D. W. CURTIS

Hyperspaces of noncompact metric spaces

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0. Introduction

For a metric space $X$, the hyperspace $2^X$ of nonempty compact subsets and the hyperspace $C(X)$ of nonempty compact connected subsets are topologized by the Hausdorff metric, defined by $\rho(A, B) = \inf\{\varepsilon : A \subseteq N_\varepsilon(B) \text{ and } B \subseteq N_\varepsilon(A)\}$. It is easily seen that the hyperspace topologies induced by $\rho$ are invariants of the topology on $X$. It is known that $2^X \approx Q$, the Hilbert cube, if and only if $X$ is a non-degenerate Peano continuum, and $C(X) \approx Q$ if and only if $X$ is a nondegenerate Peano continuum with no free arcs [6]. In this paper we obtain various characterization theorems for hyperspaces of non-compact connected locally connected metric spaces.

**Theorem 1.6:** $2^X$ is an ANR (AR) if and only if $X$ is locally continuum-connected (connected and locally continuum-connected).

**Theorem 3.3:** $2^X \approx Q\text{point}$ if and only if $X$ is noncompact, connected, locally connected, and locally compact.

**Theorem 4.2:** $X$ admits a Peano compactification $\hat{X}$ such that $(2^{\hat{X}}, 2^X) = (Q, s)$ if and only if $X$ is topologically complete, separable, connected, locally connected, nowhere locally compact, and admits a metric with Property $S$.

Analogous results are obtained for $C(X)$. Additionally, we discuss two examples relating to local continuum-connectedness, and an example relating to Property $S$.
1. Hyperspaces which are ANR's

A growth hyperspace $\mathcal{G}$ of a metric space $X$ is any closed subspace of $2^X$ satisfying the following condition: if $A \in \mathcal{G}$ and $B \in 2^X$ such that $B \supset A$ and each component of $B$ meets $A$, then $B \in \mathcal{G}$. Both $2^X$ and $C(X)$ are growth hyperspaces of $X$. Another growth hyperspace of particular interest is $\mathcal{G}_A(X)$, the smallest growth hyperspace containing $A \in 2^X$. Thus $\mathcal{G}_A(X) = \{B \in 2^X : B \supset A \text{ and each component of } B \text{ meets } A\}$. Growth hyperspaces of Peano continua were studied in [4].

**Lemma 1.1**: (Kelley [11]). Let $A, B \in 2^X$ such that $B \in \mathcal{G}_A(X)$ and $B$ has finitely many components. Then there exists a path $\sigma : I \to \mathcal{G}_A(B)$ such that $\sigma(0) = A$ and $\sigma(1) = B$.

**Definition**: A metric space $X$ is continuum-connected if each pair of points in $X$ is contained in a subcontinuum. $X$ is locally continuum-connected if it has an open base of continuum-connected subsets.

Note that in verifying the local property it is sufficient to produce, for each neighborhood $U$ of a point $x$, a neighborhood $V \subset U$ of $x$ such that each $y \in V$ is connected to $x$ by a subcontinuum in $U$. For topologically complete metric spaces, the properties of local connectedness, local continuum-connectedness, and local path-connectedness are equivalent, since every complete connected locally connected metric space is path-connected. Examples given later show that in general these properties are not equivalent.

**Lemma 1.2**: Let $A \in 2^X$, with $X$ a locally continuum-connected metric space. Then for arbitrary $\epsilon > 0$ there exists $\tilde{A} \in \mathcal{G}_A(X)$ such that $\rho(A, \tilde{A}) < \epsilon$ and $\tilde{A}$ has finitely many components.

**Proof**: For each $n \geq 1$ choose $\epsilon_n > 0$ such that, whenever $x \in A$ and $y \in X$ with $d(x, y) < \epsilon_n$, there exists a continuum in $X$ connecting $x$ and $y$ with diameter less than $\min\{1/n, \epsilon\}$. For each $n$ let $A_n \subset A$ be a finite $\epsilon_n$-net for $A$. Then for each $p \in A_{n+1}$ there exists a continuum $L_p$ in $X$ with diameter less than $\min\{1/n, \epsilon\}$, connecting $p$ and some point of $A_n$. Set $\tilde{A}_1 = A_1$ and $\tilde{A}_{n+1} = \cup \{L_p : p \in A_{n+1}\}$ for each $n \geq 1$. Then $\tilde{A} = \text{cl}(\cup \tilde{A}_n) = \cup \tilde{A}_n \cup A$ has the required properties (note that each component of $\tilde{A}$ meets the finite subset $A_1$).

**Lemma 1.3**: Let $X$ be a connected and locally continuum-con-
nected metric space. Then every compact subset is contained in a continuum.

PROOF: Let $A$ be a compact subset of $X$. There exists by Lemma 1.2 a compact set $\tilde{A} \supset A$ such that $\tilde{A}$ has finitely many components. It is easily seen that $X$ is continuum-connected. Thus the components of $\tilde{A}$ may be connected together by the addition of a finite collection of subcontinua of $X$, thereby producing a continuum $B \supset \tilde{A} \supset A$.

**Lemma 1.4:** If $X$ is a locally continuum-connected metric space, then every growth hyperspace $\mathcal{G}$ of $X$ is locally path-connected.

PROOF: Given $A \in \mathcal{G}$ and $\epsilon > 0$, choose $\delta > 0$ such that whenever $x \in A$ and $y \in X$ with $d(x, y) < \delta$, there exists a continuum in $X$ of diameter less than $\epsilon$ connecting $x$ and $y$. We claim that for any $B \in \mathcal{G}$ with $\rho(A, B) < \delta$, there exists a path $\sigma: [0, 1] \to \mathcal{G}$ between $A$ and $B$, with $\rho(A, \sigma(t)) < \epsilon$ for each $t$. We may assume by Lemmas 1.1. and 1.2 that each of $A$ and $B$ has finitely many components. Adding a finite collection of continua to $A \cup B$, which connect each component of $A$ to $B$ and each component of $B$ to $A$, and all of which have diameter less than $\epsilon$, we obtain an element $C \in \mathcal{G}_A(X) \cap \mathcal{G}_B(X)$ such that $\rho(A, C) < \epsilon$. Then paths between $A$ and $C$, and $B$ and $C$, given by Lemma 1.1, will provide the desired path.

**Lemma 1.5:** Let $\mathcal{D} \subset 2^X$ be compact and connected, and let $A \in \mathcal{D}$. Then $\cup \mathcal{D} \in \mathcal{G}_A(X)$.

PROOF: Clearly, $\cup \mathcal{D}$ is a compact subset of $X$ and contains $A$. We show that each component of $\cup \mathcal{D}$ meets $A$. Let $x \in D \in \mathcal{D}$ be given. For each $\epsilon > 0$ there exists an $\epsilon$-chain $\{D_i\}$ in $\mathcal{D}$ between $D$ and $A$, and therefore an $\epsilon$-chain $\{q_i\}$ in $\cup \mathcal{D}$ between $x$ and some point of $A$. Since $A$ is compact, there exists $a \in A$ such that for each $\epsilon > 0$, there is an $\epsilon$-chain in $\cup \mathcal{D}$ between $x$ and $a$. Then $x$ and $a$ are in the same quasi-component, hence the same component, of $\cup \mathcal{D}$.

**Theorem 1.6:** If $X$ is locally continuum-connected (connected and locally continuum-connected), then every growth hyperspace $\mathcal{G}$ of $X$ is an ANR (AR). Conversely, if there exists a growth hyperspace $\mathcal{G}$ such that $\mathcal{G} \supset C(X)$ and $\mathcal{G}$ is an ANR (AR), then $X$ is locally continuum-connected (connected and locally continuum-connected).

PROOF: We use the Lefschetz–Dugundji characterization of metric
ANR's [9]: a metric space $M$ is an ANR if and only if, for each open cover $\alpha$ of $M$, there exists an open refinement $\beta$ such that every partial $\beta$-realization in $M$ of a simplicial polytope $K$ (with the Whitehead topology) extends to a full $\alpha$-realization of $K$. Thus, let $\alpha$ be an open cover of $\mathcal{G}$, and assume that the elements of $\alpha$ are open metric balls, with respect to the Hausdorff metric on $\mathcal{G}$. Take an open star-refinement $\alpha'$ of $\alpha$. By Lemma 1.4 there exists an open refinement $\beta$ of $\alpha'$ such that each element of $\beta$ is path-connected.

Then every partial $\beta$-realization $f : L \to \mathcal{G}$ of a polytope $K$ extends to a partial $\alpha$-realization $g : L \cup K^1 \to \mathcal{G}$, where $K^1$ is the 1-skeleton of $K$. Using Lemma 1.5, we may extend $g$ to a full $\alpha$-realization $h : K \to \mathcal{G}$ by the following inductive procedure. Consider an $n$-simplex $\sigma$ of $K$, $n \geq 2$, such that $h$ has been defined over $\partial \sigma$. Let $r : \sigma \to C(\partial \sigma)$ be any extension of the natural injection $\partial \sigma \to C(\partial \sigma)$. Then define $h$ over $\sigma$ by setting $h(x) = \cup \{h(p) : p \in r(x)\}$. Thus $\mathcal{G}$ is an ANR.

If additionally $X$ is connected, then by Lemma 1.3 every compact subset of $X$ is contained in a continuum. Thus for arbitrary $A, B \in \mathcal{G}$, there exists a continuum $C$ containing $A \cup B$, and $C \in \mathcal{G}$. By Lemma 1.1. there exist paths in $\mathcal{G}_A(C)$ from $A$ to $C$ and in $\mathcal{G}_B(C)$ from $B$ to $C$, hence a path in $\mathcal{G}$ between $A$ and $B$. Thus $\mathcal{G}$ is path-connected. Since the argument of the preceding paragraph shows that $\mathcal{G}$ is always $n$-connected for $n \geq 1$, it follows that $\mathcal{G}$ is an AR.

Conversely, suppose there exists a growth hyperspace $\mathcal{G}$ of $X$ such that $\mathcal{G} \supseteq C(X)$ and $\mathcal{G}$ is an ANR. Let $x \in X$ and a neighborhood $U$ be given. Since $\mathcal{G}$ is locally path-connected, there exists a neighborhood $V$ of $x$ such that for each $y \in V$, there exists a path $f : I \to \mathcal{G}$ between $\{x\}$ and $\{y\}$ with each $f(t) \subset U$. By Lemma 1.5, $\cup \{f(t) : t \in I\} \subset U$ is a continuum. Thus $X$ is locally continuum-connected. And if $\mathcal{G}$ is an AR, and therefore connected, $X$ must also be connected.

The ANR (AR) characterizations for the hyperspaces $2^X$ and $C(X)$ of a compact metric space $X$ were obtained by Wojdyslawski [15]. These characterizations were extended to complete metric spaces by Tašmetov [13]. Independently, some partial results along these lines were announced by Borges [3].

The following examples show that for noncomplete metric spaces, the property of local continuum-connectedness lies strictly between local connectedness and local path-connectedness.

**Example 1.7:** There exists a connected and locally connected subset of the plane which is not locally continuum-connected.

**Proof:** There exist disjoint subsets $A$ and $B$ of the plane $E^2$ such
that every nondegenerate continuum in the plane meets both \( A \) and \( B \) ([10], p. 110). Thus \( A \) contains no nondegenerate subcontinuum, and is not locally continuum-connected. However, \( A \) is connected and locally connected. Suppose \( A = A_1 \cup A_2 \) is a separation. Then there exists a closed subset \( C \) of the plane separating \( A_1 \) and \( A_2 \). Since \( C \) cannot be 0-dimensional, it contains a nondegenerate subcontinuum \( D \). Then \( D \) must meet \( A \), impossible. Thus \( A \) is connected, and the same argument applied locally shows that \( A \) is locally connected.

**Example 1.8:** There exists a connected and locally continuum-connected subset of the plane which is not locally path-connected.

**Proof:** We begin with the continuum

\[
S = \{(x, \sin 1/x) : 0 < |x| \leq 1/\pi\} \cup \{(0,t) : |t| \leq 1\}.
\]

A countable collection \( \{S_i\} \) of progressively smaller copies of \( S \) is then fitted inside the individual loops of \( S \) as indicated, creating local continuum-connectedness on the limit segment \( L = \{(0,t) : |t| \leq 1\} \subset S \). Then for each \( i \), a countable collection \( \{S_{ij}\} \) of copies of \( S \) is similarly fitted inside the loops of \( S_i \). The infinite iteration of this procedure produces the desired space \( X = S \cup (\cup \{S_i : i \geq 1\}) \cup (\cup \{S_{ij} : i, j \geq 1\}) \cup \ldots \).

\( X \) is connected and locally continuum-connected. However, \( X \) is not locally path-connected at any point on a limit segment such as \( L \). It suffices to show that there exists no path in \( X \) between the endpoints \( a = (-1/\pi, 0) \) and \( b = (1/\pi, 0) \). Suppose there exists such a path \( \sigma \). Then for some \( i \) (in fact, for infinitely many \( i \)), \( \sigma \) must contain a subpath \( \sigma_i \) in \( S_i \cup (\cup \{S_{ij} : j \geq 1\}) \cup \ldots \) between the corresponding endpoints \( a_i \) and \( b_i \) of \( S_i \). By the same argument \( \sigma_i \) must contain a subpath \( \sigma_{ij} \) in some \( S_{ij} \cup (\cup \{S_{ijk} : k \geq 1\}) \cup \ldots \) between the endpoints \( a_{ij} \) and \( b_{ij} \) of \( S_{ij} \). Thus the path \( \sigma \) must pass through each
member of some nested sequence \((S_i, S_{ij}, S_{ijk}, \ldots)\). But this is impossible, since the limit point of such a sequence is not included in \(X\).

2. Peano compactifications with locally non-separating remainders

Since \(2^Y \approx Q\) for every non-degenerate Peano space \(Y\), one way to study the hyperspace of a noncompact space \(X\) is to consider, when possible, a Peano compactification \(\bar{X}\) of \(X\), and the corresponding \(Q\)-compactification \(2^\bar{X}\) of \(2^X\). The procedure works if the remainder \(\bar{X} \setminus X\) is sufficiently “nice”. In this section we specify the desired property of the remainder, and establish the conditions under which such a compactification exists.

**Definition:** A subset \(A\) of \(X\) is locally non-separating in \(X\) if, for each nonempty connected open subset \(U\) of \(X\), \(U \cap A\) is nonempty and connected.

Note that if \(A\) is locally non-separating, so is every subset of \(A\). It is easily shown that if a locally connected space \(X\) has a connected open base \(\{U_a\}\) such that each \(U_a \cap A\) is nonempty and connected, then \(A\) is locally non-separating.

The motivation for considering locally non-separating subsets comes from the following pair of results on positional properties of intersection hyperspaces. For \(A_1, \ldots, A_n \in 2^X\), we define the intersection hyperspaces \(2^X(A_1, \ldots, A_n) = \{F \in 2^X : F \cap A_i \neq \emptyset \text{ for each } i\}\) and \(C(X; A_1, \ldots, A_n) = \{F \in C(X) : F \cap A_i \neq \emptyset \text{ for each } i\}\). For any nondegenerate Peano space \(X\), \(2^X(A_1, \ldots, A_n) \approx Q\), and \(C(X; A_1, \ldots, A_n) \approx Q\) if additionally \(X\) contains no free arcs [7]. A closed subset \(F\) of a metric space \(Y\) is a \(Z\)-set in \(Y\) if, for each compact subset \(K\) of \(Y\) and \(\varepsilon > 0\), there exists a map \(\eta : K \to Y \setminus F\) with \(d(\eta, id) < \varepsilon\).

**Proposition 2.1:** Let \(A\) be a closed subset of a Peano continuum \(X\). Then \(2^X(A)\) is a \(Z\)-set in \(2^X\) if and only if \(A\) is locally non-separating in \(X\). More generally, for closed subsets \(A, B_1, \ldots, B_n\) of \(X\), \(2^X(A, B_1, \ldots, B_n)\) is a \(Z\)-set in \(2^X(B_1, \ldots, B_n)\) if and only if \(A\) is locally non-separating in \(X\) and \(B_i \setminus A\) is dense in \(B_i\) for each \(i\).

**Proof:** Suppose \(A\) satisfies the stated conditions, and let \(\varepsilon > 0\) be given. We must construct a map \(\eta : 2^X(B_1, \ldots, B_n) \to 2^X(B_1, \ldots, B_n) \setminus 2^X(A, B_1, \ldots, B_n)\) such that \(\rho(\eta, id) < \varepsilon\). For each \(i\), there exists a finite \(\varepsilon/3\)-net \(\beta_i\) for \(B_i\) such that \(\beta_i \subset B_i \setminus A\). By [7], there
exists an “expansion” map \( h : 2^X(B_1, \ldots, B_n) \to 2^X(\beta_1, \ldots, \beta_n) \) such that \( \rho(h, id) \leq \varepsilon/3 \). And by \([8]\), \( 2^X(\beta_1, \ldots, \beta_n) = \text{inv lim } (2^{\Gamma_i}(\beta_1, \ldots, \beta_n), f_i) \), where \( \{\Gamma_i\} \) is a sequence of compact connected graphs in \( X \), with each \( \Gamma_i \) containing \( \beta_1 \cup \ldots \cup \beta_n \) in its vertex set, and each bonding map \( f_i : 2^{\Gamma_{i+1}}(\beta_1, \ldots, \beta_n) \to 2^{\Gamma_i}(\beta_1, \ldots, \beta_n) \) induced by a map \( \varphi_i : \Gamma_{i+1} \to C(\Gamma_i) \) such that \( \varphi_i(b) = \{b\} \) for each \( b \in \beta_1 \cup \ldots \cup \beta_n \). Thus for some \( i \) the projection map \( p_i : 2^X(\beta_1, \ldots, \beta_n) \to 2^{\Gamma_i}(\beta_1, \ldots, \beta_n) \) satisfies \( \rho(p_i, id) < \varepsilon/3 \).

Let \( \mathcal{U} \) be a finite cover of \( \Gamma_i \) by connected open subsets of \( X \) with diameters less than \( \varepsilon/3 \). There exists a subdivision \( Sd\Gamma_i \) of \( \Gamma_i \) such that each simplex of \( Sd\Gamma_i \) is contained in a member of \( \mathcal{U} \). To each vertex \( v \) of \( Sd\Gamma_i \) we assign a point \( \kappa(v) \in \bigcap \{U \in \mathcal{U} : v \in U\} \setminus A \), with \( \kappa(b) = b \) if \( b \in \beta_1 \cup \ldots \cup \beta_n \). Then \( \kappa \) may be extended to a map \( \kappa : Sd\Gamma_i \to X \setminus A \) such that, for each simplex \( \sigma \) of \( Sd\Gamma_i \), \( \kappa(\sigma) \subset U \setminus A \) for some \( U \in \mathcal{U} \) with \( \sigma \subset U \) (we use the fact that each \( U \setminus A \) is connected, locally connected, and locally compact, therefore path-connected). Thus \( d(\kappa, id) < \varepsilon/3 \), and the induced map \( k : 2^{\Gamma_i}(\beta_1, \ldots, \beta_n) \to 2^{X \setminus A}(\beta_1, \ldots, \beta_n) \) satisfies \( \rho(k, id) < \varepsilon/3 \). The composition \( kp_i h : 2^X(B_1, \ldots, B_n) \to 2^{X \setminus A}(\beta_1, \ldots, \beta_n) \subset 2^X(B_1, \ldots, B_n) 2^X(A, B_1, \ldots, B_n) \) satisfies \( \rho(kp_i h, id) < \varepsilon \), and \( 2^X(A, B_1, \ldots, B_n) \) is a \( Z \)-set in \( 2^X(B_1, \ldots, B_n) \).

Conversely, suppose the \( Z \)-set condition is satisfied. Then each \( B_i \setminus A \) must be dense in \( B_i \) otherwise \( 2^X(A, B_1, \ldots, B_n) \) has a nonempty interior in \( 2^X(B_1, \ldots, B_n) \). For each \( i \), choose \( b_i \in B_i \setminus A \). Given a neighborhood \( U \) of a point \( y \in A \), let \( V \) be a connected open neighborhood of \( y \) such that \( V \subset U \setminus \{b_1, \ldots, b_n\} \). We show that \( V \setminus A \) is connected, thus \( A \) is locally non-separating. Suppose \( V \setminus A = V_0 \cup V_1 \) is a separation. There exists a continuum \( M \) in \( V \) such that \( M \cap V_0 \neq \emptyset \neq M \cap V_1 \). Let \( \mathcal{F} = \{F \in 2^X(M) : F \setminus M = \{b_1, \ldots, b_n\}\} \). Then \( \mathcal{F} \) is homeomorphic to the connected hyperspace \( 2^M \), and \( \mathcal{F} \subset 2^X(B_1, \ldots, B_n) \). For each \( \varepsilon > 0 \) there exists a map \( \eta : \mathcal{F} \to 2^X \setminus 2^X(A) \) with \( \rho(\eta, id) < \varepsilon \). If \( \varepsilon \) is sufficiently small, there exist elements \( F_0, F_1 \in \mathcal{F} \) such that \( \eta(F_0) \cap V_0 \neq \emptyset \) and \( \eta(F_1) \cap V_0 = \emptyset \), and \( \eta(F) \cap bd V = \emptyset \) for every \( F \in \mathcal{F} \). Then \( \eta(\mathcal{F}) = \{\eta(F) : \eta(F) \cap V_0 \neq \emptyset\} \cup \{\eta(F) : \eta(F) \cap V_0 = \emptyset\} \) is a separation of the connected space \( \eta(\mathcal{F}) \), impossible.

**Proposition 2.2:** Let \( A, B_1, \ldots, B_n \) be closed subsets of a Peano continuum \( X \). Then \( C(X; A, B_1, \ldots, B_n) \) is a \( Z \)-set in \( C(X; B_1, \ldots, B_n) \) if and only if \( A \) is locally non-separating in \( X \) and \( B_i \setminus A \) is dense in \( B_i \) for each \( i \).

**Proof:** The argument for obtaining the \( Z \)-set property is the exact
parallel of the corresponding argument in the proof of Proposition 2.1. For the converse, suppose \( C(X; A, B_1, \ldots, B_n) \) is a \( Z \)-set in \( C(X; B_1, \ldots, B_n) \). Actually, we only use the fact that \( C(X; A, B_1, \ldots, B_n) \) has empty interior in \( C(X; B_1, \ldots, B_n) \). It is immediate that each \( B_i \setminus A \) must be dense in \( B_i \), and \( X \setminus A \) must be connected. Thus there exists a connected open set \( G \) in \( X \setminus A \) such that \( G \cap B_i \neq \emptyset \) for each \( i \) and \( G \cap A = \emptyset \). Given a neighborhood \( U \) of a point \( y \in A \), let \( V \) be a connected open neighborhood of \( y \) such that \( \overline{V} \subset U \setminus \overline{G} \), and choose \( \epsilon > 0 \) such that \( N_{\epsilon}(V) \subset U \). Let \( \mathcal{W} = \{ V \setminus A, G, W_1, W_2, \ldots \} \) be an open cover of \( X \setminus A \) such that each \( W_i \) is connected and has diameter less than \( \epsilon \). By connectedness of \( X \setminus A \), we obtain a chain in \( \mathcal{W} \) between \( V \setminus A \) and \( G \), which in turn leads to connected open sets \( H \) and \( W \) in \( X \setminus A \) such that \( H \supset G, H \cap V = \emptyset, H \cap W \neq \emptyset \neq W \cap V \), and \( \text{diam } W < \epsilon \). Then \( V \cup W \subset U \) is a connected open neighborhood of \( y \), and we claim that \( (V \cup W) \setminus A \) is connected. If there exists a separation \( (V \cup W) \setminus A = V_0 \cup V_1 \), with the connected set \( W \) contained in \( V_1 \), then \( (V \cup W \cup H) \setminus A = V_0 \cup (V_1 \cup H) \) is also a separation. However, there exists a continuum \( K \) in the connected open set \( V \cup W \cup H \) which meets each \( B_i \), and also meets the open sets \( V_0 \) and \( V_1 \). Then \( K \) is in the interior of \( C(X; A, B_1, \ldots, B_n) \) in \( C(X; B_1, \ldots, B_n) \), impossible.

**Definition:** A metric \( d \) for a space \( X \) has Property S if, for each \( \epsilon > 0 \), there exists a finite connected cover of \( X \) with mesh less than \( \epsilon \).

If \( X \) admits a metric with Property S, then \( X \) is locally connected. Without added conditions, the converse is not true (see Lemma 3.2 and Example 4.3).

**Definition:** A metric \( d \) for a connected space \( X \) is strongly connected if, for each \( x, y \in X \), \( d(x, y) = \inf \{ \text{diam } M : M \text{ is a connected subset containing } x \text{ and } y \} \).

A convex metric on a Peano continuum is an example of a strongly connected metric. If \( X \) admits a strongly connected metric, then \( X \) is locally connected. Conversely, the proof of the following lemma shows that every connected, locally connected metric space admits a strongly connected metric.

**Lemma 2.3:** Let \( X \) be a connected metric space which admits a metric with Property S. Then \( X \) admits a strongly connected metric with Property S.
PROOF: Let $d$ be a metric with Property $S$. Define a topologically equivalent metric $d^*$ for $X$ by $d^*(x, y) = \inf\{\text{diam } M : M \text{ is a connected subset of } X \text{ containing } x \text{ and } y\}$. It is easily verified that $d^*$ is a metric function. Since $d^*(x, y) \geq d(x, y)$, every open set with respect to $d$ is open with respect to $d^*$. The converse is easily established, using the local connectedness of $X$. And since the diameters of connected subsets are the same with respect to $d$ and $d^*$, $d^*$ is strongly connected and has Property $S$.

PROPOSITION 2.4: A connected metric space $X$ has a Peano compactification $\hat{X}$ with a locally non-separating remainder $\hat{X} \setminus X$ if and only if $X$ admits a metric with Property $S$.

PROOF: Suppose $X$ admits a metric $d$ with Property $S$. We may assume by Lemma 2.3 that $d$ is also strongly connected. Then the completion $(\hat{X}, \hat{d})$ of $(X, d)$ is the desired Peano compactification. That $(\hat{X}, \hat{d})$ is connected and has Property $S$ follows from the same properties for $(X, d)$. And since a complete, totally bounded metric space is compact, $(\hat{X}, \hat{d})$ is a Peano compactification of $(X, d)$.

Given a nonempty connected open subset $U$ of $\hat{X}$, we show that the nonempty set $U \cap X$ is connected, thereby verifying that $\hat{X} \setminus X$ is locally non-separating in $\hat{X}$. Suppose $U \cap X = H \cup K$ is a separation. Since $U$ is open in $\hat{X}$, $U \cap X$ is dense in $U$, and $U \subset \overline{H} \cup \overline{K}$ (the closures are taken in $\hat{X}$). We must have $\overline{H} \cap \overline{K} \cap U \neq \emptyset$, otherwise $U = (\overline{H} \cap U) \cup (\overline{K} \cap U)$ is a separation. Let $p \in \overline{H} \cap \overline{K} \cap U$. Choose $\delta > 0$ such that the $3\delta$-neighborhood of $p$ lies in $U$, and choose points $h$ and $k$ of $H$ and $K$, respectively, lying in the $\delta$-neighborhood of $p$. Then $d(h, k) < 2\delta$, and since $d$ is strongly connected there exists a connected subset $M$ of $X$ containing $h$ and $k$, with $\text{diam } M < 2\delta$. Then $M$ lies in the $3\delta$-neighborhood of $p$, therefore in $U$. Thus $M \subset U \cap X$ is a connected set meeting both $H$ and $K$, and $H \cup K$ cannot be a separation of $U \cap X$.

Conversely, suppose $X$ has a Peano compactification $\hat{X}$ such that $\hat{X} \setminus X$ is locally non-separating. Take any admissible metric $\hat{d}$ on $\hat{X}$, and let $d$ be its restriction to $X$. For every connected open cover $\{U_i\}$ of $\hat{X}$, $\{U_i \cap X\}$ is a connected cover of $X$. Since $(\hat{X}, \hat{d})$ has finite connected open covers with arbitrarily small mesh, so does $(X, d)$, and $d$ has Property $S$. 
3. Hyperspaces which are homeomorphic to $Q\setminus \text{point}$

**Lemma 3.1:** Let $X$ be a connected, locally connected metric space, with compact subsets $A$ and $B$ such that $A \subset \text{int } B$. Then only finitely many components of the complement $X \setminus A$ meet $X \setminus B$.

**Proof:** Each component $U$ of $X \setminus A$ must have a limit point in $A$, otherwise $U$ is both open and closed in $X$. Thus if $U \setminus B \neq \emptyset$, we must have $U \cap bdB = \emptyset$. Suppose there exists an infinite sequence $\{U_i\}$ of distinct components of $X \setminus A$, each extending beyond $B$. Choose $y_i \in U_i \cap bdB$ for each $i$. By compactness of $bdB$, we may assume that $y_i \to y \in bdB$. Since $y$ has a connected neighborhood in $X \setminus A$, the component of $X \setminus A$ containing $y$ meets $U_i$ for almost all $i$, contradicting our supposition that the $U_i$ are distinct components.

**Lemma 3.2:** Every connected, locally connected, locally compact metric space admits a metric with Property $S$.

**Proof:** Let $\tilde{X} = X \cup \infty$ be the one-point compactification of such a space $X$. Then $\tilde{X}$ is metrizable, since $X$ is separable metric. We claim that for any admissible metric $d$ on $\tilde{X}$, the restriction of $d$ to $X$ has Property $S$ (and therefore $\tilde{X}$ is a Peano continuum). Given $\epsilon > 0$, choose a compact subset $A \subset X$ such that the complement $X \setminus A$ lies in the $\epsilon$-neighborhood of $\infty$, and let $B \subset X$ be a compact neighborhood of $A$. Then by Lemma 3.1, only finitely many components of $X \setminus A$ extend beyond $B$. Thus a finite connected cover of $B$ with mesh less than $\epsilon$, together with the finite collection of components of $X \setminus A$ extending beyond $B$, provides a finite connected cover of $X$ with mesh less than $\epsilon$.

**Theorem 3.3:** $2^X \approx Q\setminus \text{point}$ if and only if $X$ is a connected, locally connected, locally compact, noncompact metric space. Similarly, $C(X) \approx Q\setminus \text{point}$ if and only if $X$ satisfies the above conditions and contains no free arcs.

**Proof:** Suppose $X$ satisfies the stated conditions. By Lemma 3.2, $X$ admits a metric with Property $S$, and by Proposition 2.4, $X$ has a Peano compactification $\tilde{X}$ with locally non-separating remainder. Since $X$ is locally compact it must be open in its compactification $\tilde{X}$, and the remainder $\tilde{X} \setminus X$ is closed. By Proposition 2.1, the intersection hyperspace $2^{\tilde{X}}(\tilde{X} \setminus X)$ is a $Z$-set in $2^{\tilde{X}}$. Thus $(2^{\tilde{X}}, 2^{\tilde{X}}(\tilde{X} \setminus X))$ and $(Q \times [0, 1], Q \times \{0\})$ are homeomorphic as pairs, and $2^X = 2^{\tilde{X}} \setminus 2^{\tilde{X}}(\tilde{X} \setminus X)$ is
homeomorphic to $Q \times (0, 1]$, which is homeomorphic to $Q \setminus \text{point}$ (since Cone $Q = Q$).

If in addition $X$ contains no free arcs, then neither does $\tilde{X}$, and the hyperspaces $C(\tilde{X})$ and $C(\tilde{X}; \tilde{X}\setminus X)$ are copies of $Q$. By Proposition 2.2, $C(\tilde{X}; \tilde{X}\setminus X)$ is a Z-set in $C(\tilde{X})$, and it follows as above that $C(X) \approx Q \setminus \text{point}$.

Conversely, if either $2^X$ or $C(X)$ is homeomorphic to $Q \setminus \text{point}$, $X$ must be a connected, locally connected metric space by Theorem 1.6. Since $X$ has a closed imbedding into both $2^X$ and $C(X)$, $X$ must be locally compact. Obviously, $X$ is noncompact, and if $C(X) \approx Q \setminus \text{point}$, $X$ contains no free arcs (otherwise $C(X)$ contains an open 2-cell).

4. Hyperspaces which are homeomorphic to $1^2$

With the Hilbert cube $Q$ coordinatized as $\Pi^\infty [0, 1]$, let $s = \Pi^\infty (0, 1) \subset Q$. Anderson [1] showed that $s$ is homeomorphic to the Hilbert space $1^2 = \{(x_i) \in R^\infty : \sum x_i < \infty \}$. Any subspace $P$ of $Q$ such that $(Q, P) \approx (Q, s)$ is called a pseudo-interior for $Q$, and its complement $Q \setminus P$ is a pseudo-boundary. A non-trivial example of a pseudo-boundary is the subset $\Sigma = \{(x_i) \in Q : 0 < \inf x_i$ and $\sup x_i < 1\}$. Kroonenberg [12] has given the following characterization for pseudo-boundaries, based on the original characterization by Anderson [2].

**Lemma 4.1:** Let $\{K_i\}$ be an increasing sequence of subsets of $Q$ such that:

i) each $K_i \approx Q$,

ii) each $K_i$ is a Z-set in $Q$,

iii) each $K_i$ is a Z-set in $K_{i+1}$,

iv) for each $\epsilon > 0$, there exists a map $f : Q \to K_i$ for some $i$ such that $d(f, \text{id}) < \epsilon$.

Then $\bigcup K_i$ is a pseudo-boundary for $Q$.

**Theorem 4.2:** The following conditions are equivalent:

1) $X$ has a Peano compactification $\tilde{X}$ such that $(2^X, 2^X) \approx (Q, s)$,

2) $X$ has a Peano compactification $\tilde{X}$ such that $(C(\tilde{X}), C(X)) \approx (Q, s)$,

3) $X$ is a topologically complete, separable, connected, locally connected, nowhere locally compact metric space which admits a metric with Property $S$. 
PROOF: Suppose $X$ satisfies condition 3). Then by Proposition 2.4, $X$ has a Peano compactification $\tilde{X}$ with a locally non-separating remainder. Let $\tilde{d}$ be a convex metric for $\tilde{X}$. Since $X$ is topologically complete and nowhere locally compact, the remainder $\tilde{X}\setminus X$ must be a dense countable union $\bigcup_i F_i$ of closed, locally non-separating sets in $\tilde{X}$. We may assume that $F_i \subset F_{i+1}$ and $F_i$ has empty interior in $F_{i+1}$, for each $i$. This can be arranged inductively as follows. Select a dense sequence $\{x_n\}$ in $F_i$, a sequence $\{y_n\}$ in $\tilde{X}\setminus F_i$ such that $\tilde{d}(x_n, y_n) < 1/n$ for each $n$, and a sequence $\{z_n\}$ in $(\tilde{X}\setminus X)\setminus F_i$ such that $\tilde{d}(y_n, z_n) < 1/n$ for each $n$. Then replace $F_{i+1}$ by the compact set $F_i \cup F_{i+1} \cup \{z_n : n \geq 1\}$.

By Proposition 2.1, each intersection hyperspace $2^X(F_i)$ is a Z-set copy of $Q$ in $2^X$, and each $2^X(F_i) = 2^X(F_i, F_{i+1})$ is a Z-set in $2^X(F_{i+1})$. Given $\epsilon > 0$, we claim there exists a map $f : 2^X \to 2^X(F_i)$ for some $i$, such that $\rho(f, id) \leq \epsilon$. For $D \in 2^X$, define $f(D)$ to be the closed $\epsilon$-neighborhood of $D$ in $X$ (with respect to the convex metric $\tilde{d}$). Suppose $f(2^X) \cap 2^X(F_i) \neq \emptyset$ for each $i$. Then there exists a convergent sequence $y_i \to y$ in $\tilde{X}$ such that the $\epsilon$-neighborhood of $y_i$ is disjoint from $F_i$, for each $i$. It follows that the $\epsilon$-neighborhood of $y$ is disjoint from $\bigcup_i F_i = \tilde{X}\setminus X$, contrary to the fact that $\tilde{X}\setminus X$ is dense in $\tilde{X}$. Thus by Lemma 4.1, $\bigcup_i 2^X(F_i) = 2^X \setminus 2^X$ is a pseudo-boundary for $2^X$, and $(2^X, 2^X) \approx (Q, s)$.

The proof that $(C(\tilde{X}), C(X)) \approx (Q, s)$ is virtually the same as above, using Proposition 2.2.

Conversely, suppose either condition 1) or 2) is satisfied. Since $s$ is a topologically complete, separable, nowhere locally compact metric AR, $X$ must be a topologically complete, separable, connected, locally connected, nowhere locally compact metric space. We show that the remainder $\tilde{X}\setminus X$ is locally non-separating in $\tilde{X}$. For every connected open subset $U$ of $\tilde{X}$, the hyperspace $2^U$ is a connected open subset of $2^X$. Since the pseudo-boundary $Q\setminus s$ is locally non-separating in $Q$, $2^X \setminus 2^X$ is locally non-separating in $2^X$. Thus $2^U \cap 2^X = 2^U \cap X$ is connected, and $U \cap X$ is connected. It follows from Proposition 2.4 that $X$ admits a metric with Property $S$.

The first result of this type, $(2^Q, 2^s) \approx (C(Q), C(s)) \approx (Q, s)$, was obtained by Kroonenberg [12].

Using the very powerful Hilbert space characterization theorem of Torunczyk [14], the author has recently shown that $2^X \approx C(X) \approx 1^2$ for every topologically complete, separable, connected, locally connected, nowhere locally compact metric space $X$ [5]. The following example illustrates the difference between this result and Theorem 4.2.
EXAMPLE 4.3: There exists a space $X$ such that $2^X = C(X) \approx 1^2$, but $X$ does not admit a metric with Property $S$.

PROOF: The space $X$ is a countable union of copies of $1^2$ meeting at a single point $\theta$, and given the uniform topology at $\theta$. $X$ may be realized in $1^2$ as follows. Let $N = \bigcup_{i=1}^{\infty} \alpha_i$ be a partition of the positive integers, with each $\alpha_i$ infinite, and for each $i$ set $1^2_i = \{(x_n) \in 1^2 : x_n = 0$ if $n \in \alpha_i\}$. Then $X = \bigcup_{i=1}^{\infty} 1^2_i \subset 1^2$. Clearly, $X$ is a closed, connected, locally connected, nowhere locally compact subset of $1^2$, thus $2^X = C(X) \approx 1^2$.

The argument that the space $X$ does not admit a metric with Property $S$ is easy. Consider any admissible metric $d$ for $X$. For some $\delta > 0$, the $\delta$-neighborhood (with respect to $d$) of $\theta$ in $X$ must be contained in the neighborhood $\{x \in X : \|x\| < 1\}$ of $\theta$. Now consider any connected cover of $X$ with mesh less than $\delta$. For each $i$, any element of the cover intersecting $\{x \in 1^2_i : \|x\| \geq 1\}$ cannot contain $\theta$, and must therefore lie in $1^2_i \setminus \theta$. Hence the cover is infinite, and $d$ does not have Property $S$.

REFERENCES

The condition iv) of the pseudo-boundary characterization Lemma 4.1 is insufficient, and should be replaced by the following condition iv)*: there exists a deformation \( h: Q \times [0, 1] \rightarrow Q \), with \( h(q, 0) = q \) for each \( q \in Q \), such that for each \( \varepsilon > 0 \), \( h(Q \times [\varepsilon, 1]) \subseteq K_i \) for some \( i \). In the application of Lemma 4.1 contained in the proof of Theorem 4.2, this stronger condition is easily verified (the map \( f \) of \( \tilde{X} \) is replaced by the deformation \( h: \tilde{X} \times [0, 1] \rightarrow \tilde{X} \), where \( h(D, t) \) is the closed \( t \)-neighborhood of \( D \) in \( \tilde{X} \)).