

# COMPOSITIO MATHEMATICA

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*Compositio Mathematica*, tome 40, n° 2 (1980), p. 243-250

[http://www.numdam.org/item?id=CM\\_1980\\_\\_40\\_2\\_243\\_0](http://www.numdam.org/item?id=CM_1980__40_2_243_0)

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## SINGULARITY OF PARABOLIC MEASURES\*

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### Abstract

We show by example that a recent result of Dahlberg on harmonic measure for the Laplace equation can not be extended to parabolic measure for heat equation. The example is based on the non-self-adjointness of the heat operator; the methods are estimations of Green's function and construction of special boundary curves.

Let  $f(t)$  be a continuous function on  $(-\infty, \infty)$  and  $\Omega \subseteq R^2$  be the region  $\{(x, t): x > f(t)\}$ . Let  $m$  be the measure on  $\partial\Omega$  defined by  $m(E) =$  the Lebesgue measure of  $\{t: (f(t), t) \in E\}$ . If  $\Omega$  is Dirichlet regular for the heat equation (or adjoint heat equation), for a fixed point  $(y, s) \in \Omega$ , the parabolic measure (or adjoint parabolic measure) of a Borel set  $E \subseteq \partial\Omega$  at  $(y, s)$ , denoted by  $w^{(y,s)}(E)$  (or  $w^{*(y,s)}(E)$ ), is defined to be the value at  $(y, s)$  of the solution of the heat equation (or adjoint heat equation) on  $\Omega$  with boundary value 1 on  $E$  and 0 on  $\partial\Omega \setminus E$  in the Brelot-Peron-Wiener sense.

In case  $f(t) \equiv 0$ , and  $\Omega = \{x > 0\}$  it is known that  $m, w^{(x_0, t_0)}, w^{*(x_0, t_0)}$  are mutually absolutely continuous on  $\{(0, t): s_0 \leq t \leq t_0\}$ .

Let  $\lambda^{(y,s)}$  be the harmonic measure on  $\partial\Omega$  at  $(y, s)$  corresponding to the Laplace equation  $\partial^2/\partial x^2 + \partial^2/\partial t^2 = 0$ . It is known by conformal mapping that

**THEOREM A:** *If  $f$  is Lip 1, then  $\lambda^{(y,s)}$  and  $m$  are mutually absolutely continuous on  $\partial\Omega$ .*

In fact, Dahlberg [1] has proved Theorem A for  $x \in R^n, n \geq 2$ . The proof depends explicitly on the self-adjointness of the Laplace equation. Domains with Lip 1 boundaries are the most general regions on which the boundary behavior of harmonic functions has been extensively studied.

\* Research partially supported by an NSF-Grant at Illinois and an XL-Grant at Purdue.

Because the affine transformations  $\{x \rightarrow ax + b, t \rightarrow a^2t + c\}$  are the only diffeomorphisms that preserve solutions of heat equation [2], regions with  $\text{Lip} \frac{1}{2}$  boundaries are very natural for studying solutions of heat equation. In [5], Petrowski proved that if  $f(t)$  is  $\text{Lip} \frac{1}{2}$  then every point on  $\partial\Omega$  is a regular point for heat equation.

Richard A. Hunt proposed the problem whether  $m, w$  and  $w^*$  are mutually absolutely continuous on  $\partial\Omega$  if  $f(t)$  is  $\text{Lip} \frac{1}{2}$ . In [6], the first author proved the following

**THEOREM B:** *Suppose that  $f(t)$  is  $\text{Lip} \frac{1}{2}$  and  $E$  is a set on  $\partial\Omega$  with  $m$  measure zero, then  $E$  is composed of two parts, one with  $w^{(y,s)}$  measure zero, the other with  $w^{*(y,s)}$  measure zero for each  $(y, s) \in \Omega$ .*

In this note, we show by example that  $m, w$  and  $w^*$  can be mutually singular on  $\partial\Omega$ , for certain  $\text{Lip} \frac{1}{2}$  function  $f(t)$ .

### 1. Lemmas on parabolic functions

We call solutions of heat equation parabolic functions.

For fixed  $(y, s) \in R^2$ , we denote by  $W(x, t; y, s)$  the fundamental solution of heat equation defined by

$$W(x, t; y, s) = [4\pi(t - s)]^{-1/2} \exp\left[-\frac{|x - y|^2}{4(t - s)}\right] \quad \text{for } t > s$$

$$0 \quad \text{for } t \leq s.$$

Suppose  $\Omega$  is Dirichlet regular for heat equation, we say  $g$  is the Green's function for  $\Omega$ , if for each fixed  $(y, s) \in \Omega$ ,  $W(x, t; y, s) - g(x, t; y, s)$  is the bounded parabolic function in  $\Omega$  with boundary value  $W(x, t; y, s)$  for  $(x, t) \in \partial\Omega$ .

**LEMMA 1:** *Suppose  $D = \{(x, t): x > 2\sqrt{|t|}\}$ ,  $(X, -T)$  is a point in  $D$  with  $T > 0$  and  $g(x, t)$  is the Green's function on  $D$  with pole at  $(X, -T)$ . Then there are constants  $c$  and  $C$  depending on  $(X, -T)$  so that  $g(x, 0) \leq Cx^2$  for  $0 < x < c$ .*

**PROOF:** Let  $b = X - 2\sqrt{T}$  and  $S = \{(x, t); x = X - b/2 \text{ and } -T \leq t \leq 0\}$ . Let  $C_1 = \sup\{g(x, t): (x, t) \in S\}$  and

$$C_2 = \inf\left\{\frac{1}{\sqrt{8\pi T}} - W(x, t; 0, -2T): (x, t) \in S\right\}.$$

The level curve  $\gamma$  defined by  $W(x, t; 0, -2T) = 1/\sqrt{8\pi T}$  satisfies the equation:

$$\frac{1}{\sqrt{4\pi(t+2T)}} e^{-(x^2/4(t+2T))} = \frac{1}{\sqrt{8\pi T}}$$

or

$$x^2 = -2(t+2T) \log\left(1 + \frac{t}{2T}\right).$$

If  $(x, t) \in \gamma$  and  $-T \leq t < 0$  then  $x^2 < -4t$ . Therefore if  $(x, t) \in D$  and  $t \geq -T$ , then  $(1/\sqrt{8\pi T}) - W(x, t; 0, -2T) \geq 0$ . Let  $V$  be the region  $D \cap \{(x, t): t > -T\} \cap \{x < X - b/2\}$ . We observe by the definitions of  $C_1$  and  $C_2$  that

$$C_2 g(x, t) \leq C_1 \left( \frac{1}{\sqrt{8\pi T}} - W(x, t; 0, -2T) \right)$$

for  $(x, t) \in S$ . When  $(x, t)$  is on the part of  $\partial V$  with  $t = -T$  or on the part of  $\partial V$  with  $x = 2\sqrt{|t|}$ , the above inequality also holds because the left side is zero and the right side is positive. In view of the maximum principle for the solutions of heat equations [3, Chap. 2], we obtain

$$\begin{aligned} g(x, 0) &\leq \frac{C_1}{C_2} \left[ \frac{1}{\sqrt{8\pi T}} - W(x, 0; 0, -2T) \right] \\ &= \frac{C_1}{C_2} \frac{1}{\sqrt{8\pi T}} (1 - e^{-x^2/8T}) \\ &\leq Cx^2 \end{aligned}$$

for  $(x, t) \in S$ . When  $(x, t)$  is on the part of  $\partial V$  with  $t = -T$  or on the part of  $\partial V$  with  $x = 2\sqrt{|t|}$ , the above inequality also holds because the

Suppose  $f$  is  $\text{Lip } \frac{1}{2}$  satisfying  $|f(t) - f(\tau)| \leq M|t - \tau|^{1/2}$ . For  $a > 0$  we denote by  $\Delta(t, a) = \{(f(s), s): |s - t| \leq a\}$  and  $A(t, a) = (f(t) + 10M\sqrt{a}, t + 2a)$ . Under these assumptions, we may reformulate Lemma 1.4 in [4] and Lemma 2.2 in [6] as follows.

**LEMMA 2:** *There exist positive constants  $C, c$  depending on  $M$  only, so that*

$$w^{(y,s)}(\Delta(t, r)) \leq Cw^{(y,s)}(\Delta(t, a))w^{A(t,a)}(\Delta(t, r))$$

whenever  $0 < r < a/2$ ,  $(y, s) \in \Omega$  and  $|y - f(t)|^2 + |s - t| > ca$ .

LEMMA 3: Let  $(x_0, t_0), (y_0, s_0)$  be two fixed points in  $\Omega$  with  $t_0 > s_0 > a > 0$ , and  $g$  be the Green's function on  $\Omega$ . Then there are positive constants  $C, \mu, \rho$  depending on  $a, M, (x_0, t_0)$  and  $(y_0, s_0)$  so that

$$w^{(y_0, s_0)}(\Delta(t, r)) \leq Cr^{1/2}g(x_0, t_0; f(t) + \mu\sqrt{r}, t)$$

whenever  $-a < t < a$  and  $0 < r < \rho$ .

### 2. A test for singular measures

Suppose that  $\mu$  is a positive Borel measure on  $[0, 1]$ , and that  $\mu$  is not totally singular to Lebesgue measure  $m$ ; then  $d\mu/dx \geq c > 0$  on some set  $E$  of measure  $m(E) > 0$ . Thus

$$\begin{aligned} \lim \mu([x, x + h])h^{-1} &= \mu'(x) \text{ and} \\ \lim \mu([x - h, x])^{-1} &= \mu'(x) \end{aligned}$$

as  $h \rightarrow 0^+$ , when  $x \in E$ . Letting  $h = 1, \frac{1}{2}, \frac{1}{3}, \dots$  we can apply Egoroff's theorem to find a set  $E_0 \subseteq E$ , with  $m(E_0) > 0$ , and a sequence  $\epsilon_n > 0$  decreasing to 0, so that

$$\begin{aligned} |\mu([x, x + h]) - h\mu'(x)| &\leq \epsilon_n h, \text{ and} \\ |\mu([x - h, x]) - h\mu'(x)| &\leq \epsilon_n h, \end{aligned}$$

when  $(n + 1)^{-1} \leq h \leq n^{-1}$ , and  $x \in E_0$ . Let now  $r(n)$  be a polygonal function on  $(0, 1]$ , defined by the conditions  $r(n^{-1}) = \epsilon_{n-1}$  for  $n \geq 2$  and  $r$  is constant on  $[\frac{1}{2}, 1]$ . Then

$$|\mu([a, b]) - (b - a)\mu'(x)| \leq (b - a)r(b - a)$$

whenever  $0 \leq a < x < b \leq 1$  and  $x \in E_0$ .

Let  $0 < \delta < \frac{1}{4}$  and observe that by the Lebesgue density theorem,  $E_0$  must meet one of the sets  $[(k + \frac{5}{8})N^{-1}, (k + \frac{3}{4})N^{-1}]$  ( $1 \leq k \leq N - 2$ ) whenever  $N$  is sufficiently large. We apply the inequality on  $\mu$ -measures to intervals  $[a, b_1], [a, b_2], [a, b_3]$  with

$$\begin{aligned} a &= (k + \frac{1}{2})N^{-1}, \quad b_1 = (k + 1 - \delta)N^{-1}, \quad b_2 = (k + 1 + \delta)N^{-1}, \\ &\quad b_3 = (k + 1 + \frac{1}{2})N^{-1}. \end{aligned}$$

We write these inequalities as

$$E_i: |\mu([a, b_i]) - (b_i - a)\mu'(x)| \leq N^{-1}r(N^{-1}), \quad i = 1, 2, 3.$$

Combination of  $E_1$  and  $E_2$  gives

$$|\mu((b_1, b_2]) - (b_2 - b_1)\mu'(x)| \leq 2N^{-1}r(N^{-1}),$$

and comparison with  $E_3$  yields

$$|\mu((b_1, b_2]) - 2\delta\mu([a, b_3])| \leq 3N^{-1}r(N^{-1}).$$

Let us write  $I = [(k + \frac{1}{2})N^{-1}, (k + 1 + \frac{1}{2})N^{-1}]$ ,  $I_\delta = ((k + 1 - \delta)N^{-1}, (k + 1 + \delta)N^{-1}]$ , that is  $I = [a, b_3]$ ,  $I_\delta = (b_1, b_2]$ . We divide the last inequality by  $\mu(I)$ , which exceeds  $c/(2N)$  for  $N \geq N_0$ . We obtain

$$\mu(I_\delta)/\mu(I) \geq 2\delta + o(1), \quad N \rightarrow +\infty.$$

### 3. Construction of curves

Let  $h(t)$  be a function on  $[0, 1]$  subject to the following conditions

- 1)  $0 \leq h \leq 1$ ,
- 2)  $h(t) = 4t^{1/2}$  for  $0 \leq t \leq \frac{1}{3}$ ,
- 3)  $h(t) = h(1 - t)$ ,
- 4)  $h$  is of class  $C^1$  on  $[\frac{1}{4}, \frac{3}{4}]$ ,
- 5)  $|h(t) - h(s)| \leq 4|t - s|^{1/2}$  for  $0 \leq s \leq t \leq 1$ .

Let  $h_n(t)$  be the function on  $[0, 1]$  with period  $1/n$ , such that  $h_n(t) = h(nt)/n^{1/2}$  for  $0 \leq t \leq 1/n$ .

Let  $\ell_n(t)$  be a function of class  $C^1[0, 1]$ , periodic with period  $1/n$ , such that

- 6)  $0 \leq \ell_n \leq h_n$ ,
- 7)  $\ell_n(0) = 0$  and  $\ell_n(t) = h_n(t)$  for  $n^{-3} \leq t \leq n^{-1} - n^{-3}$ ,
- 8)  $|\ell_n(t) - \ell_n(s)| \leq 5|t - s|^{1/2}$ .

We shall choose a sequence  $(n_j)$  and set  $f_k(t) = \sum_1^k \ell_{n_j}(t)$ ,  $f(t) = \sum_1^\infty \ell_{n_j}(t)$ . We require the following properties of  $f$  and  $f_k$ :

- 9)  $|f(t) - f(s)| \leq 8|t - s|^{1/2}$ .
- 10)  $0 \leq f(t) - f_k(t) \leq n_k^{-3/2}$ .

11) The inequalities  $3|t - \tau|^{1/2} \leq f_k(t) - f_k(\tau) \leq 6|t - \tau|^{1/2}$  hold whenever  $\tau = i/n_k$  ( $1 \leq i \leq n_k - 1$ ) and  $n_k^{-3} \leq |t - \tau| \leq (4n_k)^{-1}$ .

To obtain 10) we observe the inequality  $0 \leq f(t) - f_k(t) \leq \sum_{k+1}^\infty n_j^{-1/2}$ , and simply choose  $n_{j+1} > 16n_j^3$ . This choice is compatible with the rest of the construction and we don't mention it again.

To obtain 11) and 9) we let  $B_{k-1}$  be an upper bound for  $|f'_{k-1}|$ , so that

$$\begin{aligned} |f_k(t) - f_k(\tau) - \ell_{n_k}(t) + \ell_{n_k}(\tau)| &\leq B_{k-1}|t - \tau| \\ &\leq B_{k-1}(4n_k)^{-1/2}|t - \tau|^{1/2}, \end{aligned}$$

for the numbers  $\tau, t$  mentioned in 11). By 7) and 2) we find

$$\ell_{n_k}(t) - \ell_{n_k}(\tau) = 4|t - \tau|^{1/2}$$

for these numbers, because  $n_k\tau$  is an integer, and we obtain 11) by taking  $B_{k-1}(4n_k)^{-1/2} < 1$ . To obtain 9) we suppose that  $|f_p(t) - f_p(s)| \leq (6 - p^{-1})|t - s|^{1/2}$ , for  $p = k - 1$ . (This is true when  $p = 1$ ). Then

$$|f_k(t) - f_k(s)| \leq |f_{k-1}(t) - f_{k-1}(s)| + |\ell_{n_k}(t) - \ell_{n_k}(s)|.$$

Since  $\ell_{n_k} \leq n_k^{-1/2}$ , we have the inequality

$$|f_k(t) - f_k(s)| \leq (6 - (k - 1)^{-1})|t - s|^{1/2} + 2n_k^{-1/2}.$$

Thus the required estimate is valid when  $(k^2 - k)^{-1}|t - s|^{1/2} \geq 2n_k^{-1/2}$  or  $|t - s| \geq 4n_k^{-1}(k^2 - k)^2$ . But when the last inequality is violated, we can use the estimation

$$|f_k(t) - f_k(s)| \leq B_{k-1}|t - s| + 5|t - s|^{1/2};$$

this yields the inequality in question provided  $B_{k-1}|t - s| \leq |t - s|^{1/2} \cdot \frac{1}{2}$ , or  $|t - s| \leq (2B_{k-1})^{-2}$ . One estimate or the other is available for large  $n_k$ .

#### 4. The theorem

We retain the notations from §3 and extend the function  $f$ , constructed in §3, to  $(-\infty, \infty)$  by defining  $f(t) = f(0)$  for  $t < 0$  and  $f(t) = f(1)$  for  $t > 1$ . We let  $\Omega$  be  $\{(x, t): x > f(t)\}$ ,  $w$  be the parabolic measure on  $\partial\Omega$  evaluated at  $(X, T)$  and  $w^*$  be the adjoint parabolic measure on  $\partial\Omega$  evaluated at  $(Y, S)$  where  $(X, T)$  and  $(Y, S)$  are two fixed points in  $\Omega$  with  $T > 1$  and  $S < 0$ .

We observe, with the aid of maximum principle, that for  $E \subseteq \{(f(t), t): 0 < t < 1\}$ , if  $w(E) = 0$  then  $w^{(x,t)}(E) = 0$  for every  $(x, t) \in \Omega$ ; and if  $w^*(E) = 0$  then  $w^{*(x,t)}(E) = 0$  for every  $(x, t) \in \Omega$ .

**THEOREM:** *None of the three measures:  $m, w$  and  $w^*$  on  $\partial\Omega$  is absolutely continuous with respect to another. In fact  $m, w$  and  $w^*$  are totally singular with respect to each other on  $\{(f(t), t): 0 < t < 1\}$ .*

We first prove the following lemma and assume as we may that  $(X, T) = (10, 100)$ .

LEMMA 4: *There are positive absolute constants  $C$  and  $\rho < \frac{1}{32}$ , so that whenever  $\tau = i/n_k$  ( $1 \leq i \leq n_k - 1$ ),  $n_k^{-1} < \delta < \rho$ ,*

$$I_k = \{(f(t), t) : |t - \tau| < (16n_k)^{-1}\} \text{ and}$$

$$E_k = \{(f(t), t) : |t - \tau| < \delta n_k^{-1}\}$$

then  $w(E_k) \leq C\delta^{3/2}w(I_k)$  for sufficiently large  $k$ .

PROOF: For a fixed  $\tau = i/n_k$ , we let  $A = (f(\tau) + 5/\sqrt{n_k}, \tau + 1/8n_k)$  and  $B$  be  $A + (0, 1/8n_k)$ . From 9) and Lemma 2 it follows that for some absolute constant  $C$ ,

$$w(E_k) \leq Cw(I_k)w^A(E_k).$$

Let  $\Phi$  be the map  $(x, t) \rightarrow (\sqrt{n_k}(x - f(\tau)), n_k(t - \tau))$  and  $G$  be the Green's function on  $\Phi(\Omega)$ . We note that  $\partial\Phi(\Omega) = \Phi(\partial\Omega)$  is the graph of a  $\text{Lip } \frac{1}{2}$  function with 8 as an upper bound for the  $\text{Lip } \frac{1}{2}$  constant and  $\Phi$  preserves parabolic functions (i.e.  $v$  is parabolic on  $\Phi(\Omega)$  if and only if  $v(\Phi)$  is parabolic on  $\Omega$ ). Let  $\bar{w}$  be the parabolic measure on  $\partial\Phi(\Omega)$ , thus  $\bar{w}^{\Phi(A)}(\Phi(E_k)) = w^A(E_k)$ . Because  $\Phi(A) = (5, \frac{1}{8})$  and  $\Phi(B) = (5, \frac{1}{4})$ , it follows from Lemma 3 that there exist absolute constants  $C$ ,  $\mu$  and  $\rho < \frac{1}{32}$ , so that

$$w^A(E_k) = \bar{w}^{\Phi(A)}(\Phi(E_k))$$

$$\leq C\sqrt{\delta}G(\Phi(B); \mu\sqrt{\delta}, 0)$$

if  $0 < \delta < \rho$ .

Let  $\alpha(t) = 2|t - \tau|^{1/2} + f(\tau) - 10n_k^{-3/2}$ . From 9), 10) and 11) it follows that  $\alpha(t) \leq f(t)$  whenever  $|t - \tau| < (4n_k)^{-1}$ . Therefore  $\Phi(\Omega) \cap \{(x, t) : |t| < \frac{1}{4}\} \subseteq \{(x, t) : x > 2|t|^{1/2} - 10n_k^{-1}\}$ . let  $\tilde{G}$  be the Green's function on  $\{(x, t) : x > 2|t|^{1/2} - 10n_k^{-1}\}$ . We recall that  $\Phi(B) = (5, \frac{1}{4})$  and  $n_k^{-1} < \delta < \rho$ , and obtain, by the maximum principle and the adjoint form of Lemma 1, that

$$G(\Phi(B); \mu\sqrt{\delta}, 0)$$

$$\leq \tilde{G}(\Phi(B); \mu\sqrt{\delta}, 0) = \tilde{G}^*(\mu\sqrt{\delta}, 0; \Phi(B))$$

$$\leq C(\mu\sqrt{\delta} + 10n_k^{-1})^2 \leq C\delta,$$

for absolute constants  $C$ . Thus  $w^A(E_k) \leq C\delta^{3/2}$ . This proves Lemma 4.

PROOF OF THE THEOREM: From the above lemma and the test for singular measures in §2 we see that  $w$  is totally singular to  $m$  on



$\{(f(t), t): 0 < t < 1\} \equiv S$ , that is, there is a set  $E \subseteq S$  of  $m$  measure zero but  $w(E) = w(S)$ . Similarly there is a set  $E^* \subseteq S$  of  $m$  measure zero but  $w^*(E^*) = w^*(S)$ . From these properties and Theorem B, we conclude that  $w$  and  $w^*$  are mutually singular. The theorem follows easily.

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(Oblatum 27-IV-1978 & 23-X-1978)

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