_v) G_{\alpha_F}(\mathcal{O}_v)$$

where $\alpha_F \in \Sigma_F$.

Let us now fix $K_f = \prod_{v < \infty} K_v$, $K_v = \prod_{v \mid \infty} K_v$, $K = K \cdot K_f$. Then $G(A) = B(A) \cdot K$.

1.7. Let $X(G)$ be the lattice of rational characters on $G$. Let $L(s, G)$ be the Artin $L$-function corresponding to the $\text{Gal}(E/F)$-module $X(G) \otimes \mathbb{Q}$ and let $L_v(s, G)$ be its $v$-component.

Let $\chi$ be a nontrivial character on $A$ trivial on $F$. $\chi$ defines a
nontrivial character $\chi_v$ of $F_v$ at each place $v$ of $F$. Let $dx_v$ be the additive Haar measure on $F_v$ self-dual with respect to $\chi_v$ and let $dx = \prod_v dx_v$. For $v$ finite, the Haar measure on $F_v^*$ is chosen so that the measure of $0_v^*$ is one.

Let $\omega$ be an $F$-rational left-invariant nowhere vanishing exterior form of highest degree on $G$. For each $v$, $\omega$ and $dx_v$ defines a measure $|\omega|_v$ on $G_v$ (cf. [23]). We put $dg_v = L_v(1,G)|\omega|_v$, for finite $v$, and $dg_v = |\omega|_v$ for infinite $v$. Then the Tamagawa measure $dg$ on $G(\mathbb{A})$ is the Haar measure on $G(\mathbb{A})$ defined by

$$ (5) \quad dg = \lim_{s \to 1} (s-1)^{-1} L(s,G) \prod_v dg_v $$

where $r$ the rank of the lattice of $F$-rational characters $X(G)_F$ of $G$ (cf. [17]). This measure is independent of choice of $\chi$ and $\omega$.

Let $\chi_1, \ldots, \chi_r$ a basis of $X(G)_F$. Then the map $g \to ([\chi_1(g)], \ldots, [\chi_r(g)])$ defines a homomorphism $G(\mathbb{A}) \to (\mathbb{R}_+)^r$. Let $G^1(\mathbb{A})$ be the kernel of this homomorphism. Also, the restriction of $\chi_1, \ldots, \chi_r$ to the split component $Z_d$ of the radical of $G$ defines an $F$-homomorphism $\delta$ from $Z_d$ to $GL(1)^r$. This defines a homomorphism $\delta_\circ$ from the identity component of $Z_d^\circ$ to $GL(1)^r$. For each $t \in \mathbb{R}_+^r$, call $\xi(t)$ the idele $(\xi(t)_v)$ such that $\xi(t)_v = 1$ for every finite place and $\xi(t)_v = t$ for every infinite place. Then $t \to \xi(t)$ is an isomorphism of $\mathbb{R}_+^r$ onto a subgroup $GL^+(1)_\circ$ of $GL(1)_\circ$. Let $Z^\circ_+$ be the identity component of inverse image of $GL^+(1)_\circ$ under $\delta_\circ$. Then $Z^\circ_+$ is isomorphic to $(\mathbb{R}_+^r)^r$ and $G(\mathbb{A}) = G(\mathbb{A})^1 \times Z^\circ_+$. If we put the measure $dt = \wedge_{i=1}^r (dt_i/t_i)$ on $\mathbb{R}_+^r$, then

$$ (6) \quad dg = dg^1 \times dt $$

defines a Haar measure on $G^1(\mathbb{A})$. This measure is independent of choice of $\chi_1, \ldots, \chi_r$. The Tamagawa number $\tau(G)$ is the finite number defined by

$$ (7) \quad \tau(G) = \int_{G(F)G^1(\mathbb{A})} dg^1 = \int_{G(F)Z^\circ_+G(\mathbb{A})} dg. $$

1.8. Let $N$ be the unipotent radical of $B$. Then we can define Tamagawa measures $da$ (resp. $dn$) on $A(\mathbb{A})$ (resp. $N(\mathbb{A})$) as in the case of $G$. We normalize the measure on $K_v$ by the condition

$$ \int_{K_v} dk_v = 1. $$
Then we have \( dk = \Pi_v dk_v \) and

\[
\int_K dk = 1.
\]

Let \( \rho \) be the half sum of the positive roots of \( G \) with respect to \( A \). To simplify notation we write \( \rho \) for the quasi-character on \( A(F) \backslash A(A) \) determined by \( \rho \). Since \( G(A) = B(A) \cdot K = N(A)A(A)K \), there exists a positive constant \( \kappa \) such that for any \( f \in C_c(G(A)) \),

\[
(8) \quad \int_{G(A)} f(g) \, dg = \kappa \int_{N(A)A(A)K} f(nak) \rho^{-1}(a) \, dn \, da \, dk.
\]

According to the Bruhat decomposition of \( G \) we have

\[
(9) \quad G_v = \bigcup_{w \in \mathcal{W}_v} N_v A_v w N_v.
\]

But except for the Weyl group element \( w_0 \) that sends all the positive roots to negative roots, the cosets \( N A w N \) has lower dimension than that of \( G \), and so \( N A w N \) has measure zero. Thus if we write \( g_v = n_v a_v w_0 n'_v \), we have

\[
(10) \quad dg_v = \rho^{-1}(a) \, dn_v \, da_v \, dn'_v.
\]

where \( da_v \) is the local measure on \( A_v \) induced by \( |\omega|_v \).

2. Eisenstein series and \( M(w, \lambda) \)

2.1. For our purposes it is sufficient to consider the contribution to the spectral decomposition of \( L^2(Z^+_\mathbb{R}G(F) \backslash G(A)/K) \) from the Borel subgroup \( B \). We can define the adelic analogue of the function spaces \( \mathcal{E}(V, W) \), \( \mathcal{D}(V, W) \) and \( \mathcal{H}(\mathcal{D}(V, W)) \) of §2 and 3 of [13] with respect to the Borel subgroup \( B \), the trivial representation of \( K \) and a character \( \lambda \) of \( Z^+_\mathbb{R}A(F) \backslash A(A) \) which is trivial on the image of \( B(A) \cap K \) in \( N(A) \backslash B(A) \).

2.2. Define \( A^+_\mathbb{R} \) (resp. \( A(A)^1 \)) in the same way as \( Z^+_\mathbb{R} \) (resp. \( G(A)^1 \)). Let \( (Z^+_\mathbb{R}A(F) \backslash A(A))^* \) be the set of characters of \( Z^+_\mathbb{R}A(F) \backslash A(A) \). Fix a basis \( \{ \chi_i \} \) of \( L_F \). Each element \( \lambda = \Sigma s_i \chi_i \) of \( L_F \otimes \mathbb{C} \) can be considered as a character of \( Z^+_\mathbb{R}A(F) \backslash A(A) \) via the formula
In this way \( L_F \hat{\otimes} \mathbb{C} \) is identified with a subset of \( (Z^+_+A(F)\backslash A(\mathbb{A}))^\ast \).
From now on we shall consider only those \( \lambda \) in \( L_F \hat{\otimes} \mathbb{C} \).

Let \( \mathcal{E}(\lambda) \) be the space of continuous functions on \( N(\mathbb{A})B(F)\backslash G(\mathbb{A})/K \) satisfying the condition

\[
\Phi(ag) = \lambda(a)\rho(a)\Phi(g)
\]

for \( a \in A(\mathbb{A}), \ g \in G(\mathbb{A}) \).

Let \( \mathcal{H}(\lambda) \) be the space of functions \( \Phi(\cdot, g) \), with values in \( \mathcal{E}(\lambda) \), which is defined and analytic in a tube in \( L_F \hat{\otimes} \mathbb{C} \) over a ball of radius \( R \) with \( R > (\rho, \rho)^{1/2} \) and which goes to zero at infinity faster than the inverse of any polynomial.

2.3. Let \( D_0 \) be the unitary characters of \( Z^+_+A(F)\backslash A(\mathbb{A}) \). Then \( (Z^+_+A(F)\backslash A(\mathbb{A}))^\ast \) is also the union of sets of the form

\[
D_\sigma = \{ \chi \in (Z^+_+A(F)\backslash A(\mathbb{A}))^\ast \mid |\chi| = \sigma \}
\]

where \( \sigma \) is a fixed character with values in \( \mathbb{R}^+ \). We equip \( D_0 \) with the dual Haar measure via Pontrjagin duality and give \( D_\sigma \) the measure obtained by transport of structure from \( D_0 \).

We write \( \mathcal{D} \) for the space spanned by functions of the form

\[
\phi(g) = \int_{\text{Re } \lambda = \lambda_0} \Phi(\lambda, g) |d\lambda|
\]

where \( \Phi \in \mathcal{H}(\lambda) \) and \( \lambda_0 \) is a character with values in \( \mathbb{R}^+ \). By means of Fourier transform we get

\[
\Phi(\lambda, g) = \int_{Z^+_+A(F)\backslash A(\mathbb{A})} \phi(ag) \lambda^{-1}(a)\rho^{-1}(a) \, da.
\]

According to Langlands [13, 14], for \( \phi \in \mathcal{D} \) the theta series

\[
\tilde{\phi}(g) = \sum_{\gamma \in P(F)G(F)} \phi(\gamma g)
\]

belongs to \( L^2(Z^+_+G(F)\backslash G(\mathbb{A})) \). Combining with (2), we get

\[
\tilde{\phi}(g) = \int_{\text{Re } \lambda = \lambda_0} E(g, \Phi, \lambda) \, d\lambda
\]
is an Eisenstein series. It converges uniformly for \( g \) in compact subsets of \( G(\mathbb{A}) \) and \( \lambda \in L_F \otimes \mathbb{C} \) such that \( \text{Re}(\lambda, \alpha) > (\rho, \alpha) \) for every positive root \( \alpha \).

We define the constant term of the Eisenstein series \( E(g, \Phi, \lambda) \) by

\[
E_0(g, \Phi, \lambda) = \int_{N(F)N(\mathbb{A})} E(ng, \Phi, \lambda) \, dn.
\]

**2.4. PROPOSITION:** The constant term is given by the following formula:

\[
E_0(g, \Phi, \lambda) = \sum_{w \in W_F} M(w, \lambda) \Phi(\lambda, g)
\]

where \( W_F \) is the \( F \)-rational Weyl group of \( G \) and

\[
M(w, \lambda) \Phi(\lambda, g) = \int_{w^{-1}B(F)w \cap N(F)N(\mathbb{A})} \Phi(\lambda, wng) \, dn.
\]

**PROOF:** We have

\[
E_0(g, \Phi, \lambda) = \int_{N(F)N(\mathbb{A})} \sum_{B(F)G(\mathbb{F})} \Phi(\lambda, \gamma ng) \, dn.
\]

The proposition is immediate once we break up the sum over \( B(F)G(\mathbb{F}) \) into a sum over \( W_F = B(F)G(\mathbb{F})/N(\mathbb{F}) \) (Bruhat decomposition) and a sum over \( (w^{-1}B(F)w \cap N(\mathbb{F}))/N(\mathbb{F}) \).

**2.5.** We can define local version of \( \mathcal{E}(\lambda) \) as the space \( \mathcal{E}_v(\lambda) \) of continuous functions \( \Phi_v \) on \( N_v \backslash G_v/K_v \) satisfying

\[
\Phi_v(a_v g_v) = \lambda(a_v) \rho(a_v) \Phi(g_v)
\]

(here \( \rho(a_v) \) is to be interpreted as \( |\rho(a_v)|_v \)).

For \( \Phi \in \mathcal{E}(\lambda) \), we let \( \Phi_v \) denote its restriction to \( G_v \). Since \( \Phi \) is right invariant under \( K = \prod K_v \) where \( K_v = G(0_v) \) almost all \( v \), and
G(A) is the direct limit of G', we can write

\[ \Phi(g) = \prod \Phi_e(g_e). \]

(Here it is understood that \( \Phi(1) = 1 \).

Furthermore, \( M(w, \lambda) \) is a linear map from \( \mathcal{E}(\lambda) \) to \( \mathcal{E}(\lambda^w) \) where \( \lambda^w(a) = \lambda(waw^{-1}) \). In fact it is just multiplication by a constant to be calculated below. Moreover, \( M(1, \lambda) = 1 \) because \( \text{vol}(N(F) \backslash N(A)) = 1 \).

2.6. **Proposition:** Let \( N^w = w^{-1}Nw \cap N \) and \( N^w = w^{-1} \bar{N}w \cap N \) where \( \bar{N} \) is the unipotent subgroup opposite to \( N \). Define a linear transform \( M_\tau(w, \lambda) : \mathcal{E}_\tau(\lambda) \to \mathcal{E}_\tau(\lambda^w) \) by

\[
M_\tau(w, \lambda) \Phi(g) = \int_{N_\tau^w} \Phi(wng) \, dn
\]

for \( g \in G_\tau \). Then we have

\[
M(w, \lambda) = \prod M_\tau(w, \lambda).
\]

(Here one regard the \( M_\tau(w, \lambda) \) as complex numbers.)

**Proof:** First we have \( N = N^w \cdot N^w \). So

\[
N(F) \backslash N(A) = (N(F) \cdot N(A)) \cdot N^w(A).
\]

It follows that, for \( \Phi \in \mathcal{E}(\lambda) \)

\[
M(w, \lambda) \Phi(g) = \int_{N(F) \backslash N(A)} \Phi(wng) \, dn
\]

\[= \int_{N(F) \cap N(A)} \int_{N^w(A)} \Phi(wn_1 w^{-1} \cdot wn_2 g) \, dn_2 \, dn_1. \]

The formula (10) now follows from the above and the fact that we have normalized our measure such that

\[\int_{N(F) \cap N(A)} \, dn_1 = 1.\]
3. $M_v(w, \lambda)$ in the rank one case

3.1. We shall compute $M_v(w, \lambda)$ for those places $v$ of $F$ satisfying the following conditions:

(i) $G$ is a connected reductive quasi-split group over $F_v$.
(ii) $G$ splits over an unramified extension of $F_v$.
(iii) $G_v = B_v K_v$ and $K_v = G(0_v)$.
(iv) $G$ is of semisimple $F_v$-rank one.

Let us write $E_v$ for the unramified extension of $F_v$ over which $G$ splits and write $\tilde{\sigma}$ for the uniformizing element of both $E_v$ and $F_v$. We denote by $\sigma$ the Frobenius element in $\text{Gal}(E_v/F_v)$.

Under the assumption, the $F_v$-rational Weyl group $W_{F_v} = \{1, w_0\}$, where $w_0$ sends all the positive roots to negative roots. We know that

$$M_v(1, \lambda) = 1.$$ 

It remains to calculate $M_v(w_0, \lambda)$. As $E_v(\lambda)$ is one dimensional it suffices to calculate

$$M_v(w_0, \lambda) = M_v(w_0, \lambda) \Phi(\lambda, 1) = \int_{N_{\tilde{w}_0}} \Phi(\lambda, w_0 n) \, dn$$

where $\Phi(\lambda)$ is $E(\lambda)$ is chosen to satisfy

$$\Phi(\lambda, 1) = 1.$$ 

$G$ has $F_v$-rational rank 1 also implies that $L_{F_v} \otimes \mathbb{C}$ is isomorphic to $\mathbb{C}$ and hence can be replaced by the set $\{\rho^s \mid s \in \mathbb{C}\}$. Thus it suffices to consider $M(w_0, \rho^s)$. We define $\Phi(\rho^s)$ by:

$$\Phi(\rho^s, a) = |\rho(a)|_0^{s+1} \quad \text{if } a \in A_v,$$

$$\Phi(\rho^s, ngk) = \Phi(\rho^s, g) \quad \text{if } n \in N_v, k \in K_v.$$ 

Let us write $M(s)$ for $M(w_0, \rho^s)$. Then (1) becomes

$$M(s) = \int_{N_{\tilde{w}_0}} \rho^{s+1}(w_0 n) \, dn.$$ 

We can further assume that $w_0 \in K_v$, then changing variable by the map $n \rightarrow w_0 n w_0^{-1}$, we have

$$M(s) = \int_{\tilde{N}_v} \rho^{s+1}(\tilde{n}) \, d\tilde{n},$$
and
\[ \rho'(a) = (|\hat{\omega}|_{F_v}^{(\sigma(a))})'. \]

3.2. PROPOSITION: Let \( \hat{n} \) be the subspace of the Lie algebra of \( \hat{G} \) spanned by the positive root vectors. Then

\[
M(s) = \frac{\det(I - |\hat{\omega}|_{F_v}^{\sigma} \text{ Ad } \hat{t}_{|\hat{n}})}{\det(I - \sigma \text{ Ad } \hat{t}_{|\hat{n}})}
\]

where \( \hat{t} = \hat{t}_{sp} \).

Let \( G' \) be the derived subgroup of \( G \). Then the unipotent radical of the Borel subgroup of \( G' \) is the same as that of the corresponding Borel subgroup \( B \) of \( G \). Thus we only need to compute the integral \( M(s) \) for connected semisimple quasi-split groups of \( F_v \)-rank one. Henceforth, in this subsection we shall assume \( G \) to be of such type.

According to Steinberg’s variation of Chevalley’s theme, the quasi-split form of \( G \) is determined up to \( F_v \)-isomorphism by its Dynkin diagram and the twisted action of galois group (modulo inner twisting). As a result, up to central isogeny, \( G \) can only be one of the following types:

(I) \( G \) splits over \( G_v \) and has a connected Dynkin diagram, i.e. \( G = SL_2 \).

(II) \( G \) is a twisted form of a \( F_v \)-split group whose Dynkin diagram is type \( A_2 \), i.e. \( G(F_v) = SU_3(E_v/F_v) = \{ g \in SL_3(E_v) \mid 'gJg = J \} \) where \( E_v/F_v \) is a quadratic extension; the conjugation by the nontrivial element of the Galois group \( \text{Gal}(E_v/F_v) \) is denoted by \( x \rightarrow \bar{x} \); \( 'g \) is the conjugate-transpose of the matrix \( g : J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) is the matrix of the Hermitian form with respect to the nontrivial element of \( \text{Gal}(E_v/F_v) \).

(III) \( G \) is a twisted form of a \( F_v \)-split group whose Dynkin diagram consists of \( n \) copies of \( A_1 \), i.e. there exists an extension \( E_v/F_v \) of degree \( n \) and \( G(F_v) = SL_2(E_v) \).

(IV) \( G \) is a twisted form of \( F_v \)-split group whose Dynkin diagram consists of \( n \) copies of \( A_2 \); there exists field extensions \( E_v, E'_v \) of \( F \) such that \( [E_v : E'_v] = 2, [E'_v : F_v] = 2n \). If \( x \rightarrow \bar{x} \) is the nontrivial action of the Galois group \( \text{Gal}(E_v/E'_v) \) then \( G(F_v) = SU_3(E_v/E'_v) = \)
It is obvious that it suffices to calculate (2) up to isogeny (see for example [18] §4.3). Moreover Rapoport [18] pointed out that it is possible to avoid the calculation of (2) for the cases (III) and (IV) by proving a general lemma on the behaviour of (2) under restriction of ground field.

3.3. When $G$ is $\text{SL}_2$, it is well known that

$$M(s) = \frac{1 - q^{-(s+1)}}{1 - q^{-s}}.$$ 

The Lie algebra $\hat{\mathfrak{g}}$ in this case is one dimensional and it is trivial to check the formula (3). We shall omit the details.

3.4. PROPOSITION: Let $E_v/F_v$ be an unramified quadratic extension of local fields such that $2$ is a unit in $E_v$. Then for the quasi-split group $\text{SU}_3(E_v/F_v)$ we have

$$M(s) = \frac{(1 - q^{-2(s+1)})(1 + q^{-2s-1})}{(1 - q^{-2s})(1 + q^{-2s})} = \frac{\det(I - |\omega|_{F_v}\sigma \text{ Ad } \hat{t}|_{\hat{\mathfrak{g}}})}{\det(I - \sigma \text{ Ad } \hat{t}|_{\mathfrak{g}})}$$

PROOF: First we have

$$A(F_v) = \left\{ \begin{pmatrix} a & b \\ \bar{a}^{-1} \end{pmatrix} \bigg| \begin{array}{c} a, b \in E_v^* \\ b\bar{b} = 1, ab\bar{a}^{-1} = 1 \end{array} \right\},$$

$$N(F_v) = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & \bar{x} \\ 0 & 0 & 1 \end{pmatrix} \bigg| \begin{array}{c} y + \bar{y} + x\bar{x} = 0 \\ x, y \in E_v \end{array} \right\},$$

$$K = \text{SU}_3(0_{E_v}).$$

$E$ is an unramified quadratic extension of $F$, so there exists an element $c \in 0_{F_v} - \omega 0_{F_v}$ such that its image in $0_{F_v}/\omega 0_{F_v}$ is not a square and $E_v = F_v(\sqrt{c})$. Let the map $\text{ord}_{E_v} : F_v^* \to \mathbb{Z}$ be defined by the condition

$$|x|_{F_v} = |\omega|_{F_v}^{\text{ord}_{E_v} x} \quad \text{for} \ x \in F_v^*.$$
Similar condition defines $\text{ord}_{E_v}$. Note if $x \in F_v$, then $|x|_{E_v} = |x|_{F_v}^2$ implies $\text{ord}_{F_v} x = \text{ord}_{E_v} x$.

Next, let us determine the measure $\text{dn}$ on the nilpotent group $N(F_v)$. Let $x, y \in E_v$ such that $y + \bar{y} + xx = 0$. Then we can write $y = y_1 \sqrt{c} - \frac{xx}{2}$ where $y_1 \in F_v$. Note that $xx = N_{E_v/F_v}(x)$ also belongs to $F_v$.

A typical element of $N(F_v)$ can now be written as

\[
\begin{pmatrix}
1 & x & y \\
1 & -\bar{x} & 1
\end{pmatrix} = \begin{pmatrix}
1 & x & -\frac{xx}{2} \\
1 & -\bar{x} & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & y_1 \sqrt{c} \\
1 & 0 & 1
\end{pmatrix}.
\]

Thus we can write $N(F_v) = N_1 N_2$ (as sets) and take $\text{dn}$ to be the image of the product of the measure on $E_v$ and $F_v$ respectively under the maps;

\[
x \mapsto n_1 = \begin{pmatrix}
1 & x & -\frac{xx}{2} \\
1 & -\bar{x} & 1
\end{pmatrix}, \quad x \in E_v,
\]
\[
y_1 \mapsto n_2 = \begin{pmatrix}
1 & 0 & y_1 \sqrt{c} \\
1 & 0 & 1
\end{pmatrix}, \quad y_1 \in F_v.
\]

We normalize the measures on $E_v$ and $F_v$ by the condition that the volume of the respective maximal compact subrings is one.

The nontrivial element of the Weyl group corresponds to the matrix

\[
w_0 = \begin{pmatrix}
1 \\
-1 \\
1
\end{pmatrix}.
\]

We have

\[
\tilde{N}_v = \left\{ \begin{pmatrix}
1 & -\bar{x} & y \\
y & x & 1
\end{pmatrix} \middle| \begin{array}{l}
y + y + xx = 0 \\
x, y \in E_v
\end{array} \right\}.
\]

If $\bar{n} \in \tilde{N}_v$, then by Iwasawa decomposition of $\text{SU}_3(E_v/F_v)$, we get
\[ \tilde{n} = \begin{pmatrix} 1 & \tilde{x} & 1 \\ -\tilde{x} & 1 & 1 \end{pmatrix} = n \begin{pmatrix} \tilde{a}^{-1} & b & 0 \\ 0 & a & 0 \end{pmatrix} k \]

for some \( n \in N_v, k \in K_v \).

As noted we can write \( y = y_1\sqrt{c - \frac{xx}{2}} \) for some \( y_1 \in F \).

Then \( \text{ord}_E y = \inf(\text{ord}_E y_1, 2 \text{ord}_E x) \) and

\[ |a|_{E_v} = |\tilde{a}|_{E_v} \inf(0, \text{ord}_E x, \text{ord}_E y). \]

The zero in the "inf" is put into account for the case when both \( x \) and \( y \) are integral, and \( \tilde{n} \in K_v \).

Direct calculation using the definition of \( \rho^s \) gives

\[ \rho^s \begin{pmatrix} a & b \\ \tilde{a}^{-1} \end{pmatrix} = |a|_{E_v}, \quad s \in \mathbb{C}. \]

To calculate the value of \( \rho^{s+1}(\tilde{n}) \), we have to consider four cases:

1. \( \text{ord}_E x \geq 0 \) and \( \text{ord}_E y_1 \geq 0 \)
   \[ \Rightarrow \text{ord}_E y \geq 0 \]
   \[ \Rightarrow \inf(0, \text{ord}_E x, \text{ord}_E y) = 0 \]
   \[ \Rightarrow \rho^{s+1}(\tilde{n}) = 1. \]

2. \( 2 \text{ord}_E x \geq \text{ord}_E y_1, \text{ord}_E y_1 < 0, \text{ord}_E y_1 \) is even.
   If \( \text{ord}_E x \geq 0 \) then \( \text{ord}_E y_1 < \text{ord}_E x \).
   If \( \text{ord}_E x < 0 \) then \( \text{ord}_E y_1 = 2 \text{ord}_E x < \text{ord}_E x \).
   Thus \( \inf(0, \text{ord}_E x, \text{ord}_E y) = \text{ord}_E y_1 \) and
   \[ \rho^{s+1}(\tilde{n}) = |\tilde{a}^{-1}|_{E_v}^{s+1} = q^{2(s+1)\text{ord}_E y_1}. \]

Note: if \( \text{ord}_E y_1 = -2m \) then

\[ \text{ord}_E x \geq \frac{\text{ord}_E y_1}{2} = -m. \]

3. \( 2 \text{ord}_E x \geq \text{ord}_E y_1 < 0, \text{ord}_E y_1 \) is odd
   \[ \Rightarrow \inf(0, \text{ord}_E x, \text{ord}_E y) = \text{ord}_E y_2 \]
   \[ \Rightarrow \rho^{s+1}(\tilde{n}) = q^{2(s+1)\text{ord}_E y_1}. \]
Note: if \( \text{ord}_{E_v} y_1 = -(2m - 1), m \geq 1 \) then
\[
\text{ord}_{E_v} x = -m + \frac{1}{2} \quad \text{or} \quad \text{ord}_{E_v} x \geq -(m - 1).
\]

4. \( \text{ord}_{E_v} x < \text{ord}_{E_v} y_1, \text{ord}_{E_v} x < 0 \Rightarrow \text{ord}_{E_v} y = 2 \text{ord}_{E_v} x \Rightarrow \rho^{s+1}(\bar{n}) = q^{2(s+1)2\text{ord}_{E_v} x} \).

Note: if \( \text{ord}_{E_v} x = -m \) then \( \text{ord}_{E_v} y_1 > -2m \geq -(2m - 1) \).

Now we are ready to calculate the integral \( M(s) \). We break the integral up into four pieces corresponding to the four cases above and transfer the integral over \( \bar{N}(F_v) \) to those over \( E_v \times F_v \), viz.,

\[
M(s) = \int_{\bar{N}(F_v)} \rho^{s+1}(\bar{n}) \, d\bar{n} = \int_{\bar{N}_1} \int_{\bar{N}_2} \rho^{s+1}(\bar{n}_1, \bar{n}_2) \, d\bar{n}_1 \, d\bar{n}_2
\]

\[
= \int_{0_{E_v}} \int_{0_{F_v}} dx \, dy_1 + \sum_{m=1}^{\infty} \int_{P_{E_v}^{2m}} \int_{P_{F_v}^{2m}} q^{(s+1)2(2m)} \, dx
\]

\[
+ \sum_{m=1}^{\infty} \int_{P_{E_v}^{m}P_{F_v}^{2m-2}} \int_{P_{E_v}^{2m-2}P_{F_v}^{2m-2}} q^{(s+1)(2m-1)} \, dx \, dy_1
\]

\[
+ \sum_{m=1}^{\infty} \int_{P_{E_v}^{2m-2}P_{F_v}^{2m-2}} \int_{P_{E_v}^{2m-1}} q^{(s+1)(2m-2)} \, dx \, dy_1
\]

where \( P_{E_v} \) (resp. \( P_{F_v} \)) is the maximal prime ideal of \( E_v \) (resp. \( F_v \)). We normalized measure on \( E_v, F_v \) by \( \int_{0_{E_v}} dx = 1 \) and \( \int_{0_{F_v}} dy_1 = 1 \).

Further calculation gives

\[
\int_{0_{E_v}} \int_{0_{F_v}} dx \, dy_1 = 1.
\]

\[
\sum_{m=1}^{\infty} \int_{P_{E_v}^{m}P_{E_v}^{2m-2}} q^{(s+1)(2m-1)} \, dx \, dy_1
\]

\[
= \sum_{m=1}^{\infty} q^{2m} (q^{2m} - q^{2m-1})q^{-4(s+1)m},
\]

\[
= (1 - q^{-1}) \sum_{m=1}^{\infty} (q^{-4s})^m = \frac{(1 - q^{-1})q^{-4s}}{1 - q^{-4s}}.
\]
Adding all the terms, we have

$$\sum_{m=1}^{\infty} \int_{\mathbb{P}^{m-1}} \int_{\mathbb{P}^{m-1}} q^{2m-2} (q^{2m-1} - q^{2m-2}) q^{-2(s+1)(2m-1)} \, dx \, dy$$

$$= \sum_{m=1}^{\infty} q^{2m-2} (q^{2m-1} - q^{2m-2}) q^{-2(s+1)(2m-1)}$$

$$= (q^{-1} - q^{-3}) q^{2s} \sum_{m=1}^{\infty} (q^{-4s})^m$$

$$= \frac{(q^{-1} - q^{-3}) q^{-2s}}{1 - q^{-4s}}.$$

Adding all the terms, we have

$$M(s) = \frac{(1 - q^{-2s-2})(1 + q^{-2s-1})}{(1 - q^{-2s})(1 + q^{-2s})}.$$

To complete the proof of the proposition, let us look at the Lie algebra $\hat{\mathfrak{g}}$ of the analytic group $\hat{G}$ associated with $G$. We can take $\hat{\mathfrak{g}}$ to be $\mathfrak{sl}(2, \mathbb{C})$ and let $\Sigma^+ = \{\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3\}$, $\hat{\alpha}_3 = \hat{\alpha}_1 + \hat{\alpha}_2$. There exists root vectors $X_{\hat{\alpha}_1}, X_{\hat{\alpha}_2}, X_{\hat{\alpha}_3}$ such that

$$[X_{\hat{\alpha}_1}, X_{\hat{\alpha}_2}] = X_{\hat{\alpha}_3}.$$

$\hat{\mathfrak{g}}$ has a Dynkin diagram of type $A_2$

the arrows indicate the action of $\sigma \in \text{Gal}(E/F)$, i.e. $\sigma(X_{\hat{\alpha}_1}) = X_{\hat{\alpha}_2}$.

Since this action is to be extended to a Lie algebra isomorphism, i.e. $\sigma[X_{\hat{\alpha}_1}, X_{\hat{\alpha}_2}] = [\sigma X_{\hat{\alpha}_1}, \sigma X_{\hat{\alpha}_2}]$, so $\sigma X_{\hat{\alpha}_3} = [X_{\hat{\alpha}_2}, X_{\hat{\alpha}_1}] = -X_{\hat{\alpha}_3}$. 
Also, we have

$$(\text{Ad } \hat{t})X_{\hat{\alpha}} = \hat{\alpha}(\hat{t})X_{\hat{\alpha}} = |\hat{\omega}|_{F_v}^{(\varphi, \hat{\alpha})}X_{\hat{\alpha}}$$

$$= |\hat{\omega}|_{F_v}^{\hat{\alpha}}X_{\hat{\alpha}} \quad \text{if } \hat{\alpha} = \hat{\alpha}_1 \text{ or } \hat{\alpha}_2,$$

or

$$= |\hat{\omega}|_{F_v}^{\hat{\alpha}_3}X_{\hat{\alpha}} \quad \text{if } \hat{\alpha} = \hat{\alpha}_3,$$

because $\langle \rho, \hat{\alpha} \rangle = \frac{2(\rho, \alpha)}{\langle \alpha, \alpha \rangle} = 1$ if $\alpha$ simple and

$$\langle \rho, \hat{\alpha}_3 \rangle = \langle \rho, \alpha_1 \rangle + \langle \rho, \alpha_2 \rangle = 2.$$

We take $\hat{n} = CX_{\hat{\alpha}_1} + CX_{\hat{\alpha}_2} + CX_{\hat{\alpha}_3}$. Then

$$\text{det}(I - \sigma \text{ Ad } \hat{t}|_{\hat{n}})$$

$$= \text{det} \left( I - \begin{pmatrix} 0 & |\hat{\omega}|_{F_v}^{\hat{\alpha}} & 0 \\ |\hat{\omega}|_{F_v}^{\hat{\alpha}} & 0 & 0 \\ 0 & 0 & -|\hat{\omega}|_{F_v}^{\hat{\alpha}_3} \end{pmatrix} \right),$$

$$= (1 - q^{-2s})(1 + q^{-2s}),$$

and

$$\text{det}(I - |\hat{\omega}|_{F_v}^{\alpha} \sigma \text{ Ad } \hat{t}|_{\hat{n}})$$

$$= (1 - q^{-2s-1})(1 + q^{-2s-1}).$$

This completes the proof of the proposition.

3.5. Let us now consider the case (III). $G$ is a connected semi-simple quasi-split algebraic group defined over $F_V$ splits over an unramified extension $E_v/F_v$ of degree $n$.

The absolute Dynkin diagram of $G$ consists of $n$ copies of $A_1$, and the action of the Frobenius $\sigma$ in $\text{Gal}(E_v/F_v)$ is the cyclic permutation as indicated

![Diagram](image)

The action has only one orbit; $G$ is of $F$-rank 1 and $G(F_v) = \text{SL}_2(E_v)$. The integral that we are interested in becomes $M(s) = \int_{\hat{N}_e} \rho^{s+1}(\hat{n}) \, d\hat{n}$.
where

\[ \tilde{N}_e = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \middle| x \in E_e \right\}, \]

\[ A_e = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a \in E_v^\ast \right\}, \]

and

\[ |a|_{E_v}^{s+1} = \rho^{s+1} \begin{pmatrix} a \\ a^{-1} \end{pmatrix}. \]

So by §3.3

\[ M(s) = \frac{1 - q_{E_v}^{-(s+1)}}{1 - q_{E_v}^{-s}} \quad \text{where } q_{E_v} = \text{modulus of } E_v = q^n, \]

\[ = \frac{1 - q^{-n(s+1)}}{1 - q^{-ns}} . \]

But on the other hand \( \hat{n} = \hat{s}l_2 \times \ldots \times \hat{s}l_2 \). Let \( X_{\hat{\alpha}_i} \) be the root vector corresponding to the positive root \( \hat{\alpha}_i \) of the \( i \)th copy of \( \hat{s}l_2 \) in the product. Then

\[ (\text{Ad } \hat{i}) X_{\hat{\alpha}_i} = \hat{\alpha}_i (\hat{i}) X_{\hat{\alpha}_i} = |\hat{\alpha}|_{E_v}^{s \langle \rho \hat{\alpha}_i \rangle} X_{\hat{\alpha}_i} = q^{-s} X_{\hat{\alpha}_i} \]

because \( \rho = \frac{1}{2} \sum \alpha_i \) and as the diagram is disconnected \( \langle \alpha_j, \hat{\alpha}_i \rangle = 0 \) if \( i \neq j \), and \( \langle \frac{\alpha_i}{2}, \hat{\alpha}_i \rangle = 1 \). So,

\[ \det(I - \text{Ad } \hat{i}|_{\hat{\alpha}}) = \begin{pmatrix} 1 & & & -q^{-s} & 1 \\ & 1 & & \ddots & \ddots \\ & & \ddots & 1 \\ -q^{-s} & & \ddots & -q^{-s} & 1 \\ 1 & -q^{-s} & & & 1 \end{pmatrix}, \]

\[ = 1 - q^{-ns}. \]

Similarly,

\[ \det(I - |\hat{\alpha}|_{E_v} \sigma \text{ Ad } \hat{i}|_{\hat{\alpha}}) = 1 - q^{-n(s+1)}, \]

and we are done.
3.6. Finally, let us look at the last case IV. Here \( G \) is a \( F_v \)-form of a split group with a Dynkin diagram consisting of \( n \) copies of \( A_2 \). \( G \) is defined over \( F_v \) splits over an unramified extension \( E_v \) of degree \( 2n \); there exists a field \( E'_v \) in \( E_v/F_v \) such that \([E'_v:F_v] = n\); the non-trivial element of \( \text{Gal}(E_v/E'_v)(\subset \text{Gal}(E_v/F_v)) \) give rise to the twisting; the action of this element is shown in the diagram

\[
\begin{array}{cccc}
\alpha_1 & \beta_1 & \cdots & \alpha_n \\
\alpha_2 & \beta_2 & & \\
& & \cdots & \\
& & & \beta_n
\end{array}
\]

This determines a special unitary group \( SU_3(E_v/E'_v) \) with respect to the form

\[
J = \begin{pmatrix} 
1 \\
1
\end{pmatrix}
\]

such that

\[
G(F) = SU_3(E_v/E'_v) = \{g \in SL_3(E_v) \mid ^t\bar{g}g = J\}.
\]

Thus, using the result in §3.4, we get

\[
M(s) = \frac{(1 - q^{-2n(s+1)})(1 + q^{-n(2s+1)})}{(1 - q^{-2ns})(1 + q^{-2ns})}
\]

(Note: modulus of \( E_v = q^{2n} \).

To establish the formula

\[
M(s) = \frac{\det(I - [\bar{\omega}]_F \sigma \ Ad \hat{\tau} |_\delta)}{\det(I - \sigma \ Ad \hat{\tau} |_\delta)}
\]

we shall evaluate the determinants directly.

Let us denote the simple root system \( \Delta \) by \( \{\alpha_1, \beta_1; \ldots; \alpha_n, \beta_n\} \). We calculate

\[
(Ad \hat{\tau})X_{\alpha_i} = \hat{\alpha}(\hat{\tau})X_{\alpha_i} = [\bar{\omega}]_F \hat{\tau}(\bar{\alpha})X_{\alpha_i} = q^{-s}X_{\alpha_i}.
\]
Here \( \rho = \frac{1}{2} \sum_{i=1}^{n} (\alpha_i + \beta_i + (\alpha_i + \beta_i)), \)

\[
\langle \rho, \hat{\alpha}_i \rangle = \sum_{j=1}^{n} \langle \rho_j, \hat{\alpha}_i \rangle \quad \text{where} \quad \rho_j = \alpha_j + \beta_j,
\]

because \( i \neq j \)

\[
\langle \rho_j, \hat{\alpha}_i \rangle = 0,
\]

and

\[
\langle \rho_i, \hat{\alpha}_i \rangle = 1.
\]

Similarly

\[
(\text{Ad } \hat{i})X_{\hat{\alpha}_i} = q^{-s}X_{\hat{\beta}_i},
\]

and

\[
(\text{Ad } \hat{g})X_{\hat{\alpha}_i + \hat{\beta}_i} = q^{-2s}X_{\hat{\alpha}_i + \hat{\beta}_i}.
\]

Next we write down the effect of the Galois action as indicated by the arrows in the above diagram. For \( 1 \leq i \leq n - 1, \)

\[
\sigma X_{\hat{\alpha}_i} = X_{\hat{\alpha}_{i+1}},
\]

\[
\sigma X_{\hat{\beta}_i} = X_{\hat{\beta}_{i+1}},
\]

\[
\sigma X_{\hat{\alpha}_i + \hat{\beta}_i} = \sigma[X_{\hat{\alpha}_i}, X_{\hat{\beta}_i}] = [\sigma X_{\hat{\alpha}_i}, \sigma X_{\hat{\beta}_i}]
\]

\[
= [X_{\hat{\alpha}_{i+1}}, X_{\hat{\beta}_{i+1}}] = X_{\hat{\alpha}_{i+1} + \hat{\beta}_{i+1}},
\]

and

\[
\sigma X_{\hat{\delta}_n} = X_{\hat{\beta}_1},
\]

\[
\sigma X_{\hat{\beta}_n} = X_{\hat{\delta}_1},
\]

\[
\sigma X_{\hat{\delta}_n + \hat{\beta}_n} = [\sigma X_{\hat{\delta}_n}, \sigma X_{\hat{\beta}_n}] = [X_{\hat{\beta}_1}, X_{\hat{\delta}_1}] = -X_{\hat{\delta}_1 + \hat{\beta}_1}.
\]

If we take the basis of \( \mathfrak{n} \) to be \( X_{\hat{\alpha}_1}, X_{\hat{\beta}_1}, X_{\hat{\alpha}_1 + \hat{\beta}_1}, \ldots, X_{\hat{\alpha}_n}, X_{\hat{\beta}_n}, X_{\hat{\delta}_n + \hat{\beta}_n} \) (in that order), then it is trivial to show that

\[
\det(I - \sigma \text{ Ad } \hat{g}|_{\mathfrak{n}})
\]

\[
= (1 - q^{-2s})(1 + q^{-2s}),
\]
and

$$\det(I - \sigma \ Ad \ \hat{\mathfrak{t}}_{\lambda})$$

$$= (1 - q^{-2n(s+1)})(1 + q^{-n(2s+1)}).$$

Thus the required formula is proved. With this we complete the proof of Proposition 3.2.

4. Reduction to rank one

To determine the local factor $M_v(w, \lambda)$ for almost all $v$ for $G$ of arbitrary $F$-rank, we use the method of reduction to $F$-rank one which was first studied by Bhanu-Murti [1] and was extended by Gindikin and Karpelevich [6]. This method has also been used in Langlands’ Euler Product (Yale, 1971) and in the thesis of Jacquet (Paris) and Lai (Yale). Here we shall follow Shiffmann [19].

4.1. We want to calculate the integral (9) of §2. For $\lambda \in L_F \otimes \mathbb{C}$, $\mathcal{E}(\lambda) \neq 0$ and so $\mathcal{E}_v(\lambda) \neq 0$ for all $v$. We have $W_F \subseteq W_F$. We can consider $w$ as an element of $W_F$ and do the rest of the calculation over $F_v$. Moreover for almost all $v$, $\mathcal{E}_v(\lambda)$ is one dimensional. It is sufficient to evaluate the integral for the following function in $\mathcal{E}_v(\lambda)$:

(1) $$\Phi(g_v) = |\lambda(a_v)\rho(a_v)|_v$$

where $g_v = n_v a_v k_v \in G_v$. The linear transformation $M_v(w, \lambda)$ is just multiplication by the following constant which we also denoted by $M_v(w, \lambda)$:

$$M_v(w, \lambda) = \int_{N_v^w} \Phi(wn) \ dn.$$

Changing the variable by $n \to w^{-1}nw$ and writing $N_v^w = wN^w w^{-1} = wNw^{-1} \cap N$, we have

(2) $$M_v(w, \lambda) = \int_{N_v^w} \Phi(nw) \ dn.$$  

Recall that the length $\ell(w)$ of $w$ is the smallest integer $g$ of such that there exists $g$ simple $F_v$-roots $\beta_1, \ldots, \beta_g$ with
(3) \[ w = s_{\beta_1}, \ldots, s_{\beta_g} \]

\((s_{\alpha_j})\) is the symmetry with respect to \(\alpha_j\). Moreover the \(F_v\)-roots \(\alpha_j = s_{\beta_{\ell(w)}}, \ldots, s_{\beta_{\ell+1}(\beta_j)}\) \(j = 1, \ldots, \ell(w)\) are positive and if we write

\[ 0\Sigma^{\ell(w)}_F(w) = \{ \alpha \in 0\Sigma^+ F | \alpha < 0 \} \]

then

\[ 0\Sigma^{\ell(w)}_F(w) = \{ \alpha_1, \ldots, \alpha_{\ell(w)} \}. \]

We quote the following lemma from Schiffmann ([19], Prop. 1.3).

4.2. **Lemma:** Let \( w, w', w'' \) be three elements of \( w_F \) such that \( w = w'w'' \) with \( \ell(w) = \ell(w') + \ell(w'') \). Then the map \((n', n'') \rightarrow n'(w'n''w'^{-1})\) defines a variety isomorphism \( \bar{N}^w \times \bar{N}^{w'} \rightarrow \bar{N}^{w''} \).

4.3. Using the above lemma, and assuming the integrals involve converges, we have

\[ M_v(w, \lambda) = \int_{\bar{N}^w} \Phi(n'w'n''w'^{-1}w) \, dn' \, dn'', \]

\[ = \int_{\bar{N}^w} M_v(w', \lambda) \Phi(n''w'') \, dn'', \]

and so

\[ M_v(w, \lambda) = M_v(w', \lambda''') M_v(w'', \lambda). \]

If we write \( w \) as a product of symmetries (as in (3)) then formula (5) allows us to reduce the calculation to the case \( \ell(w) = 1 \), i.e. the \( F \)-rank one case, and in this case the convergence follows from the explicit formula given in §3. To summarize we have

4.4. **Proposition:** Let \( N_\alpha = G_\alpha \cap N \) for \( \alpha \in 0\Sigma^+_F \) and \( \bar{N}_\alpha \) the unipotent subgroup of \( G_\alpha \) opposite to \( N_\alpha \). Then the integral (2) converges for \( \lambda \in L_F \otimes \mathbb{C} \) with \( \text{Re}(\lambda, \hat{\alpha}) > 0 \) for all \( \alpha \in 0\Sigma^+_F(w) \),

\[ M_v(w, \lambda) = \prod_{\alpha \in 0\Sigma^+_F(w)} \int_{\bar{N}_\alpha(F_v)} \Phi_\alpha(\bar{n}) \, d\bar{n} \]

where \( \Phi_\alpha \) is the restriction of \( \Phi \) to \( G_\alpha \).
4.5. As each $G_a$ has $F_v$-rank one we can apply Proposition 3.2 to get

\[ \int_{\hat{h} \in (F_v)_{\hat{n}}(w)} \Phi_{\alpha}(\hat{h}) \, d\hat{h} = \frac{\det(I - |\hat{\omega}|_v \sigma \text{ Ad } \hat{\ell}|_{\hat{h}_a})}{\det(I - \sigma \text{ Ad } \hat{\ell}|_{\hat{h}_a})}. \]

Let $\hat{h}$ be the nilpotent subalgebra of $\hat{a}$ spanned by $\hat{a}_a$ for $\alpha \in \omega \Sigma^+_{F_v}(w)$. The action of $\sigma \text{ Ad } \hat{\ell}$ on $\hat{h}_w$ preserves the subspaces $\hat{h}_a$. Hence

\[ \frac{\det(I - |\hat{\omega}|_v \sigma \text{ Ad } \hat{\ell}|_{\hat{h}_w})}{\det(I - \sigma \text{ Ad } \hat{\ell}|_{\hat{h}_w})} = \prod_{\alpha \in \omega \Sigma^+_{F_v}(w)} \frac{\det(I - |\hat{\omega}|_v \sigma \text{ Ad } \hat{\ell}|_{\hat{h}_a})}{\det(I - \sigma \text{ Ad } \hat{\ell}|_{\hat{h}_a})}. \]

The following proposition follows immediately from (6), (7) and (8).

4.6. **Proposition:** For almost all $v$, we have

\[ M_v(w, \lambda) = \frac{\det(I - |\hat{\omega}|_v \sigma \text{ Ad } \hat{\ell}|_{\hat{h}_w})}{\det(I - \sigma \text{ Ad } \hat{\ell}|_{\hat{h}_w})} \]

where $\sigma$ is the Frobenius and $\hat{\ell} = \hat{\ell}_\lambda$.

5. Value of the local factor at one

5.1. Let $\mathcal{S}$ be a finite set of places of $F$ containing all the infinite place of $F$, all the ramified places of $F$ and all the places at which the conditions (i) to (iii) of §3.1 are not satisfied. Let us write

\[ M_{\mathcal{S}}(s) = \prod_{v \in \mathcal{S}} M_v(w_0, \rho^s) \]

where $s \in \mathbb{C}$ and $w_0 \in W_F$ sends all positive roots to negative roots. Then $M_{\mathcal{S}}(1)$ can be considered as a linear map $E_{\mathcal{S}}(\rho) \to E_{\mathcal{S}}(\rho^{-1})$ and

\[ M_{\mathcal{S}}(1) \Phi(g) = \int_{N_{\mathcal{S}}} \Phi(w_0 n g) \, dn \]

for $\Phi \in E_{\mathcal{S}}(\rho)$, $g \in G_{\mathcal{S}}$. Now $G_{\mathcal{S}} = B_{\mathcal{S}} K_{\mathcal{S}}$ implies that $E_{\mathcal{S}}(\rho)$ is one dimensional and $M_{\mathcal{S}}(1)$ is just multiplication by a constant which we also
denoted by $M_{\mathfrak{g}}(1)$. We have

\begin{equation}
M_{\mathfrak{g}}(1) = \int_{N_{\mathfrak{g}}} \rho^2(w_0 n) \, dn.
\end{equation}

5.2. Let $L(s, G)$ be the Artin $L$-function of the Galois action on the rational characters of $G$, $L_v(s, G)$ be the local factor at $v$ of $L(s, G)$ and

\[ \mu_G = \lim_{s \to 1} (s - 1)^{r_G} L(s, G) \]

where $r_G$ is the rank of $X(G)_F$. Similar definitions are made with $A$ replacing $G$.

**PROPOSITION:** For $\mathfrak{S}$ sufficiently large we have

\begin{equation}
M_{\mathfrak{S}}(1) = \kappa \frac{\mu_G}{\mu_A} \prod_{v \in \mathfrak{S}} \frac{L_v(1, A)}{L_v(1, G)} \prod_{v \not\in \mathfrak{S}} \text{vol } K_v
\end{equation}

where the $\text{vol } K_v$ is calculated by the local measure $d_{g_v}$.

**PROOF:** Let $h$ be an integrable function on $N_{\mathfrak{S}} + A_{\mathfrak{S}}$. Let $f$ be a function on $G(\mathbb{A})$ which vanishes at $g$ except if $g_v \in K_v$ for all $v \not\in \mathfrak{S}$ and if the latter condition is satisfied, we have

\[ f(g) = f(g_{\mathfrak{S}}) = h(n, a) \]

for $g = n a k$. First of all we have

\begin{equation}
\int_{G(\mathbb{A})} f(g) \, dg = \kappa \int_{N_{\mathfrak{S}} A_{\mathfrak{S}}} h(n_2, a_2) \rho^{-2}(a_2) \, dn_2 \, da_2.
\end{equation}

On the other hand, suppose that $g_{\mathfrak{S}}$ lies in the large cell $N_{\mathfrak{S}} S_{\mathfrak{S}} w_0 N_{\mathfrak{S}}$ of the Bruhat decomposition: $g_{\mathfrak{S}} = n_2 a_2 w_0 n_1$ where $a_2 \in A_{\mathfrak{S}}$ and $n_1, n_2 \in N_{\mathfrak{S}}$ and if we write $w_0 n_1 = n(n_1) a(n_1) k$ with $n(n_1) \in N_{\mathfrak{S}}$ and $a(n_1) \in A_{\mathfrak{S}}$, then $g_{\mathfrak{S}} = n_2 a_2 n(n_1) a_2^{-1} a_2 a(n_1) k$ and

\begin{equation}
\int_{G(\mathbb{A})} f(g) \, dg = \prod_{v \not\in \mathfrak{S}} \text{vol } K_v \int_{N_{\mathfrak{S}} A_{\mathfrak{S}} N_{\mathfrak{S}}} h(n_2 a_2 n(n_1) a_2^{-1}, a_2 a(n_1)) \rho^{-2}(a_2) \, dn_2 \, da_2 \, dn_1.
\end{equation}
After changing the measures, the integral in the above formula becomes

\[ \int_{N_p A \cap N_p} \rho^2(a(n_1)) h(n_2, a_2) \rho^{-2}(a_2) \, dn_2 \, \overline{d a_1} \, dn_1. \]

Substitute this and

\[ d a_2 = \left( \prod_{v \in S} L_v(1, A) \right) \overline{d a_2} \]

into (5). Comparing the result with (4), we obtain (3) by noting that the choice of \( h \) is arbitrary.

5.3. **Corollary:** For \( v \not\in S \), if we write

\[ M_v(1) = M_v(w_0, \rho) = \int_{N_v} \rho^2(w_0 n) \, dn \]

then

\[ M_v(1) = \text{vol}(K_v) \cdot L_v(1, A)/L_v(1, G). \tag{6} \]

**Proof:** Apply the proposition to \( S' = S \cup \{v\} \). The corollary then follows immediate form

\[ M_{S'}(1) = M_v(1)M_{S}(1). \]

5.4. **Remark:** We have followed Rapoport [18] in the proof of corollary 5.3. An alternative approach is given in my thesis (Yale 1974) in which (6) is deduced from (9) of §4 by calculating directly \( \text{vol}(K_v) \) via reduction mod \( v \).

6. The constant functions

We calculate in this section the projection of \( \mathcal{E} \) into the subspace of constant functions in \( L^2(Z^*_v G(F)) \backslash G(A) \).

6.1. Let \( \mathcal{L} \) be the closed subspace of \( L^2(Z^*_v G(F)) \backslash G(A) \) generated by \( \tilde{\phi} \) for \( \phi \in \mathcal{B} \). Write \( \mathcal{H} \) for the union of \( \mathcal{H}(\lambda) \) for all \( \lambda \) in \( L_F \otimes \mathbb{C} \).

Suppose that \( f \) is a complex valued function defined, bounded and
analytic in a tube in $L_F \otimes \mathbb{C}$ over a ball of radius $R$ with centre at zero and $R > (\rho, \rho)^{1/2}$. Assume also that $f(w, \lambda) = f(\lambda)$ for all $w \in W_F$. Then

$$\Phi \rightarrow \Psi = f\Phi$$

where $\Psi(\lambda, g) = f(\lambda)\Phi(\lambda, g)$, defines a linear map on $H$ and induces a bounded linear operator

$$\Lambda(f) : \phi \rightarrow \psi$$

on $L$. If $a > (\rho, \rho)$ and $f(\lambda) = (a - (\lambda, \lambda))^{-1}$, then $\Lambda(f)$ is self-adjoint. We define

$$A = a - \Lambda(f)^{-1}.$$  

It is an unbounded self-adjoint operator on $L$ ($A$ is introduced in Langlands [14] §6 and [15]). It is obvious that if $\Psi(\lambda, g) = (\lambda, \lambda)\Phi(\lambda, g)$ then $A\Phi = \psi$. The following two lemmas and the corollary are easy to prove.

6.2. **Lemma:** Let $(, )$ be the inner product on $L^2(Z^*G(F)) \backslash G(A)$ and $1$ be the constant function. For $\tilde{\phi} \in L$, we have

$$(\tilde{\phi}, 1) = \kappa \Phi(\rho, 1).$$

6.3. **Lemma:** For $\tilde{\phi} \in L$ and $A$ as defined above we have

$$(A\tilde{\phi}, 1) = (\rho, \rho)(\tilde{\phi}, 1).$$

6.4. **Corollary:** $A1 = (\rho, \rho)1$.

6.5. For $z \in \mathbb{C}$, let $R(z, A) = (z - A)^{-1}$ be the resolvent of $A$. For $\lambda_0 \in L_F \otimes R$ if $Re z > (\lambda_0, \lambda_0)$, then it is easy to show that

$$(R(z, A)\tilde{\phi}, \psi) = \kappa \sum_{w \in W_F} \int_{|\lambda| = \lambda_0} \frac{M(w, \lambda)\Phi(\lambda)\overline{\Psi(-w\lambda)}}{z - (\lambda, \lambda)} d\lambda.$$  

Let $E(x)$, $-\infty < x < \infty$ be a right continuous spectral resolution of the self-adjoint operator $A$. It is obvious that $(\rho, \rho)$ belongs to the point spectrum of $A$ and corollary 6.4 implies that the constant functions are in the range of the projection $E((\rho, \rho)) - E((\rho, \rho) - 0) = E(say)$. Suppose $a > (\rho, \rho) > b$, and $a - b$ is small, then $(E\tilde{\phi}, \psi)$ is
given by Stieltjes inversion,

\[
\int \frac{1}{2\pi i} \oint_{C(a,b,c,e)} (R(z, \mathcal{A}) \tilde{\phi}, \tilde{\psi}) \, dz
\]

where \( C(a, b, c, e) \) is the following contour:

\[\begin{array}{c}
\begin{array}{c}
\text{Re } z
\end{array}
\end{array}\]

6.6. Next we want to determine the dual measure for the Fourier transform on \( A \).

We have put on \( A(A) \) the Tamagawa measure \( da \) which can be written as \( da = da1 dt \) corresponding to the decomposition \( A(A) = A1(A)A_+^* \). In §2.3 we put a measure on \( (Z_+ A(F) A(A))^* \) via Pontryagin duality. But

\[(Z_+ A(F) | A(A))^* = (A(F) | A(A)) \times Hom(Z_+ A(A), C^*)\]

and \( (A(F) | A(A))^* \) is discrete, \( Hom(Z_+ A(A), C^*) \) is a vector space over \( C \). Thus we can give \( (Z_+ A(F) | A(A))^* \) the structure of a complex manifold; as such, it has a natural measure which gives the measure 1 to the identity element of the Pontryagin dual of the compact abelian group \( A(F) \times A_1(A) \); while the dual measure to \( da1 \) gives the measure \( 1/\text{vol}(A(F)|A(A)) \) to the identity element.

The measure on \( A_+^* \) (resp. \( A_1^*, Z_+^* \)) is fixed by identifying it with a power of \( R^* \) by means of a basis of the lattice \( LF \) (resp. \( 1LF, 0LF \)). Since \( A_+^* = Z_+ A_1^* \), we see that the dual measure to \( da \) gives the measure \( 1/f \) to the identity element of \( 1LF \), where \( f = [1LF \oplus 0L_F^*: L_F] / [0L_F^*: 0L_F] \).

Now \( A_+^* \) is identified with \( 1\hat{L}_F \otimes R \). Let \( \{\mu_j\} \) be a basis of \( 1L_F \) and let
\{\mu_k\} be a dual basis in $\hat{L}_F \otimes R$ defined by $\langle \mu_j, \mu_k \rangle = \delta_{jk}$. Take the Euclidean measure $d\lambda$ on $\hat{L}_F \otimes R$ to be the one induced by identification of $\hat{L}_F \otimes R$ with $R'$ via the basis $\{\mu_j\}$, where $r$ is the rank of $\hat{L}_F$. Suppose we change the basis of $\hat{L}_F \otimes R$, namely, we use the Euclidean measure $d\lambda^+$ with respect to $\hat{L}_F \otimes R$. Then $d\lambda^+ = e \, d\lambda$ where $e = [\hat{L}_F^+ : \hat{L}_F]$. Choose a basis $\{\mu_j^+\}$ of $\hat{L}_F^+$ such that $\langle \mu_j^+, \alpha_k \rangle = \delta_{jk}$, where $\{\alpha_k\}$ is the set of simple $F$-roots. Let

$$\lambda : C' \to \hat{L}_F \otimes C$$

be the isomorphism defined by

$$\langle \lambda(s_1, \ldots, s_r), \alpha_k \rangle = s_k, \quad 1 \leq k \leq r.$$ 

That is we identify $\hat{L}_F \otimes C$ with $C'$ via the basis $\{\mu_j^+\}$. Then $e \, d\lambda = ds_1, \ldots, ds_r$. Finally we remark that for Fourier inversion in Euclidean space, the dual measure to $\hat{L}_F \otimes R = R'$ is $(2\pi i)^{-r}$ times the measure on $\hat{L}_F \otimes R$.

To summarize we have the following lemma.

6.7. **Lemma:** The measure induced on $\hat{L}_F \otimes C$ by that of $(Z^+_\infty A(F)A(A))^*$ is

$$ds_1 \ldots ds_r/c \, \text{vol}(A(F) \backslash A^1(A))(2\pi i)^r$$

where

$$c = ef = [\hat{L}_F^+ : L_F]/[0\hat{L}_F^+ : 0L_F].$$

6.8. **Remark:** In the remainder of this section we essentially reproduce Langlands [15] in adelic form. We follow Rapoport [18] in the proofs of lemma 6.9 and 6.10.

6.9. **Lemma:** All the local factors $M_v(w, \lambda(s))$ are holomorphic in $s$ in an open half space of $C'$ containing the point $(1, \ldots, 1)$.

**Proof:** Rewriting the formula (6) of §4 as

$$M_v(w, \lambda(s)) = \prod_{\alpha \in \Delta^+_F(w)} M^G_\alpha(\langle \lambda(s), \alpha \rangle)$$
we see that it is sufficient to consider the $F$-rank 1 case. And in this case, if $\phi$ is a locally constant function with compact support on $F_v$, then the integral of $\phi(\rho(a(\tilde{n})))$ over $\tilde{N}_V$ exists.

Thus there exists a non-negative measure $d\mu$ on $F_v$ such that

$$\int_{\tilde{N}_v} \phi(\rho(a(\tilde{n}))) \, d\tilde{n} = \int_{F_v} \phi(t) \, d\mu$$

for all reasonable functions $\phi$ on $F_v$. In particular, for $\phi : t \rightarrow |t|^{s+1}$ ($\text{Re} \, s > t$), we get

$$M_v(s) = \int_{F_v} |t|^{s+1} \, d\mu.$$ 

That is $M_v(s)$ is the Mellin transform of a non-negative measure and is continuous at 1 ($\S 5$). 6.9 now results from a variant of Landau's lemma.

6.10. Lemma: $M(w, \lambda(s))$ is meromorphic in $s$. There exists a positive number $\epsilon$ such that the only singularities of $M(w, \lambda)$ in the region $1 - \epsilon < \text{Re} \, s_i < 1 + \epsilon \quad (i = 1, \ldots, r)$ are simple poles in the hyperplane $s_i = 1$ for $i$ corresponding to a simple positive root in $0 \sum \tilde{f}(w)$.

Proof: By the preceding lemma, we can leave out a finite number of factors $M_v(w, \lambda)$ from $M(w, \lambda)$. In the relative rank 1 case, up to a finite number of factors, there are four cases:

(I) $M(s) = \frac{\zeta_F(s)}{\zeta_F(s + 1)}$

(II) $M(s) = \zeta_F(2s) \prod_{v < \infty} \frac{(1 - |\tilde{\omega}_v|^{2(s+1)})(1 + |\tilde{\omega}_v|^{2(s+1)})}{(1 + |\tilde{\omega}_v|^{2})}$

(III) $M(s) = \frac{\zeta_E(s)}{\zeta_E(s + 1)}$

(IV) $M(s) = \zeta_E(2s) \prod_{v < \infty} \frac{(1 - |\tilde{\omega}_v|^{2(s+1)})(1 + |\tilde{\omega}_v|^{2(s+1)})}{(1 + |\tilde{\omega}_v|^{2})}$

where $\zeta_F$ (resp. $\zeta_E$) is the Dedekind zeta function of $F$ (resp. $E$). It is clear that in the cases (I) and (III) $M(s)$ has a simple pole at $s = 1$ and
in cases (II) and (IV) $M(s)$ is holomorphic in an open half-space of $\mathbb{C}$ containing 1. The higher rank case now follows immediately from (7).

6.11. **Proposition:** For $\Phi, \Psi \in \mathcal{H}$, we have

$$\left( \tilde{E}_\Phi, \tilde{\Psi} \right) = \frac{\kappa \mu_A}{\mu_G c(A)} \lim_{s \to 1} \frac{L(s, G)}{L(s, A)} M(w_0, s \rho) \Phi(s \rho) \tilde{\Psi}(\bar{s} \rho)$$

where $w_0 \in \mathcal{W}_F$ is the unique element which sends all the positive roots to negative roots.

First we introduce some functions:

$$f_q(w; s) = M(w, \lambda(s)) \Phi(\lambda(s)) \tilde{\Psi}(-w \lambda(s))$$

$$f_q(w; s_1, \ldots, s_q) = \text{Res}_{s_{q+1} = 1} f_{q+1}(w; s_1, \ldots, s_{q+1}) \text{ for } 0 \leq q \leq \tau - 1$$

$$Q_q(s) = (\lambda(s), \lambda(s))$$

$$Q_q(s_1, \ldots, s_q) = Q_q(s_1, \ldots, s_q, 1, \ldots, 1).$$

We also write $s^q$ for $(s_1, \ldots, s_q)$.

6.12. **Lemma:** (i) For $0 \leq q \leq r$, the functions $f_q(w, s^q)$ are meromorphic in all the $s^q$-spaces. In the region

$$\{ s^q \mid \text{Re } s_i > 1, 1 \leq i \leq q \}$$

$f_q(w, s^q)$ is holomorphic, goes to zero faster than the inverse of all polynomials as the imaginary part of $s^q$ goes to infinity and the real part stays in a compact subset of this region.

(ii) There exists a positive number $\epsilon$ such that the only singularities of $f_q(w; s^q)$ in the region

$$\{ s^q \mid 1 - \epsilon < \text{Re } s_i < 1 + \epsilon; i = 1, \ldots, q \}$$

are simple poles lying the hyperplane $s_i = 1$.

**Proof:** (i) is just a restatement of the corresponding property of property of $M(w, \lambda)$ which is a consequence of the global theory of Eisenstein series (cf. [14]). (ii) follows from lemma 6.10.

6.13. It follows from §6.4 and 6.5 that
provided each of these limits exists. We shall show by induction that there exists the limit

\[ \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{C(a,b,c,e)} \frac{1}{(2\pi i)^q} \int_{\Re s = s_0} f_q(w; s) \, ds_1 \ldots ds_q \, dz \]

if \( s_0^q = s(s_0,1, \ldots, s_0,q) \) with \( s_{0,i} > 1, \, 1 \leq i \leq q \). Note that analyticity implies that expression is independent of the actual value of \( s_0^q \), provided its coordinates are strictly greater than one.

Take two small positive real numbers \( u \), and \( v \) such that \( u \) is much smaller than \( v \). Set \( s_0^q = (1 + u, \ldots, 1 + u, 1 + v) \) and \( s_0^{q-1} = (1 + u, \ldots, 1 + u) \). Then \( Q_q(1 + u, \ldots, 1 + u, 1 - v) < (\rho, \rho) \). Pick \( b \) such that \( Q(1 + u, \ldots, 1 + u, 1 - v) < b < (\rho, \rho) \). Then, we can find a constant \( \tau \) such that if either

\[ \begin{align*}
\Re s_i &= 1 + u, \quad 1 \leq i \leq q - 1 \\
\Re s_q &= 1 - v
\end{align*} \]

or

\[ \begin{align*}
\Re s_i &= 1 + u, \quad 1 \leq i \leq q - 1 \\
1 - v &\leq \Re s_q \leq 1 + v \\
|\Im s_q| &\geq \tau
\end{align*} \]

then

\[ \Re Q_q(s^q) < b - \frac{1}{\tau}. \]

We integrate

\[ \frac{1}{(2\pi i)^q} \int_{\Re s = s_0^q} \frac{f_q(w; s^q)}{z - Q_q(s^q)} \, ds_1 \ldots ds_q \]

first with respect to \( s_q \); we change the contour \( \Re s_q = s_{0,q} \) to
For $s^{q-1}$ fixed and $s_q$ in $C^q(\nu, \tau)$, the image in the $Z$-plane of $C = C^q(\nu, \tau)$ under $Q_q$ is given in the following diagram.

It follows that for $\text{Re}~s^{q-1} = s^{q-1}_\beta$ and $s_q \in C$ the function $1/(z - Q_q(s^q))$ is holomorphic in a region containing $C(a, b, c, \epsilon)$ such that

$$\lim_{\epsilon \to 0} \int_{C(a,b,c,\epsilon)} \frac{dz}{z - Q_q(s^q)} = 0$$

and (10) becomes
Finally, we get, for $q = 0$

$$\lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{C(a,b,c,e)} dz \frac{1}{(2\pi i)^{q-1}} \times \int_{\operatorname{Res} s^{-1} = s^{-1}} f_{q-1}(w; s^{q-1}) \, ds_1 \ldots ds_{q-1}.$$  

Finally, we compute

$$\lim_{\epsilon \to 0^+} f_{0}(w) \int_{C(a,b,c,e)} \frac{dz}{z - (\rho, \rho)} = f_{0}(w).$$

But it follows from lemma 6.10 that $f_{0}(w)$ is zero unless $w = w_0$ and $w_0$ takes $\rho$ to $-\rho$. We have

$$f_{0}(w_0) = \lim_{s \to i} (s - 1)' M(w_0, s\rho) \Phi(s\rho) \tilde{\Psi}(w_0(-\tilde{\rho})).$$

Hence

$$(E\tilde{\phi}, \tilde{\psi}) = \frac{\kappa \lim_{s \to i} (s - 1)' M(w_0, s\rho) \Phi(s\rho) \tilde{\Psi}(\tilde{s}\rho)}{c \operatorname{vol}(A(F)\backslash A^1(A))}$$

and (8) now follows from Ono’s formula for Tamagawa number of the torus $A$ (cf. [17]).

Using the formula

$$M(w_0, \lambda) = M_{\varphi}(w_0, \lambda) \prod_{\nu \in \mathcal{S}} M_{\nu}(w_0, \lambda)$$

and the result in §5 for the values of $M$, we see immediately that

$$(E\tilde{\phi}, \tilde{\psi}) = \kappa^2 (c\tau(A))^{-1} \Phi(\rho) \tilde{\Psi}(\rho).$$

7. Computation of Tamagawa number

7.1. Theorem: Let $G$ be a connected reductive quasi-split group defined over an algebraic number field $F$. Let $A$ be a maximal torus of $G$ defined over $F$ lying inside the Borel subgroup of $G$ defined over $F$. Then

$$\tau(G) = c\tau(A).$$
where $\tau(G)$ (resp. $\tau(A)$) denotes the Tamagawa number of $G$ (resp. $A$), and $c = [L_F^+ : L_F]/[0L_F^+ : 0L_F]$.

**Proof**: In the Hilbert space $L^2(Z \cdot G(F) \cdot G(A))$ we have

\[(\phi, 1)(1, \tilde{\psi}) = (1, 1)(\mathcal{P}\phi, \mathcal{P}\tilde{\psi}).\]

According the last formula of §6, the dimension of the image of $E$ is at most one. As we have already pointed out that the constant functions are in the image of $E$, we get $E = \phi$ and so

\[(\mathcal{P}\phi, \mathcal{P}\tilde{\psi}) = \kappa^2(\tau(A))^{-1}\Phi(\rho)\overline{\Psi}(\rho).\]

Since $(\phi, 1) = \kappa\Phi(\rho),(1, \tilde{\psi}) = \kappa\overline{\Psi}(\rho)$ and $\tau(G) = (1, 1)$ the theorem is proved.

7.2. Weil conjectured that the Tamagawa number of a semi-simple simply-connected connected algebraic group is one [17]. This conjecture holds for all classical groups ($\not\cong D_4, 6D_4$) (Tamagawa, Weil, Mars), for some exceptional groups (Mars, Demazure) and for Chevalley groups (Langlands), but it is not yet completely solved. We shall show that the Weil conjecture is true for simply-connected connected semi-simple quasi-split group $G$. This in fact follows immediately from our formula

\[\tau(G) = c\tau(A)\]

where $A$ is a maximal torus of $G$.

First, we observe that $G$ is simply-connected implies $L_F^+ = L_F$, i.e. $c = 1$; and the representation of the Galois group in the lattice of weights in a direct sum of permutation representation. Thus by duality theory of algebraic tori, we have

\[A \cong \prod_{i=1}^{n} R_{E_i/F}(G_m)\]

where $E_i$ are finite separable extension of $F$ which is the field of definition of $G$, and $G_m$ is the 1-dimensional multiplicative group. Now we have (by Ono [17])

\[\tau_F(A) = \prod_{i=1}^{n} \tau_F(R_{E_i/F}(G_m)) = \prod_{i=1}^{n} \tau_E(G_m) = 1,

where $T(G)$ (resp. $\tau(A)$) denotes the Tamagawa number of $G$ (resp. $A$), and $c = [L_F^+ : L_F]/[0L_F^+ : 0L_F]$.  

**Proof**: In the Hilbert space $L^2(Z \cdot G(F) \cdot G(A))$ we have

\[(\phi, 1)(1, \tilde{\psi}) = (1, 1)(\mathcal{P}\phi, \mathcal{P}\tilde{\psi}).\]

According the last formula of §6, the dimension of the image of $E$ is at most one. As we have already pointed out that the constant functions are in the image of $E$, we get $E = \phi$ and so

\[(\mathcal{P}\phi, \mathcal{P}\tilde{\psi}) = \kappa^2(\tau(A))^{-1}\Phi(\rho)\overline{\Psi}(\rho).\]

Since $(\phi, 1) = \kappa\Phi(\rho),(1, \tilde{\psi}) = \kappa\overline{\Psi}(\rho)$ and $\tau(G) = (1, 1)$ the theorem is proved.
because $\tau(G_m) = 1$ (which follows from the value of the residue of zeta function $\zeta_E$ at 1).

Thus by the formula of the preceeding subsection $\tau(G) = c\tau(A) = 1$ for a simply-connected semi-simple quasi-split connected algebraic group.

REFERENCES