W. CASSELMAN

The unramified principal series of p-adic groups. I. The spherical function

Compositio Mathematica, tome 40, nº 3 (1980), p. 387-406

<http://www.numdam.org/item?id=CM_1980__40_3_387_0>

COMPOSITIO MATHEMATICA, Vol. 40, Fasc. 3, 1980, pag. 387–406

THE UNRAMIFIED PRINCIPAL SERIES OF $\mathfrak{p}$-ADIC GROUPS I. THE SPHERICAL FUNCTION

W. Casselman

It will be shown in this paper how results from the general theory of admissible representations of $\mathfrak{p}$-adic reductive groups (see mainly [7]) may be applied to give a new proof of Macdonald's explicit formula for zonal spherical functions ([9] and [10]). Along the way I include many results which will be useful in subsequent work.

Throughout, let k be a non-archimedean locally compact field, $\mathfrak{o}$ its ring of integers, $\mathfrak{p}$ its prime ideal, and q the order of the residue field.

If $\mathbf{H}$ is any algebraic group defined over k, H will be the group of its k-rational points.

For any k-analytic group H, let $C_c^\infty(H)$ be the space of locally constant functions of compact support: $H \to \mathbf{C}$. For any subset X of H, let ch_X or $ch(X)$ be its characteristic function (which lies in $C_c^\infty(H)$ if X is compact and open).

Fix a connected reductive group $\mathbf{G}$ defined over k. Let $\tilde{\mathbf{G}}$ be the simply connected covering of its derived group $\mathbf{G}^{\mathrm{der}}$, $\mathbf{G}^{\mathrm{adj}}$ the quotient of $\mathbf{G}$ by its centre, and $\psi : \tilde{\mathbf{G}} \to \mathbf{G}$ the canonical homomorphism. If $\mathbf{H}$ is any subgroup of $\mathbf{G}$, let $\tilde{\mathbf{H}}$ be its inverse image in $\tilde{\mathbf{G}}$.

Fix also a minimal parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. Let $\mathbf{A}$ be a maximal split torus contained in $\mathbf{P}$, $\mathbf{M}$ the centralizer of $\mathbf{A}$, $\mathbf{N}$ the unipotent radical of $\mathbf{P}$, and $\mathbf{N}^-$ the unipotent radical of the parabolic opposite to $\mathbf{P}$. Let Σ be the roots of $\mathbf{G}$ with respect to $\mathbf{A}$, ${}^{nd}\Sigma$ the subset of nondivisible roots, Σ^+ the positive roots determined by $\mathbf{P}$, Δ the simple roots in Σ^+, W the Weyl group. For any $\alpha \in \Sigma$, let $\mathbf{N}_\alpha$ be the subgroup of $\mathbf{G}$ constructed in §3 of [2] (its Lie algebra is $\mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$).

Let δ be the modulus character of $P : mn \to |\det Ad_{\mathfrak{n}}(m)|$. Let w_ℓ be the longest element of W.

0010–437X/80/03/0387–20$00.20/0

If H is a compact group, $\mathcal{P}_H$ is the projection operator onto H-invariants.

In §1 I shall give an outline of the results from Bruhat-Tits that I shall need. Complete proofs have not yet appeared, but the necessary facts are not difficult to prove when **G** is split (see [8]) or even unramified – i.e. split over an unramified extension of k. There is no serious loss if one restricts oneself to unramified **G**, since any reductive group over a global field is unramified at almost all primes, and important applications will be global. As far as understanding the main ideas is concerned, one may assume **G** split. This will simplify both arguments and formulae considerably.

Since the first version of this paper was written, Matsumoto's book [12] has appeared with another proof of Macdonald's formula, in a more general form valid not just for the spherical functions on $\mathfrak{p}$-adic groups but for those related to more general Hecke algebras.

1. The structure of G

Let $\mathcal{B}$ be the Bruhat-Tits building of $\tilde{G}$. (Refer to [6], Chapter II of [10], and [13].)

There exists in $\mathcal{B}$ a unique apartment $\mathcal{A}$ stabilized by $\tilde{A}$. The stabilizer $\tilde{\mathcal{N}}$ of $\mathcal{A}$ in $\tilde{G}$ is equal to the normalizer $N_{\tilde{G}}(\tilde{A})$; let $\nu : \tilde{\mathcal{N}} \to \mathrm{Aut}(\mathcal{A})$ be the corresponding homomorphism. The dimension of $\mathcal{A}$ over $\mathbf{R}$ is equal to that of $\tilde{A}$ over k, say r, and the image of $\tilde{A}$ with respect to ν is a free group of rank r. Therefore the translations are precisely those elements of $\mathrm{Aut}(\mathcal{A})$ commuting with $\nu(\tilde{A})$, so that the inverse image of the translations is $\tilde{M}$. The kernel of ν is the maximal compact open subgroup $\tilde{M}_0$ of $\tilde{M}$. Let $\tilde{A}_0$ be $\tilde{A} \cap \tilde{M}_0$, which is maximal compact and open in $\tilde{A}$.

There exists on $\mathcal{A}$ a canonical affine root system Σ_{aff}. Let W_{aff} be the associated affine Weyl group. Choose once and for all in this paper a special point $x_0 \in \mathcal{A}$, let Σ_0 be the roots of Σ_{aff} vanishing at x_0, and let W_0 be the isotropy subgroup of W_{aff} at x_0. Then Σ_0 is a finite reduced root system and W_0 its Weyl group. The homomorphism ν is a surjection from $\tilde{\mathcal{N}}$ to W_{aff}, and therefore induces isomorphisms of $\tilde{\mathcal{N}}/\tilde{M}_0$ with W_{aff} and of $\tilde{\mathcal{N}}/\tilde{M}$ with W_0. It also induces an injection of $\tilde{A}/\tilde{A}_0$ into $\mathcal{A} : a \to \nu(a)x_0$, and one may therefore identify Σ_0 with a root system in the vector space $\mathrm{Hom}(\tilde{A}/\tilde{A}_0, \mathbf{R})$. The map taking the rational character α to the function $a \mapsto -\mathrm{ord}_{\mathfrak{p}}(\alpha(a))$ allows one also to identify Σ with a root system in $\mathrm{Hom}(\tilde{A}/\tilde{A}_0, \mathbf{R})$. The two root systems one thus obtains are not necessarily the same or even homothetic, but what is true is that each $\alpha \in \Sigma$ is a positive multiple of a unique root $\lambda(\alpha)$ in Σ_0. The map λ is a bijection between ${}^{nd}\Sigma$ and Σ_0. Let Σ_0^+, Δ_0 correspond to Σ^+, Δ. Let $\mathscr{C}$ be the vectorial chamber $\{\alpha(x) > 0$ for all $\alpha \in \Sigma_0^+\}$, and let C be the affine chamber of $\mathscr{A}$ contained in $\mathscr{C}$ which has x_0 as vertex.

Let $\tilde{B}$ be the Iwahori subgroup fixing the chamber C. It also fixes every element of C.

For each $\alpha \in \Sigma_{\mathrm{aff}}$, let $\tilde{N}(\alpha)$ be the group $\{n \in \tilde{N} \mid nx = x$ for all $x \in \mathscr{A}$ with $\alpha(x) \geq 0\}$. Then:

(1) $$\tilde{N}(\alpha+1) \subsetneq \tilde{N}(\alpha);$$

(2) For any $g \in \tilde{\mathscr{N}}$, $g\tilde{N}(\alpha)g^{-1} = \tilde{N}(\nu(g)\alpha)$;

(3) For any $\alpha \in {}^{nd}\Sigma$, the group $\tilde{N}_\alpha$ is the union of the $\tilde{N}(\lambda(\alpha)+i)$ $(i \in Z)$;

(4) $$\tilde{N}(-\alpha) - \tilde{N}(-\alpha+1) \subseteq \tilde{N}_\alpha \nu^{-1}(w_\alpha)\tilde{N}_\alpha;$$

(5) If $\tilde{N}_0 = \Pi\tilde{N}(\alpha)(\alpha \in \Sigma_0^+)$ and $\tilde{N}_1^- = \Pi\tilde{N}(-\alpha+1)(\alpha \in \Sigma_0^+)$ then one has the *Iwahori factorization* $\tilde{B} = \tilde{N}_1^- \tilde{M}_0\tilde{N}_0$.

As a consequence of (2):

(6) For $m \in \tilde{M}$ and $\alpha \in \Sigma_0$, $m\tilde{N}(\alpha+i)m^{-1} = \tilde{N}(\alpha + i - \alpha(\nu(m)x_0))$.

Let $\tilde{\alpha}$ be the dominant root in Σ_0, and let S_{aff} be $\{w_\alpha \mid \alpha \in \Delta_0$ or $\alpha = \tilde{\alpha} - 1\}$. Then $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ is a Coxeter group, and in fact $(\tilde{G}, \tilde{B}, \tilde{\mathscr{N}}, S_{\mathrm{aff}})$ is an affine Tits system.

Recall that the Hecke algebra $\mathscr{H}(\tilde{G}, \tilde{B})$ is the space of all compactly supported functions $f: \tilde{G} \to \mathrm{C}$ which are right- and left-$\tilde{B}$-invariant, endowed with the product given by convolution. (Here $\tilde{B}$ is assumed to have measure 1, so that $ch(\tilde{B})$ is the identity of this algebra.) As a linear space it has the basis $\{ch(\tilde{B}w\tilde{B})\ (w \in W_{\mathrm{aff}})\}$.

(7) If $w \in W_{\mathrm{aff}}$ has the reduced expression $w = w_1 \cdots w_p$ $(w_i \in S_{\mathrm{aff}})$ then $ch(\tilde{B}w\tilde{B}) = \Pi ch(\tilde{B}w_i\tilde{B})$.

For any $w \in W_{\mathrm{aff}}$, define $q(w)$ to be $[\tilde{B}w\tilde{B} : \tilde{B}]$. Then

(8) $$ch(\tilde{B}w_\alpha\tilde{B})^2 = (q(w_\alpha)-1)ch(\tilde{B}w_\alpha\tilde{B}) + q(w_\alpha)ch(\tilde{B})\quad (\alpha \in S_{\mathrm{aff}})$$

For any $\alpha \in \Sigma_0$, define

(9) $$a_\alpha = w_\alpha \circ w_{\alpha-1}.$$

It is a translation of $\mathscr{A}$ whose inverse image in $\tilde{M}$ is a coset of $\tilde{M}_0$, and I shall often treat it as if it were an element of this coset. Because of (6),

$$a_\alpha \tilde{N}(\alpha + i) a_\alpha^{-1} = \tilde{N}(\alpha + i + 2) \tag{10}$$

or, in other words, $a_\alpha(\alpha) = \alpha - 2$.

1.1. REMARK: There is another way to consider a_α which may be more enlightening. If $\tilde{G}$ is of rank one, then $\tilde{M}/\tilde{M}_0$ is a free group of rank one over Z, and a_α is the coset of $\tilde{M}_0$ which generates this group and takes $-\mathscr{C}$ into itself. If $\tilde{G}$ is not necessarily of rank one and $\alpha \in \Delta_0$, then the standard parabolic subgroup associated to $\Delta - \{\lambda^{-1}(\alpha)\}$ has the property that its derived group is of rank one and again simply connected ([3] 4.3) and a_α for $\tilde{G}$ is the coset of $\tilde{M}_0$ containing the a_α for this group. If α is not necessarily in Δ_0, there will exist $w \in W_0$ such that $\beta = w^{-1}\alpha \in \Delta_0$; let $a_\alpha = w a_\beta w^{-1}$. If G is split, the construction is even simpler; let a_α be the image of a generator of $\mathfrak{p}$ with respect to the *co-root* $\alpha_* : \mathbf{G}_m \to \tilde{\mathbf{G}}$.

It is always true that:

$$\text{For any } w \in W_0,\ w a_\alpha w^{-1} = a_{w\alpha}. \tag{11}$$

For each $\alpha \in \Sigma_{\text{aff}}$, let

$$q_\alpha = [\tilde{N}(\alpha - 1) : \tilde{N}(\alpha)]. \tag{12}$$

Because of (10), $q_{\alpha+2}$ is always the same as q_α, but it is not necessarily the same as $q_{\alpha+1}$. Macdonald ([10] III) defines the subset Σ_1 with $\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_0 \cup \frac{1}{2}\Sigma_0$; $\alpha/2$ (for $\alpha \in \Sigma_0$) lies in Σ_1 if and only if $q_{\alpha+1} \neq q_\alpha$. He proves that Σ_1 is a root system, and for each $\alpha \in \Sigma_0$ defines $q_{\alpha/2}$ to be $q_{\alpha+1}/q_\alpha$. Then:

$$\text{For } \alpha \in \Sigma_0, [\tilde{N}(\alpha + 1) : \tilde{N}(\alpha + m + 1)] = q_{\alpha/2}^{[m/2]} q_\alpha^m; \tag{13}$$

$$\text{For } \alpha \in \Delta_0,\ q(w_\alpha) = q_{\alpha/2} q_\alpha; \tag{14}$$

When $\tilde{G}$ has rank one and $\alpha > 0$,

$$\delta(a_\alpha) = 1/[\tilde{N}(\alpha) : a_\alpha \tilde{N}(\alpha) a_\alpha^{-1}] = q_{\alpha/2}^{-1} q_\alpha^{-2}. \tag{15}$$

It may happen that $q_{\alpha/2} < 1$. For example, if $\tilde{G}$ has rank one then there are two possible inequivalent choices of the special point, and if q_α is not always equal to $q_{\alpha+1}$ then for one of these choices $q_{\alpha/2}$ will be <1, for the other >1. The second choice is better in some sense; the corresponding maximal compact subgroup is what Tits [13] calls hyperspecial. In general, a simple argument on root hyperplanes will show that there is always some choice of x_0 which assures $q_{\alpha/2} \geq 1$ for all $\alpha > 0$.

This completes my summary of the simply connected case.

The algebraic group of automorphisms of **G** contains $\mathbf{G}^{\mathrm{adj}}$, and therefore there is a canonical homomorphism from **G** to Aut($\tilde{\mathbf{G}}$). Thus G acts on $\tilde{G}: x \mapsto {}^g x$. If X is a compact subset of $\tilde{G}$, so is ${}^g X$, so that this action of G preserves what [6] calls the *bornology* of G. By [6], 3.5.1. the morphism $\psi: \tilde{G} \to G$ is $\tilde{B}$*-adapted*. This means ([6] 1.2.13) that for each $g \in G$ the subgroup ${}^g\tilde{B}$ is conjugate in $\tilde{G}$ to $\tilde{B}$, or that there exists $h \in \tilde{G}$ such that $hBh^{-1} = \psi^{-1}(g\psi(\tilde{B})g^{-1}) = {}^g\tilde{B}$. The action of G on $\tilde{G}$ therefore induces one of G on $\mathcal{B}$.

The stabilizer of $\mathcal{A}$ in G is $\mathcal{N} = N_G(A)$. Let here, too, ν be the canonical homomorphism: $\mathcal{N} \to \mathrm{Aut}(\mathcal{A})$. The inverse image of the translations is M.

Theorem 3.19 of [4] and its proof assert that the inclusion of M into G induces an isomorphism of $M/\psi(\tilde{M})Z_G$ with $G/\psi(\tilde{G})Z_G$, hence that every $g \in G$ may be expressed as $m\psi(\tilde{g})$ with $m \in M$, $\tilde{g} \in \tilde{G}$. Since $m\mathcal{A} = \mathcal{A}$, this implies that one may choose the h above so that simultaneously $h\tilde{B}h^{-1} = {}^g\tilde{B}$ and $h\tilde{\mathcal{N}}h^{-1} = {}^g\tilde{\mathcal{N}}$. Therefore ψ is $\tilde{B}-\tilde{\mathcal{N}}$-adapted ([6] 1.2.13).

Since $\tilde{\mathcal{N}}/\tilde{M} \cong \mathcal{N}/M \cong W$, ψ is of *connected type* ([6] 4.1.3). Let $G_1 = \{g \in G \mid |\chi(g)| = 1$ for all rational characters $\chi: G \to G_m\}$. If G^{der} is the derived group of G, then $\psi(\tilde{G}) \subseteq G^{\mathrm{der}} \subseteq G_1$; [4] 3.19 implies that $\psi(\tilde{G})$ is closed in G and $G^{\mathrm{der}}/\psi(\tilde{G})$ compact, while it is clear that G_1/G^{der} is compact. Therefore $G_1/\psi(\tilde{G})$ is compact.

Let

$$B = \{g \in G_1 \mid gx = x \text{ for all } x \in C\}$$

$$K = \{g \in G_1 \mid gx_0 = x_0\}.$$

Since $\tilde{B}$ is compact, so is $\psi(\tilde{B})$ and furthermore $B \cap \psi(\tilde{G}) = \psi(\tilde{B})$. Therefore since $G_1/\psi(\tilde{G})$ is compact, so is B. Since $B \subseteq K$ and K/B is finite, K is also compact. The subgroup K is what [6] calls a *special, good, maximal bounded* subgroup of G.

Let $\mathcal{N}_K = \mathcal{N} \cap K$ and $M_0 = M \cap K = M \to B$. The injection of $\mathcal{N}_K/M_0$ into W is an isomorphism ([6] 4.4.2). *From now on I assume every representative of an element of W to lie in K.* Such a representative is determined up to multiplication by an element of M_0.

The triple $(K, B, \mathcal{N}_K)$ form a Tits system with Weyl group W, and therefore

(16) K is the disjoint union of the BwB $(w \in W)$;

(17) $$[K:B] = \Sigma\,[BwB:B] = \Sigma\, q(w)\ (w \in W).$$

The group G has the *Iwasawa decomposition* ([6] 4.4.3)

(18) $$G = PK$$

and a refinement:

(19) G is the disjoint union of the PwB $(w \in W)$.

Let

$$M^- = \{m \in M \mid m^{-1}\mathscr{C} \subseteq \mathscr{C}\};$$
$$A^- = A \cap M^-.$$

The group A^- is also $\{a \in A \mid |\alpha(a)| \leq 1\}$ for $\alpha \in \Delta\}$, so that this terminology agrees with that of [7].

The group G has the *Cartan decomposition* ([6] 4.4.3):

(20) $$G = KM^-K.$$

Let ξ be the canonical homomorphism ([6] 1.2.16) from G to the group of automorphisms of $\mathscr{A}$ taking C to itself, and let $G_0 = G_1 \cap \ker(\xi)$. The triple $(G_0, B, \mathcal{N} \cap G_0)$ form a Tits system with affine Weyl group isomorphic to W_{aff}, and ψ induces an isomorphism between the Hecke algebras $\mathscr{H}(\tilde{G}, \tilde{B})$ and $\mathscr{H}(G_0, B)$ ([6] 1.2.17). Define Ω to be the subgroup of $\mathcal{N}/M_0$ of elements taking C to itself. Then elements of Ω normalize B, and hence for any $\omega \in \Omega$, $w \in W_{\text{aff}}$

(21) $$ch(B\omega B)ch(BwB) = ch(B\omega wB)$$

in $\mathscr{H}(G, B)$. Furthermore the group $\mathcal{N}/M_0$ is a semi-direct product of Ω and W_{aff}, and

(22) G is the disjoint union of the BxB $(x \in \mathcal{N}/M_0)$.

(In fact, $(G, B, \mathcal{N})$ form a *generalized* Tits system – see [8].) As a corollary of (7), (8), (21), and the isomorphism between $\mathscr{H}(\tilde{G}, \tilde{B})$ and $\mathscr{H}(G_0, B)$:

1.2. Proposition: *In any finite-dimensional module over $\mathcal{H}(G, B)$ each $ch(BxB)$ $(x \in \mathcal{N})$ is invertible.*

For $\alpha \in \Sigma_{\text{aff}}$, define $N(\alpha)$ to be $\psi(\tilde{N}(\alpha))$. Since $\psi \mid \tilde{N}$ is an isomorphism with N, all the properties stated earlier for the $\tilde{N}(\alpha)$ hold also for the $N(\alpha)$. In particular, for example:

(23) B has the Iwahori factorization $B = N_1^- M_0 N_0$.

From now on let $P_0 = M_0 N_0$.

There is a nice relationship between the Bruhat decompositions of G and K:

1.3. Proposition: *For any $w \in W$*
(a) $BwB \subseteq \bigcup PxP$ $(x > w)$;
(b) $BwB \cap PwP = P_0 w N_0$.

Proof: I first claim that $BwB = BwN_0$. To see this, observe that the Iwahori factorization of B gives

$$BwB = BwN_1^- M_0 N_0 = BwN_1^- N_0$$

but then $wN_1^- = wN_1^- w^{-1} \cdot w \subseteq Bw$.

Next,

$$BwN_0 = P_0 N_1^- wN_0$$

and

$$\begin{aligned} N_1^- w &= w_\ell N_1 w_\ell^{-1} \cdot w \\ &\subseteq Pw_\ell P \cdot Pw_\ell wP \\ &\subseteq \bigcup Pw_\ell yP (y < w_\ell w) \end{aligned}$$

by Lemma 1, p. 23, of [5]. But according to the Appendix, $y < w_\ell w$ if and only if $w_\ell y > w$, and this proves 1.3(a).

For (b), it suffices to show that for $n^- \in N_1^-$, if $n^- w \in PwP$ then $n^- \in wPw^{-1}$. But if $n^- w \in PwP = PwN$, one has $n^- w = pwn$ with $p \in P$, $n \in N$ and then $n^- = p \cdot wnw^{-1}$. As is well known, elements of the group wNw^{-1} factor uniquely according to $wNw^{-1} = (wNw^{-1} \cap N)(wNw^{-1} \cap N^-)$. Hence $n^- \in wNw^{-1} \cap N^-$.

In the rest of this paper, the notation will be slightly different. The main point is that it is clumsy to have to refer to both the Bruhat-Tits system Σ_0 and the system Σ arising from the structure of G as a reductive algebraic group. Therefore I shall often confound $\alpha \in {}^{\text{nd}}\Sigma$

with $\lambda(\alpha) \in \Sigma_0$ – referring for example to q_α instead of $q_{\lambda(\alpha)}$, etc. Also I shall write $N_{\alpha,i}$ (for $\alpha \in {}^{\mathrm{nd}}\Sigma$) instead of $N(\alpha+i)$, and refer to a_α as an element of G or a coset of M_0, when what I really mean is $\psi(a_\alpha)$.

2. Elementary properties of the principal series

If σ is a complex character of M – i.e. any continuous homomorphism from M to $\mathbf{C}^\times$ – it is said to be *unramified* if it is trivial on M_0. Because the group M/M_0 is a free group of rank r, the group $X_{nr}(M)$ of all unramified characters of M is isomorphic to $(\mathbf{C}^\times)^r$. This isomorphism is non-canonical, but the induced structure of a complex analytic group is canonical.

I assume all characters of M to be unramified from now on.

The character χ of M determines as well one of P, since $M \cong P/N$. The *principal series* representation of G induced by this (which is itself said to be unramified) is the right regular representation R of G on the space $I(\chi) = \mathrm{Ind}(\chi \mid P, G)$ of all locally constant functions $\phi : G \to C$ such that $\phi(pg) = \chi\delta^{1/2}(p)\phi(g)$ for all $p \in P$, $g \in G$. This representation is admissible ([7] §3).

Define the G-projection $\mathcal{P}_\chi$ from C_c^∞ onto $I(\chi)$:

$$\mathcal{P}_\chi(f)(g) = \int_P \chi^{-1}\delta^{1/2}(p) f(pg)\,\mathrm{d}p$$

Here and elsewhere I assume P to have the left Haar measure according to which meas $P_0 = 1$.

For each $w \in W$, let $\phi_{w,\chi} = \mathcal{P}_\chi(ch_{BwB})$, and let $\phi_{K,\chi} = \mathcal{P}_\chi(ch_K)$. (I shall often omit the reference to χ). Thus ϕ_w is identically 0 off PwB and $\phi_w(pwb) = \chi\delta^{1/2}(p)$ for $p \in P$, $b \in B$.

2.1. Proposition: *The functions $\phi_{w,\chi}(w \in W)$ form a basis of $I(\chi)^B$.*

This is because G is the disjoint union of the open subsets PwB (1.9)).

2.2. Corollary: *The function $\phi_{K,\chi}$ is a basis of $I(\chi)^K$.*

Of course this also follows directly from the Iwasawa decomposition.

Recall from [7] §3 that if (π, V) is any admissible representation of

G then $V(N)$ is the subspace of V spanned by $\{\pi(n)v - v \mid n \in N, v \in V\}$, and that the *Jacquet module* V_N is the quotient $V/V(N)$. If V is finitely generated as a G-module then V_N is finite-dimensional ([7] Theorem 3.3.1). Since $V(N)$ is stable under M, there is a natural smooth representation π_N of M on V_N.

According to [7] Theorem 6.3.5, if $V = I(\chi)$ then V_N has dimension equal to the order of W. This suggests:

2.3. PROPOSITION: *The canonical projection from $I(\chi)^B$ to $I(\chi)_N$ is a linear isomorphism.*

I shall give two proofs of this. The first describes the relationship between $I(\chi)^B$ and $I(\chi)_N$ in more detail, but the second shows this proposition to be a corollary of a much more general result.

The first: it is shown in §6.3 of [7] that one has a filtration of $I(\chi)$ by P-stable subspaces I_w $(w \in W)$, decreasing with respect to the partial order on W mentioned in the Appendix. The space I_w consists of the functions in $I(\chi)$ with support in $\cup\, PxP$, $(x > w)$ and clearly $I_x \subseteq I_y$ when $y < x$. According to Proposition 1.3(a), ϕ_w lies in I_w. Each space $(I_w)_N/\Sigma\,(I_x)_N$ $(x > w,\ x \neq w)$ is one-dimensional ([7] 6.3.5), and the map on I_w which takes ϕ to

$$\int_{w^{-1}Nw\cap N\setminus N} \phi(wn)\,\mathrm{d}n$$

induces a linear isomorphism of this space with $\mathbf{C}$. It is easy to see, then, from Proposition 1.3(b) that the image of ϕ_w with respect to this map is non-trivial, and this proves 2.3.

For the second proof:

2.4. PROPOSITION: *If (π, V) is any admissible representation of G, then the canonical projection from V^B to $V_N^{M_0}$ is a linear isomorphism.*

PROOF: Because B has an Iwahori factorization with respect to P, Theorem 3.3.3 of [7] implies surjectivity.

For injectivity, suppose $v \in V^B \cap V(N)$. Then Lemma 4.1.3 of [7] implies the existence of $\epsilon > 0$ such that $\pi(ch_{BaB})v = 0$ for $a \in A^-(\epsilon)$ (where $A^-(\epsilon) = \{a \in A \mid |\alpha(a)| < \epsilon$ for all $\alpha \in \Delta\}$). Apply Proposition 1.2.

This proof of injectivity is Borel's (see Lemma 4.7 of [1]).

Proposition 2.4 may be strengthened to give as well a relationship between the structure of V^B as a module over the Hecke algebra $\mathcal{H}(G, B)$ and that of V_N as a smooth representation of M:

2.5. PROPOSITION: *Let (π, V) be an admissible representation of G, $v \in V$ with image $u \in V_N$. Then for any $m \in M^-$ the image of $\pi(ch_{BmB})v$ in V_N is equal to* $\text{meas}(BmB)\pi_N(m)v$.

PROOF: If $v \in V^B$, then because $m^{-1}N_1^- m \subseteq N_1^-$ (1.6), $\pi(m)v \in V^{M_0N_1^-}$. Jacquet's First Lemma ([7] 3.3.4) implies that $v_0 = \text{meas}(BmB)^{-1}\pi(ch_{BmB})v = \mathcal{P}_B(\pi(m)v)$ and $\pi(m)v$ have the same image in V_N.

There are two more results one can derive from Proposition 2.4.

2.6. PROPOSITION: *If (π, V) is any irreducible admissible representation of G with $V^B \neq 0$, then there exists a G-embedding of V into some unramified principal series. Conversely, if V is any non-trivial G-stable subspace of an unramified principal series, then $V^B \neq 0$.*

PROOF: Recall the version of Frobenius reciprocity given as 3.2.4 in [7]:

$$\text{Hom}_G(V, \text{Ind}(\chi \mid P, G)) \cong \text{Hom}_M(V_N, \chi\delta^{1/2}).$$

If V is a subspace of $I(\chi)$ then the left-hand side is non-trivial, hence the right-hand side. This means that $V_N^{M_0} \neq 0$, and by 2.4 neither is V^B trivial. If $V^B \neq 0$ on the other hand, then 2.4 implies that $V_N^{M_0} \neq 0$. Since it is finite-dimensional, there exists some one-dimensional M-quotient, hence by Frobenius reciprocity a G-morphism into an unramified principal series.

2.7. PROPOSITION: *The G-module $I(\chi)$ is generated by $I(\chi)^B$.*

PROOF: If U is the quotient of $I(\chi)$ by the G-space generated by $I(\chi)^B$, then $U^B = 0$. The linear dual of U^B is canonically isomorphic to $\tilde{U}^B$, where $\tilde{U}$ is the space of the admissible representation contragredient to U (see §2 of [7]), and hence $\tilde{U}^B = 0$ as well. But since U is a quotient of $I(\chi)$, $\tilde{U}$ is a subspace of $I(\chi^{-1})$, which is the contragredient of $I(\chi)$ ([7] 3.1.2). Proposition 2.6 implies that $\tilde{U}$ is trivial and therefore also U.

3. Intertwining operators

Assume in this section that all characters χ of M are *regular* – i.e. that whenever $w \in W$ is such that $w\chi = \chi$ then $w = 1$.

With this condition satisfied, it is shown in §6.4 of [7] that for each $x \in K$ representing $w \in W$ there exists a unique G-morphism $T_x : I(\chi) \to I(w\chi)$ such that for all $\phi \in I(\chi)$ with support in $\cup PyP$ $(y \not< w^{-1}) \cup Pw^{-1}P$

$$(1) \qquad T_x\phi(1) = \int_{wNw^{-1} \cap N \setminus N} \phi(x^{-1}n)\, dn.$$

Here $wNw^{-1} \cap N \setminus N$ is assumed to have the Haar measure such that the orbit of $\{1\}$ under N_0 has measure 1. Since χ is unramified, one sees easily that T_x is independent of the choice of $x \in K$ representing w, and one may call it T_w. Furthermore, it is shown in §6.4 of [7] that T_w varies holomorphically with χ in the sense that for a fixed $f \in C_c^\infty(G)$ and $g \in G$, $T_w(\mathcal{P}_\chi f)(g)$ is a holomorphic function of χ. Finally, every G-morphism from $I(\chi)$ to $I(w\chi)$ is a scalar multiple of T_w.

The operator T_w is in particular a B-morphism and a K-morphism, so it takes $I(\chi)^B$ to $I(w\chi)^B$ and $I(\chi)^K$ to $I(w\chi)^K$. Therefore it takes $\phi_{K,\chi}$ to a scalar multiple of $\phi_{K,w\chi}$.

For each $\alpha \in \Sigma$, define

$$c_\alpha(\chi) = \frac{(1 - q_{\alpha/2}^{-1/2} q_\alpha^{-1} \chi(a_\alpha))(1 + q_{\alpha/2}^{-1/2} \chi(a_\alpha))}{1 - \chi(a_\alpha)^2}.$$

3.1. THEOREM: *One has*

$$T_w(\phi_{K,\chi}) = c_w(\chi)\phi_{K,w\chi}$$

where

$$c_w(\chi) = \Pi c_\alpha(\chi) \quad (\alpha > 0,\ w\alpha < 0).$$

PROOF: Step (1). Assume G to be of semi-simple rank one, α the single non-multipliable positive root, and $w = w_\alpha$ the single non-trivial element of W. Since $\phi_K(1) = 1$, and one knows $T_w(\phi_K)$ to be a multiple of ϕ_K, it suffices to calculate $T_w(\phi_K)(1)$. Since $K = B \cup BwB$, $\phi_K = \phi_1 + \phi_w$, and one only need evaluate $T_w(\phi_1)(1)$ and $T_w(\phi_w)(1)$ separately.

Evaluating the second is simple, since ϕ_w has support in PwP, and in fact $\phi_w(wn) = 1$ if $n \in N_0$ and 0 if $n \in N - N_0$:

$$T_w(\phi_w)(1) = \int_N \phi(wn)\,\mathrm{d}n$$
$$= \int_{N_0} \mathrm{d}n = 1.$$

As for the first, since T_w varies holomorphically with χ it suffices to calculate $T_w(\phi_1)(1)$ for all χ in some open set of $X_{nr}(M)$. Define $\Phi = \Phi_\chi$ on PwP:

$$\Phi(n_1 m w n_2) = \chi^{-1}\delta^{1/2}(m).$$

For $f \in C_c^\infty(PwP) \subseteq C_c^\infty(G)$,

$$T_w(\mathscr{P}_\chi f(1) = \int_{PwP} \Phi(x) f(x)\,\mathrm{d}x.$$

Here the measure adopted on PwP is the restriction of a Haar measure on G with the normalization condition that meas $P_0 w N_0 = 1$ (note that PwP is open in G). This formula actually makes sense for all $f \in C_c^\infty(G)$ under certain conditions on χ:

3.2. Lemma: *If* $|\chi(a)| < 1$ *for all regular elements of* A^-, *then for every* $f \in C_c^\infty(G)$ *the integral*

$$\int_{PwP} \Phi_\chi(x) f(x)\,\mathrm{d}x$$

converges absolutely and is equal to $T_w(\mathscr{P}_\chi(f))(1)$. *If* $f = ch_B$, *then it is equal to* $c_\alpha(\chi) - 1$.

Proof: It suffices to let f be the characteristic function of a set of the form $N_n^- X$, where X is an open subgroup of P_0 and N_n^- $(n \geq 1)$ is the subgroup of §1. This is because every function in $C_c^\infty(G)$ is a linear combination of (1) a function in $C_c^\infty(PwP)$ and (2) right P-translates of such characteristic functions. For $f = ch(N_n^- x)$, the above integral is equal to

$$\int_{N_n^- X} \Phi_\chi(x)\,\mathrm{d}x = [P_0 : X]^{-1} \int_{N_n^-} \phi_\chi(x)\,\mathrm{d}x$$

where the measure on N_n^- is such that meas $N_1^- = [BwB : B]^{-1} = (q_\alpha q_{\alpha/2})^{-1}$. This may be not quite obvious – it is because the Haar measure adopted on G is $(q_\alpha q_{\alpha/2})^{-1}$ times the one in which meas $B = 1$, $B = N_1^- P_0$, and $\Phi(xp) = \Phi(x)$ for $p \in P_0$.

Recall from 1.(1), 1.(3), and 1.(4) that

$$N_n^- = (N_n^- - N_{n+1}^-) \cup (N_{n+1}^- - N_{n+2}^-) \cup \cdots$$

and

$$N_m^- - N_{m+1}^- \subseteq N a_\alpha^{-m} w_\alpha N.$$

Therefore the integral above is equal to

$$\sum_n^\infty [\operatorname{meas}(N_m^- - N_{m+1}^-)] \chi(a_\alpha)^m \delta^{1/2}(a_\alpha)^{-m}.$$

From (1.(13) one sees that

$$\operatorname{meas} N_m^- = q_{\alpha/2}^{-[m+1/2]} q_\alpha^{-m} \quad (m \geq 1)$$

and from (1.(15) that

$$\delta(a_\alpha) = q_{\alpha/2}^{-1} q_\alpha^{-2}.$$

When $|\chi(a_\alpha)| < 1$, therefore, it is easy to deduce that the above sum is dominated by an absolutely convergent geometric series.

When $f = ch_B$, $m = 1$. The sum may be calculated explicitly by breaking it up into even and odd terms, thus concluding the proof.

For χ such that $|\chi(a_\alpha)| < 1$, the functional Λ induces a functional λ on $I(\chi)$ such that

$$\lambda(R(p)f) = \chi^{-1}\delta^{1/2}(p)\lambda(f).$$

By Frobenius reciprocity, it corresponds to a G-morphism from $I(\chi)$ to $I(w\chi)$. This must be a scalar multiple of T_w, and since for $f \in C_c^\infty(PwP)$

$$\Lambda(f) = T_w(f)(1)$$

it corresponds exactly to T_w. Therefore when $|\chi(a_\alpha)| < 1$, and by analytic continuation for all regular χ,

$$T_w(\phi_1)(1) = c_\alpha(\chi) - 1 \quad \text{and} \quad T_w(\phi_K)(1) = c_\alpha(\chi).$$

Step (2). Let G be arbitrary, but $w = w_\alpha$, $\alpha \in \Delta$, again. In this case, each ϕ_w with $w \neq 1$, w_α lies in the complement of $P \cup P w_\alpha P$

$$T_{w_\alpha}(\phi_w)(1) = \int_{w_\alpha N w_\alpha^{-1} \cap N = N} \phi_w(w_\alpha n)\, dn = 0$$

and $T_{w_\alpha}(\phi_1)(1)$ and $T_{w_\alpha}(\phi_{w_\alpha})(1)$ may be calculated exactly as in Step (1). Since $\phi_K = \Sigma\, \phi_w$, the theorem is proven in this case.

Step (3). Proceed by induction on the length of w. Let $\Psi_w = \{\alpha > 0 \mid w\alpha < 0\}$. Then if $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ (a) $\Psi_{w_1 w_2} = w_2^{-1}\Psi_{w_1} \cup \Psi_{w_2}$ and (b) $T_{w_1 w_2} = T_{w_1} T_{w_2}$, and applying these will conclude the proof.

3.3. REMARK: When G is split, each $q_\alpha = q$ and each $q_{\alpha/2} = 1$. In this case,

$$c_\alpha(\chi) = \frac{1 - q^{-1}\chi(a_\alpha)}{1 - \chi(a_\alpha)}.$$

I won't use it in this paper, but it will be useful elsewhere to have this partial generalization:

3.4. THEOREM: *If* $\alpha \in \Delta$ *and* $\ell(w_\alpha w) > \ell(w)$, *then*

$$T_{w_\alpha}(\phi_w) = (c_\alpha(\chi) - 1)\phi_w + q_\alpha^{-1} q_{\alpha/2}^{-1} \phi_{w_\alpha w}$$
$$T_{w_\alpha}(\phi_{w_\alpha w}) = \phi_w + (c_\alpha(\chi) - q_\alpha^{-1} q_{\alpha/2}^{-1})\phi_{w_\alpha w}.$$

PROOF: One has

$$\begin{aligned} T_{w_\alpha}(\phi_1)(w_\alpha) &= [B w_\alpha B : B]^{-1} R(ch(B w_\alpha B)) T_{w_\alpha}(\phi_1))(1) \\ &= q_\alpha^{-1} q_{\alpha/2}^{-1} T_{w_\alpha}(\phi_{w_\alpha})(1) \\ &= q_\alpha^{-1} q_{\alpha/2}^{-1}. \end{aligned}$$

Since in the rank one case $T_{w_\alpha}(\phi_K) = c_\alpha(\chi)\phi_K$, one also has

$$T_{w_\alpha}(\phi_{w_\alpha})(w_\alpha) = c_\alpha(\chi) - q_\alpha^{-1} q_{\alpha/2}^{-1}$$
$$T_{w_\alpha}(\phi_{w_\alpha})(1) = 0 \quad \text{for } w \neq 1, w_\alpha.$$

Therefore, since T_{w_α} takes any ϕ_w into a linear combination of ϕ_w's:

$$T_{w_\alpha}(\phi_1) = (c_\alpha(\chi) - 1)\phi_1 + q_\alpha^{-1} q_{\alpha/2}^{-1} \phi_{w_\alpha}$$
$$T_{w_\alpha}(\phi_{w_\alpha}) = \phi_1 + (c_\alpha(\chi) - q_\alpha^{-1} q_{\alpha/2}^{-1})\phi_{w_\alpha}.$$

The theorem follows from this because $R(ch(BwB))\phi_1 = \phi_w$ and $R(ch(BwB))\phi_{w_\alpha} = \phi_{w_\alpha w}$.

This result tells the effect of T_{w_α} on $I(\chi)^B$, but to find a reasonable way to describe the effect of every T_w on $I(\chi)^B$ seems rather difficult.

As a consequence of Theorem 3.1 one has:

3.5. PROPOSITION: (a) *The operator T_w is an isomorphism if and only if $c_{w^{-1}}(w\chi)c_w(\chi) \neq 0$.*

(b) $\mathrm{Ind}(\chi)$ *is irreducible if and only if $c_{w_\ell}(w_\ell\chi)c_{w_\ell}(\chi) \neq 0$.*

PROOF: The operator $T_{w^{-1}} \circ T_w$ is a scalar multiple of the identity on $I(\chi)$. This scalar must be $c_{w^{-1}}(w\chi)c_w(\chi)$ by Theorem 3.1. If it is not 0, then T_w has an inverse. If it is 0, then either $T_w(\phi_K)$ or $T_{w^{-1}}(\phi_K) = 0$. If the first, T_w clearly has no inverse. If the second, then the image of T_w cannot be all of $\mathrm{Ind}(w\chi)$, and again has no inverse.

For (b), apply (a) and [7] 6.4.2.

3.6. PROPOSITION: *Assume that $q_{\alpha/2} \geq 1$ for all $\alpha > 0$. If $|\chi(a_\alpha)| < 1$ for all $\alpha > 0$, then ϕ_K generates $I(\chi)$.*

As I have mentioned earlier, the assumption $q_{\alpha/2} \geq 1$ amounts to restricting the initial choice of the special point x_0 – or, in other words, the subgroup K. When G is simply connected and of rank one, for example, and $q_{\alpha/2} \neq 1$ then the Proposition is true for one choice of K but not the other.

PROOF: Let U be the quotient of $I(\chi)$ by the G-space generated by ϕ_K. If $U \neq 0$, it will have an irreducible G-quotient (since it is finitely generated by Proposition 2.7). According to [7] 6.3.9 there will exist a G-embedding of this irreducible quotient into some $I(w\chi)$, and the composite map from $I(\chi)$ to $I(w\chi)$ must be a non-zero multiple of T_w. Since $U^K = 0$, $T_w(\phi_K) = 0$. Therefore $c_w(\chi) = 0$, and for some $\alpha > 0$ either $\chi(a_\alpha) = q_\alpha q_{1/2}$ or $\chi(a_\alpha) = -q_{1/2}$, contradicting the assumption.

This is the 𝔭-adic analogue of a well known result of Helgason on real groups.

I want now to introduce a new basis of $I(\chi)^B$ (still under the assumption that χ is regular). Recall from Proposition 2.3 that $I(\chi)^B \cong I(\chi)_N$, and again from §6.4 of [7] that $I(\chi)$ is isomorphic to the direct sum $\bigoplus (w\chi)\delta^{1/2}$. Explicitly, the maps

$$\Lambda_w : \phi \to T_w(\phi)(1)$$

form a basis of eigenfunctions of the dual of $I(\chi)_N$ with respect to the action of U. Let $\{f_w\} = \{f_{w,\chi}\}$ be the basis of $I(\chi)^B$ dual to this – thus

$$\Lambda_w(f_x) = \begin{cases} 0 & (x \neq w) \\ 1 & (x = w) \end{cases}$$

It is an unsolved problem and, as far as I can see, a difficult one to express the bases $\{\phi_w\}$ and $\{f_w\}$ in terms of one another. This is directly related to the problem I mentioned at the end of the proof of Theorem 3.4. The only fact which is simple is:

3.7. PROPOSITION: *One has* $f_{w_\ell} = \phi_{w_\ell}$.

PROOF: For $w \neq w_\ell$,

$$T_w(\phi_{w_\ell})(1) = \int_{wNw^{-1}\cap N\setminus N} \phi_{w_\ell}(w^{-1}n)\,dn = 0$$

because $\operatorname{supp}(\phi_{w_\ell}) \subseteq Pw_\ell P$, while

$$\begin{aligned} T_{w_\ell}(\phi_{w_\ell})(1) &= \int_N \phi_{w_\ell}(w_\ell n)\,dn \\ &= \int_{N_0} dn = 1. \end{aligned}$$

Also, by the definition of the $\{f_w\}$ and Theorem 3.1:

3.8. LEMMA: *One has*

$$\phi_K = \Sigma\, c_w(\chi) f_w.$$

It follows immediately from the definition of the $\{f_w\}$ and Proposition 2.5 that:

3.9. LEMMA: *One has* $\pi(ch_{BmB})f_w = \operatorname{meas}(BmB)(w\chi)\delta^{1/2}(m)f_w$ *for all* $m \in M^-$.

4. The spherical function

As I have mentioned earlier, the contragredient of $I(\chi)$ is $I(\chi^{-1})$. Consider the matrix coefficient

$$\Gamma_\chi(g) = \langle R(g)\phi_{K,\chi}, \phi_{K,\chi^{-1}}\rangle.$$

According to [7] 3.1.3 this is also equal to

$$\int_K \phi_{K,\chi}(gk)\phi_{K,\chi^{-1}}(k)\,\mathrm{d}k = \int_K \phi_{K,\chi}(gk)\,\mathrm{d}k$$

where meas $K = 1$. The function Γ_χ is called the zonal spherical function corresponding to χ. It satisfies

(1) $$\Gamma_\chi(1) = 1;$$

(2) $$\Gamma_\chi(k_1 g k_2) = \Gamma_\chi(g) \quad \text{for all } k_1, k_2 \in K \quad \text{and} \quad g \in G.$$

4.1. Proposition: *For any $w \in W$, $\Gamma_{w\chi} = \Gamma_\chi$.*

Proof: The matrix coefficient Γ_χ is the only matrix coefficient of $I(\chi)$ satisfying (1) and (2). As such, it is determined by the isomorphism class of $I(\chi)$. But since by Proposition 3.5 the representations $I(\chi)$ and $I(w\chi)$ are generically isomorphic, $\Gamma_\chi = \Gamma_{w\chi}$ generically as well; since Γ_χ clearly depends holomorphically on χ, $\Gamma_\chi = \Gamma_{w\chi}$ for all χ.

Define

$$\gamma(\chi) = c_{w_\ell}(w_\ell \chi) = \prod_{\alpha>0} \frac{(1 - q_{\alpha/2}^{-1/2} q_\alpha^{-1}\chi(a_\alpha)^{-1})(1 + q_{\alpha/2}^{-1/2}\chi(a_\alpha)^{-1})}{1 - \chi(a_\alpha)^{-2}}$$

Note that because of the Cartan decomposition, Γ_χ is determined by its restriction to M^-.

4.2. Theorem (Macdonald): *If χ is regular then for all $m \in M^-$*

$$\Gamma_\chi(m) = Q^{-1}\Sigma\gamma(w\chi)((w\chi)\delta^{1/2})(m) \quad (w \in W)$$

where

$$Q = \Sigma q(w)^{-1} \quad (w \in W).$$

Proof: One has

$$\phi_K = \Sigma c_w(\chi) f_w,$$

therefore

$$\begin{aligned} \Gamma_\chi(m) &= \mathcal{P}_K(R(m)\phi_K)(1) \\ &= \Sigma c_w(\chi)\mathcal{P}_K(R(m)f_w)(1) \\ &= \Sigma c_w(\chi)\mathcal{P}_K(\mathcal{P}_B R(m)f_w)(1) \end{aligned}$$

(since $B \subseteq K$)

$$= \Sigma c_w(\chi)(w\chi)\delta^{1/2}(m)\mathcal{P}_K f_w(1)$$

(by Proposition 3.9).

By Proposition 3.7,

$$\begin{aligned} \mathcal{P}_K f_{w_\ell} = \mathcal{P}_K \phi_{w_\ell} &= \mathrm{meas}(Bw_\ell B)\phi_K \\ &= Q^{-1}\phi_K \end{aligned}$$

(by (1.9) and the remarks preceding it). Therefore the term in the sum above corresponding to w_ℓ is $Q^{-1}c_{w_\ell}(w_\ell\chi)$. By the W-invariance of Γ_χ (Proposition 4.1) and the linear independence of the χ's ([10] 4.5.7) this implies the theorem.

4.3. Remark: The general theory of the asymptotic behavior of matrix coefficients (§4 in [7]) asserts the existence of $\epsilon > 0$ such that ϕ_K is a linear combination of the characters $(w\chi)\delta^{1/2}$ on $A^-(\epsilon)$. Macdonald's formula makes this explicit.

Appendix

Let Σ be a root system, Σ^+ a choice of positive roots, and (W, S) the corresponding Coxeter group. For $x, y \in W$, define $x < y$ to mean that y has a reduced decomposition $y = s_1 \cdots s_n$, where s_i is the elementary reflection associated to the simple root α_i, and $x = s_{i_1} \cdots s_{i_m}$ with $1 \leq i_1 < \cdots < i_m \leq n$. According to Lemma 3.7 of [3] (an easy application of the exchange condition of [5] Chapter IV, §1.5) one may take m to be the length of x in W. If $x < y$, then $\ell(x) \leq \ell(y)$, and $\ell(x) = \ell(y)$ if and only if $x = y$.

Let w_ℓ be the longest element in W. The following is, I believe, essentially due to Steinberg ([11] Exercise (a) on p. 128).

A.1. Proposition: *Let $x, y \in W$ be given. The following are equivalent:*

(a) $x < y$;
(b) $x^{-1} < y^{-1}$;
(c) *One has* $y = xw_1 \cdots w_r$, *where* w_i *is the re fection associated to the root* $\theta_i > 0$, *and* $xw_1 \cdots w_{i-1}(\theta_i) > 0$;
(d) $w_\ell x > w_\ell y$.

PROOF: (a)$\Leftrightarrow$(b) is immediate.

For (c)$\Rightarrow$(a): Suppose that y has the reduced decomposition $y = s_1 \cdots s_n$, and assume at first that $y = xw$, where w is the reflection corresponding to the root $\theta > 0$, and $x(\theta) > 0$. Then $y(\theta) = x(-\theta) < 0$, so that according to [5] Cor. 2, p. 158, there exists i such that $\theta = s_n \cdots s_{i+1}(\alpha_i)$. Then $w = (s_n \cdots s_{i+1})s_i(s_n \cdots s_{i+1})^{-1}$ and $x = s_1 \cdots s_{i-1}s_{i+1} \cdots s_n$, so that indeed $x < y$.

In the general case, let $y = xw_1 \cdots w_v$ as in (c), and let $y_i = xw_1 \cdots w_{i-1}$ for each i. By what I have just shown, $y = y_r > y_{r-1} > \cdots > x$, and since $<$ is clearly transitive, $x < y$.

(a)$\Rightarrow$(c): Proceed by induction on the length of x. If $\ell(x) = 0$, then $x = 1$ and $y = s_1 \cdots s_n$, where by [5] Cor. 2, p. 158, one has $s_1 \cdots s_{i-1}(\alpha_i) > 0$.

In general, say $x = s_{i_1} \cdots s_{i_m}$ is a reduced decomposition of x. Let $x' = s_{i_2} \cdots s_{i_m}$, $y' = s_{i_1+1} \cdots s_n$. Then $\ell(x') < \ell(x)$ and $x' < y'$, so that by the induction hypothesis $y' = x'w'_1 \cdots w'_r$ as in (c), say w'_i corresponding to θ'_i. One now has

$$\begin{aligned} y &= s_1 \cdots s_{i_1} y' \\ &= s_1 \cdots s_{i_1} x' w'_1 \cdots w'_r \\ &= s_1 \cdots s_{i_1-1} x w'_1 \cdots w'_r \end{aligned}$$

Let $k = i_1 - 1$ for convenience. Then

$$\begin{aligned} y &= s_1 \cdots s_k x \\ &= x \cdot (x^{-1}s_k x)((s_k x)^{-1} s_{k-1}(s_k x)) \cdots ((s_2 \cdots s_k x)^{-1} s_1 (s_2 \cdots s_k x)). \end{aligned}$$

Let θ_j be the root $(s_{j+1} \cdots s_k x)^{-1}(\alpha_j)$, w_j correspond to θ_j. One has

$$y = xw_k w_{k-1} \cdots w_1$$

and further (1) $\theta_j = (x^{-1}s_k \cdots s_{j+1})(\alpha_j) > 0$ according to [5] Cor. 2, p. 158, since by assumption on the original y one has $\ell(s_j \cdots s_k x) > \ell(s_{j+1} \cdots s_k x)$; (2) $xw_k \cdots w_{j+1}(\theta_j) = s_{j+1} \cdots s_k x(\theta_j) = \alpha_j > 0$.

(c)$\Leftrightarrow$(d): One has $y = xw_1 \cdots w_r$ as in (c) $\Leftrightarrow x < y \Leftrightarrow x^{-1} < y^{-1} \Leftrightarrow y^{-1} = x^{-1}w_1' \cdots w_s'$ as in (c) $\Leftrightarrow y = w_s' \cdots w_1'x \Leftrightarrow w_\ell y = w_\ell w_s' w_\ell^{-1} \cdots w_\ell x \Leftrightarrow (w_\ell y)^{-1} = (w_\ell x)^{-1}(w_\ell w_1' w_\ell^{-1}) \cdots (w_\ell w_s' w_\ell^{-1})$. Note that $w_\ell w_i' w_\ell^{-1}$ is the reflection associated to $\bar{\theta}_i' = w_\ell(-\theta_i')$.

REFERENCES

[1] A. Borel: Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup. *Inventiones Math. 35* (1976) 233–259.
[2] A. Borel and J. Tits: Groupes réductifs, Publ. Math. I.H.E.S. 27 (1965) 55–151.
[3] A. Borel and J. Tits: Compléments à l'article "Groupes réductifs", Publ. Math. I.H.E.S. 41 (1972) 253–276.
[4] A. Borel and J. Tits: Homomorphismes "abstraits" de groupes algebriques simples. *Annals of Math. 97* (1973) 499–571.
[5] N. Bourbaki: *Groupes et algèbres de Lie.* Chapitres IV, V, et VI. Hermann, Paris, 1968.
[6] F. Bruhat and J. Tits: Groupes réductifs sur un corps local, Publ. Math. I.H.E.S. 41 (1972) 1–251.
[7] W. Casselman: Introduction to the theory of admissible representations of $\mathfrak{p}$-adic reductive groups (to appear).
[8] N. Iwahori: Generalized Tits systems on $\mathfrak{p}$-adic semi-simple groups, in *Algebraic Groups and Discontinuous Subgroups.* Proc. Symp. Pure Math. IX. A.M.S., Providence, 1966.
[9] I.G. Macdonald: Spherical functions on a $\mathfrak{p}$-adic Chevalley group. *Bull. Amer. Math. Soc. 74* (1968) 520–525.
[10] I.G. Macdonald: *Spherical functions on a group of $\mathfrak{p}$-adic type.* University of Madras, 1971.
[11] R. Steinberg: *Lectures on Chevalley groups.* Yale University Lecture Notes, 1967.
[12] H. Matsumoto: *Analyse Harmonique dans les Système de Tits Bornologiques de Type Affine.* Springer Lecture Notes #590, Berlin, 1977.
[13] J. Tits: Reductive groups over local fields. Proc. Symp. Pure Math. XXXIII, Amer. Math. Soc., Providence, 1978.

(Oblatum 13–XI–1978)

Department of Mathematics
The University of British Columbia
2075 Westbrook Place
Vancouver, B.C. V6T 1W5
Canada