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ON ISOSPECTRAL DEFORMATIONS OF RIEMANNIAN METRICS

Ruishi Kuwabara

1. Introduction

Let M be an n -dimensional compact orientable C^∞ manifold, and g be a C^∞ Riemannian metric on M . It is known that the Laplace-Beltrami operator $\Delta_g = -g^{ij}\nabla_i\nabla_j$ acting on C^∞ functions on M has an infinite sequence of eigenvalues (denoted by $\text{Spec}(M, g)$)

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots \uparrow +\infty$$

each eigenvalue being repeated as many as its multiplicity.

Consider the following problem [1, p. 233]: Let $g(t)$ ($-\epsilon < t < \epsilon$, $\epsilon > 0$) be a 1-parameter C^∞ deformation of a Riemannian metric on M . Then, is there a deformation $g(t)$ such that $\text{Spec}(M, g(t)) = \text{Spec}(M, g(0))$ for every t ? Such a deformation is called an *isospectral deformation*.

First, we give some definitions to state the results of this article. The deformation $g(t)$ is called *trivial* if for each t there is a diffeomorphism $\eta(t)$ such that $g(t) = \eta(t)^*g(0)$ (the pull-back of $g(0)$ by $\eta(t)$). For a deformation $g(t)$ the symmetric covariant 2-tensor $h \equiv g'(0)$ is called the *infinitesimal deformation* (*i-deformation*, for short) [2]. By Berger-Ebin [3] h is decomposed as

$$h = \tilde{h} + L_X g(0),$$

where $\nabla^i \tilde{h}_{ij} = 0$ (∇ being the connection induced by $g(0)$) and L_X is the Lie derivative with respect to X . The *i-deformation* h is called *trivial* if $\tilde{h} = 0$.

The main result of this article is the following.

THEOREM A: *There is no non-trivial isospectral i -deformation of a metric of flat torus.*

REMARK: Concerning the isospectral deformation of a metric of constant curvature, we can easily get the following: *There is no non-trivial isospectral deformation of a metric of constant curvature K if $2 \leq \dim M \leq 5$, or $\dim M = 6$ and $K > 0$.* This result is obtained by combining the results of spectral geometry [1], [4] and those concerning the non-deformability of a metric of constant curvature, the latter being directly derived from the results of Berger-Ebin [3], Mostow [5] and Koiso [2]. (See Tanaka [6], for the case $\dim M = 2$ and $K < 0$.) In the case of flat metrics Sunada [7] showed that there are only finitely many isometry classes of flat manifolds with a given spectrum.

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2. Proof of Theorem A

The main part of the proof is to study the variation of the coefficients of Minakshisundaram's expansion:

$$\sum_{k=0}^{\infty} \exp(-\lambda_k s) \underset{s \rightarrow +0}{\sim} \left(\frac{1}{4\pi s} \right)^{n/2} \sum_{j=0}^{\infty} a_j s^j.$$

If $g(t)$ is an isospectral deformation, the coefficients $a_j(t)$ must be constants.

For a Riemannian metric g on M , $dV(g)$, R_{jkm}^i , R_{ij} and τ denote the volume element, the curvature tensor, the Ricci tensor and the scalar curvature, respectively. The coefficients a_j ($j = 0, 1, 2$) are given by

$$a_0 = \text{vol}(M, g) = \int_M dV(g), \quad a_1 = \frac{1}{6} \int_M \tau dV(g),$$

$$a_2 = \frac{1}{360} \int_M (2|R|^2 - 2|\rho|^2 + 5\tau^2) dV(g),$$

where $|R|^2 = R_{ijkm}R^{ijkm}$ and $|\rho|^2 = R_{ij}R^{ij}$.

Let $g(t)$ be a deformation and $h(t) \equiv g'(t)$ the i -deformation of $g(t)$

at t . Let $a_j(t)$ be the Minakshisundaram's coefficients for $(M, g(t))$. Then the following formulas are obtained by straightforward calculation.

$$(2.1) \quad a'_0(t) = \frac{1}{2} \int_M h_s^s \, dV(g(t)),$$

$$(2.2) \quad a'_1(t) = \frac{1}{6} \int_M (\frac{1}{2} \tau h_s^s - R_{ij} h^{ij}) \, dV(g(t)),$$

$$(2.3) \quad a'_2(t) = \frac{1}{360} \int_M [12(\nabla_j \nabla_i \tau) h^{ij} - 6(\nabla_k \nabla^k R_{ij}) h^{ij} + 8R_{jk} R_i^k h^{ij} - 4R_{kijm} R^{km} h^{ij} - 4R_{jkms} R_i^{kms} h^{ij} + 9(\Delta \tau) h_s^s - 10\tau R_{ij} h^{ij} + |R|^2 h_s^s - |\rho|^2 h_s^s + \frac{5}{2} \tau^2 h_s^s] \, dV(g(t)).$$

Let (M, g_0) be a flat manifold. Then we have

LEMMA 2.1: *If $g(t)$ is an isospectral deformation with $g(0) = g_0$, and $\nabla^i h_{ij} = 0$ holds at $t = 0$, then*

$$(2.4a, b) \quad h_s^s = 0, \quad \nabla_k h_{ji} = 0$$

hold at $t = 0$.

PROOF: Starting from (2.3), we have by tedious calculation,

$$(2.5) \quad a''_2(0) = \frac{1}{120} \int_M [(\nabla_k \nabla^k h^{ij})(\nabla_m \nabla^m h_{ji}) + 3(\Delta h_s^s)^2] \, dV(g_0).$$

(Note that $\nabla^i h_{ij} = 0$ and $R^i_{jkm} = 0$ at $t = 0$.) From (2.1) and (2.5), $a'_0(0) = a''_2(0) = 0$ holds if and only if (2.4a, b) hold good. Q.E.D.

LEMMA 2.2: *Let $g(t)$ be as in Lemma 2.1. Then*

$$(2.6) \quad \int_M h_{ij} \phi(\nabla^i \nabla^j \phi) \, dV(g_0) = 0$$

holds for each eigenfunction ϕ of $\Delta_{g(0)}$.

PROOF: For the eigenvalue $\lambda_k(t)$ of $\Delta_{g(t)}$, the following was obtained by Berger [8]:

$$(2.7) \quad \lambda'_k(0) = \int_M [h_{ij} \phi(\nabla^i \nabla^j \phi) + (\nabla^i h_{ij} - \frac{1}{2} \nabla_j h_s^s) \phi(\nabla^j \phi)] \, dV(g_0),$$

where ϕ is the eigenfunction for $\lambda_k(0)$. Therefore, (2.6) follows from (2.4). Q.E.D.

Now, let us prove Theorem A. Let (M, g_0) be a flat torus given by R^n/L , where L is a lattice, i.e., a discrete abelian subgroup of the group of Euclidean motions in R^n . Let L^* denote the dual lattice, consisting of all $x \in R^n$ such that $(x, y) = \sum_{i=1}^n x^i y^i$ is an integer for all $y \in L$. Then the sets of eigenfunctions and eigenvalues are given by $\{\phi_x(y) = \cos 2\pi(x, y), \psi_x(y) = \sin 2\pi(x, y); x \in L^*\}$ and $\{4\pi^2(x, x); x \in L^*\}$, respectively. Recalling (2.4b), we see that h_{ij} are constants in the coordinates induced from R^n . Therefore, from (2.6) we have

$$4\pi^2 h_{ij} x^i x^j \int_{R^n/L} \{\cos 2\pi(x, y)\}^2 dy = 0, \quad x \in L^*,$$

hence $h_{ij} x^i x^j = (hx, x) = 0$ for $x \in L^*$. This leads to $h = 0$, because the set $\{x/\|x\|; x \in L^*\}$ is obviously dense in $\{x \in R^n; \|x\| = 1\}$. Q.E.D.

3. Conformal deformations

In this section we restrict our study to the conformal deformation,

$$(3.1) \quad g(t) = e^{2\rho(t)} g_0, \quad \text{with } \rho(0) \equiv 0.$$

Set $\sigma(t) = \rho'(t)$, and straightforward calculation gives

$$(3.2) \quad a_1''(t) = \frac{1}{3}(n-1)(n-2) \int_M \sigma \Delta \sigma \, dV(g(t)) + \frac{1}{6}(n-2)^2 \int_M \tau \sigma^2 \, dV(g(t)) \\ + \frac{1}{6}(n-2) \int_M \tau \frac{\partial \sigma}{\partial t} \, dV(g(t)).$$

LEMMA 3.1: *Assume that (M, g_0) has a constant scalar curvature. If $g(t)$ is a volume-preserving deformation, we have*

$$(3.3) \quad a_1''(0) = \frac{1}{3}(n-1)(n-2) \int_M \sigma \Delta \sigma \, dV(g_0) - \frac{1}{3}(n-2)\tau \int_M \sigma^2 \, dV(g_0).$$

PROOF: It is easy to see that $a_0''(0) = 0$ is written as

$$\int_M \frac{\partial \sigma}{\partial t} \, dV(g_0) + n \int_M \sigma^2 \, dV(g_0) = 0.$$

By this equation and $\tau = \text{const.}$, (3.2) is led to (3.3).

Q.E.D.

From now on, we assume $n = \dim M > 2$.

By virtue of the above lemma we have the following theorem.

THEOREM B: *Let (M, g_0) have a constant scalar curvature and λ_1 be the non-zero first eigenvalue of $\Delta_{g(0)}$. If*

$$(3.4) \quad \lambda_1 > \frac{\tau}{n-1}$$

holds, there is no conformal isospectral deformation of g_0 .

PROOF: Let $g(t)$ be a conformal isospectral deformation of g_0 . Since $a'_0(0) = 0$, we have $\int_M \sigma \, dV(g_0) = 0$. Therefore,

$$\int_M \sigma \Delta \sigma \, dV(g_0) \geq \lambda_1 \int_M \sigma^2 \, dV(g_0)$$

holds (see [1, p. 186], for example). Accordingly, if (3.4) holds, we have $a''_0(0) > 0$ unless $\sigma = 0$, i.e., $h = 0$. Q.E.D.

The condition (3.4) is obviously satisfied if $\tau \leq 0$. Further, as shown by Obata [9], it is also satisfied for an Einstein space not isometric with a sphere. In the case of a sphere, $\lambda_1 = \tau/(n-1)$, hence $a''_0(0) \geq 0$ holds. The equality holds only when σ is the eigenfunction for λ_1 , which is equivalent to $\nabla_i \nabla_j \sigma + \{\tau/n(n-1)\} \sigma g_{ij} = 0$ (see [9]). Therefore,

$$h_{ij} = 2\sigma g_{ij} = -\frac{2n(n-1)}{\tau} \nabla_i \nabla_j \sigma.$$

Thus h is trivial.

As a consequence we have the following theorem.

THEOREM C: *Suppose (M, g_0) is an Einstein space, or a space of non-positive constant scalar curvature. Then there is no non-trivial conformal isospectral i -deformation of g_0 .*

REMARK: In the case of $\dim M = 2$, it follows from the Gauss-Bonnet formula that the coefficient $a_1(t)$ is invariant.

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