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THETA CONSTANTS OF GENUS THREE

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Introduction

In this paper we will examine theta constants of genus three and the polynomial equations that they satisfy. The totality of these (homogeneous) relations defines a projective variety and we will show that this variety can be defined by a certain collection of quartic polynomials. In the process we will gain a better understanding of this variety and, in particular, relate points on the variety to curves of genus three and limits of these. The chief tool used will be the symplectic group Sp(3, Z) and its action on theta constants. We will also make use of Riemann's calculations on the moduli of non-hyperelliptic curves of genus three. In the main, the results depend on long and complicated calculations, most of which have been omitted. The reader is referred to [7] for details. Finally, this work would not have been possible without the help of P. Swinnerton-Dyer (my research supervisor), J.H. Conway and S. Norton.[1]

1. Preliminaries

We shall only be considering theta functions of genus three with half-integer characteristics. That is, infinite series of the form:

$$(1.1)\quad \theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z,\tau)=\sum_{N\in Z^3}\exp \pi\, \mathrm{i}[{}^t(N+\tfrac{1}{2}\epsilon)\tau(N+\tfrac{1}{2}\epsilon)+2{}^t(N+\tfrac{1}{2}\epsilon)(z+\tfrac{1}{2}\epsilon')]$$

where $z\in \mathbf{C}^3$; $\tau\in \mathcal{H}_3$ the Siegel upper half plane of dimension three; ϵ, $\epsilon'\in Z^3$; exp is the exponential function; and ^{t} denotes transpose. This function is analytic in both the z and the τ variables. In

[1] This paper is essentially a summary of the author's Ph.D. thesis [7].

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particular, we shall be interested in theta constants, which are theta functions evaluated at $z = 0$. The symbol $[{\epsilon \atop \epsilon'}]$ is called the characteristic of the function and is designated odd or even according to the parity of $\epsilon \cdot \epsilon'$. The following reduction formula is valid:

$$\theta\begin{bmatrix} \epsilon + 2\delta \\ \epsilon' + 2\delta' \end{bmatrix}(z, \tau) = (-1)^{{}^t\epsilon \cdot \delta'}\theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(z, \tau) \tag{1.2}$$

As a result we can restrict our attention to characteristics which consist of only zeroes and ones. Then there will be 36 even theta functions and 28 odd ones. These functions are even or odd according to the parity of their characteristic. Hence the 28 odd theta constants all vanish identically in τ and we are left with the 36 even ones, none of which vanishes identically. It is this set of 36 functions which will be at the centre of our investigations.

There is an important group called $\mathrm{Sp}(3, Z)$ and defined to be be:

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}: a^tb = b^ta, c^td = d^tc, a^td - b^tc = I\right\}$$

where a, b, c, d are 3×3 integral matrices. This group has its principal congruence subgroups; for n a positive integer:

$$\Gamma(n) = \{M \in \mathrm{Sp}(3, Z): M \equiv I (\mathrm{mod}\, n)\}$$

as well as Igusa's group:

$$\Gamma(n, 2n) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(n): (a^tb)_0 \equiv (c^td)_0 \equiv 0 (\mathrm{mod}\, 2n)\right\}$$

where for a matrix A, A_0 denotes its principal diagonal. It is easy to see that when n is even these two conditions are equivalent to:

$$(b)_0 \equiv (c)_0 \equiv 0 (\mathrm{mod}\, 2n).$$

These definitions lead to the transformation formula for theta constants ([1] p. 227). In fact in its most general form it applies to theta functions and involves a transformation of the z variable, but we will not require this. The formula is:

$$\theta[M \circ \epsilon](M \cdot \tau) = K(M) \exp(\phi_\epsilon(M)) \det(c\tau + d)^{1/2}\theta[\epsilon](\tau) \tag{1.3}$$

where $M = ({a \atop c}{b \atop d}) \in \mathrm{Sp}(3, Z)$, ϵ, $\epsilon' \in Z^3$, $\tau \in \mathscr{H}_3$, and in the obvious context, ϵ denotes the characteristic $[^{\epsilon}_{\epsilon'}]$; the rest of the notation of (1.3) being explained below:

(a) $\mathrm{Sp}(3, \mathbb{Z})$ acts as a group of biholomorphic transformations on $\mathcal{H}_3$ by: $M \cdot \tau = (a\tau + b)(c\tau + d)^{-1}$

(b)

$$M \circ \epsilon = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}\begin{pmatrix} \epsilon \\ \epsilon' \end{pmatrix} + \begin{pmatrix} (c^t d)_0 \\ (a^t b)_0 \end{pmatrix} \tag{1.4}$$

giving an action of the group on the characteristics.

(c) $K(M)$ is an eighth root of unity, depending on M.

(d) $\phi_\epsilon(M) = -\frac{1}{4}\pi\, \mathrm{i}(\epsilon^t bd\epsilon + \epsilon''ac\epsilon' - 2\epsilon^t bc\epsilon' - 2(a^t b)_0(d\epsilon - c\epsilon'))$ where the ϵ and ϵ' on the left of the terms are transposed – we will be consistently negligent about this, because it is always unambiguous.

(e) $\theta[\epsilon](\tau)$ is the theta constant $\theta[^{\epsilon}_{\epsilon'}](0, \tau)$.

The operation defined by (1.4) has the property that M preserves evenness and oddness of the characteristic ϵ. We shall often write theta characteristics as ϵ instead of the official notation $[^{\epsilon}_{\epsilon'}]$.

The 36 even theta constants (with reduced characteristic) define a projective embedding of the quotient manifold $\mathcal{H}_3/\Gamma(4, 8)$ (see [2] p. (189). The group $\Gamma(1)$ acts on the set of reduced characteristics by formula (1.4) because of ([3] p. 89):

$$M \circ (M_1 \circ \epsilon) \equiv (MM_1) \circ \epsilon (\mathrm{mod}\, 2) \quad \text{for } M, M_1 \in \Gamma(1). \tag{1.5}$$

We would like to be able to define an action of $\Gamma(1)$ on the theta constants as a result of the transformation law (1.3). This is not possible in any obvious sense because of the sign ambiguity inherent in the square root of $\det(c\tau + d)$. However we can find an action on the $\mathbb{P}^{35}$ having the theta constants as coordinates; by putting the coordinates of $\mathbb{P}^{35}$ and the 36 even theta constants in a fixed correspondence and writing:

$$\begin{aligned} g^{-1} : (s_1, \ldots, s_{36}) \to (K(g)^{-1} \exp(-\phi_1(g)) s_{g \cdot 1}, \ldots, K(g)^{-1} \\ \exp(-\phi_{36}(g)) s_{g \cdot 36}) \end{aligned} \tag{1.6}$$

where, of course, if the reduction of a $g \cdot i \bmod 2$ changes the sign of the appropriate theta, then we introduce that change of sign. This action corresponds to that of $\Gamma(1)$ on $\mathcal{H}_3$. The proof that this action is genuine is not difficult (see [7] Chap. 2). The following theorem picks out two important subgroups.

THEOREM 1.1:

(a) *If* $M = \binom{a\ b}{c\ d} \in \Gamma(2)$ *then* $\exp(\phi_\epsilon(M)) = \pm 1$ *for all reduced even characteristics* ϵ *if and only if* $M \in \Gamma(2,4)$.

(b) *Let* $M \in \Gamma(2)$, *then we have the identity*

$$\theta[M \circ \epsilon](M \cdot \tau) = CK(M) \exp(\phi_\epsilon(M)) \theta[\epsilon](\tau)$$

for all reduced even characteristics ϵ *and for some non-zero constant* C *if and only if* $M \in \pm\Gamma(4,8)$ (*i.e. either* M *or* $-M$ *is in* $\Gamma(4,8)$).

The proof is essentially contained in [1] part II, p. 232.

2. The permutation representation

It is a well-known fact that $\Gamma(1)/\Gamma(2)$ is isomorphic to the symplectic group $\mathrm{Sp}(3, \mathrm{F}_2)$, which we shall refer to as $S_6(2)$. By formulae (1.4) and (1.5), the group $\Gamma(1)/\Gamma(2)$ acts on the 28 odd, reduced characteristics and the 36 even, reduced characteristics. It can be shown that these two permutation representations are equivalent to representations of $S_6(2)$ to be described below. The proof is a consequence of the fact that $S_6(2)$ has unique subgroups of indices 28 and 36 up to conjugacy. (See [7] Chap. 2 for details).

The group $S_6(2)$ can be thought of in the following way: the symmetric group S_8 sits inside as a subgroup of index 36. A convenient notation for a set of coset representatives is obtained by taking the numbers $(1, 2, \ldots, 8)$ and picking out subsets of four with their complements, so that $\frac{1234}{5678}$ is the same as $\frac{5678}{1234}$ (and these are independent of order), thus giving $\frac{1}{2}\binom{8}{4} = 35$ objects. These objects are called bifid maps (this terminology seems to have originated with Cayley) in the group and form a set of coset representatives of S_8 in $S_6(2)$.

Once again consider the set $(1, 2, \ldots, 8)$ and also the objects ij where $i \neq j$ both belong to the set, of which there are 28. Then $S_6(2)$ acts on the 28 objects by the recipe:

(a) a permutation $\sigma \in S_8$ sends ij to $i^\sigma j^\sigma$

(b) a bifid map $\frac{abcd}{efgh}$ fixes ij if i and j lie on opposite sides of the bar in the bifid map, and otherwise complements ij in either the numerator or the denominator (whichever is appropriate).

Now construct 36 objects from $(1, 2, \ldots, 8)$ by taking the $\frac{1}{2}\binom{8}{4}$ objects $\frac{abcd}{efgh}$ together with the symbol ∞. There is a deliberate confusion here between the bifid maps and these objects because the action to be defined here is really $S_6(2)$ on its cosets modulo S_8. Define the action of $S_6(2)$ on the 36 objects as follows:

(a) ∞ is fixed by all of S_8, but the bifid map $\frac{abcd}{efgh}$ send ∞ to the object $\frac{abcd}{efgh}$.

(b) $\sigma \in S_8$ sends the object $\frac{abcd}{efgh}$ to $\frac{\sigma a\sigma b\sigma c\sigma d}{\sigma e\sigma f\sigma g\sigma h}$; whereas if $\frac{\alpha\beta\gamma\delta}{\alpha'\beta'\gamma'\delta'}$ is a bifid map and $\frac{abcd}{efgh}$ an object, in the case that $\{\alpha, \beta, \gamma, \delta\}$ splits as $3:1$ over the bar in $\frac{abcd}{efgh}$ the latter is fixed, but if it splits $2:2$ we have generically

$$\frac{\alpha\beta\gamma\delta}{\alpha'\beta'\gamma'\delta'}: \frac{\alpha\beta xy}{\gamma\delta zt} \to \frac{\alpha\beta zt}{\gamma\delta xy}$$

We now have permutation representations of $S_6(2)$ on both 28 and 36 letters in this highly convenient form. Corresponding to these are the actions of $\Gamma(1)/\Gamma(2)$ on the odd, reduced theta characteristics and even, reduced theta characteristics respectively. As mentioned before, these actions are isomorphic in their respective pairs.

3. Equations between theta constants

The book [3] (p. 49) gives a procedure for finding degree four relations between theta constants, and we shall follow this. There are three basic types of relations, involving linear combinations of the following objects, respectively (1) fourth powers of theta constants (2) products of squares of theta constants (3) tetrads of distinct theta constants. The coefficients are always ± 1. The permutation part of the action (1.6) now has explicit form when we identify the 36 even, reduced characteristics with the objects ∞, $\frac{abcd}{efgh}$. We will write down the relations as equations in these latter-mentioned objects.

$$\infty^4 \pm abcd^4 \pm abce^4 \pm abcf^4 \pm abcg^4 \pm abch^4 = 0 \tag{1}$$

$$\infty^2 \frac{abcd^2}{efgh} \pm \frac{abef^2}{cdgh} \frac{abgh^2}{cdef} \pm \frac{abgf^2}{cdeh} \frac{abeh^2}{cdgf} \pm \frac{abhf^2}{cdeg} \frac{abeg^2}{cdfh} = 0 \tag{2}$$

$$\begin{aligned} &\infty \frac{aebf}{cgdh} \frac{aecg}{bfdh} \frac{aedh}{bfcg} \pm \frac{ebcd}{afgh} \frac{afcd}{ebgh} \frac{abgd}{efch} \frac{abch}{efgd} \\ &\pm \frac{abcd}{efgh} \frac{abgh}{cdef} \frac{afgd}{bceh} \frac{afch}{ebgd} = 0 \end{aligned} \tag{3}$$

Now recall that the theta constants embed $\mathcal{H}_3/\Gamma(4,8)$ in $\mathbb{P}^{35}$. Hence the equations above, taken together, define a variety which we shall call V_3. Also, the totality of homogeneous relations in the theta constants defines a smaller variety, which we shall call S_3. It is clear that the group action (1.6) preserves relations between the theta constants. Moreover, it is obvious from the original definition of the quartics listed above that they also are preserved by $\Gamma(1)$. (See [3] p. 49 for the original form of these quartics). This implies that $\Gamma(1)$ preserves both S_3 and V_3. We wish to investigate the possible configurations of vanishing coordinates on V_3 and for this purpose we only need the permutation part of the $\Gamma(1)$ action.

DEFINITION:

(a) A triple of distinct even characteristics is called even (resp. odd) if the vector sum of the three is even (resp. odd) in the usual dot-product sense. This is equivalent to the existence of an element of $S_6(2)$ sending the triple to (∞, X, Y) where the numerators of X and Y intersect evenly (resp. oddly).

(b) A set of coordinates in $\mathbb{P}^{35}$ is called a vanishing set if there exists a point on V_3 having precisely those coordinates equal to zero.

It is clear that $S_6(2)$ is doubly-transitive on the 36-set (S_8 is the stabiliser of ∞) hence the first two vanishing coordinates may be chosen in a single way.

LEMMA 3.1:

(a) *The vanishing of* ∞ *and* $\frac{abcd}{efgh}$ *together implies the vanishing of a coordinate oddly-related to these two in the sense of the definition.*

(b) *The vanishing of* ∞ *and* $\frac{abcd}{efgh}$ *and* $\frac{abce}{dfgh}$ *implies the vanishing of either* $\{abc\}$ *or* $\{fgh\}$; *where* $\{abc\}$ *stands for the sextet of coordinates*: $\{\infty, abcd, abce, abcf, abcg, abch\}$.

(c) *If a vanishing set contains more than six coordinates but doesn't contain a set (with nine elements) of the form* $\{abc\} \cup \{fgh\}$ *then it contains a complete set of the form* $\{ab\}$; *where* $\{ab\}$ *stands for the set* $(\infty, abxy)$ *of* 16 *coordinates for* a, b *fixed and* x, y *variable.*

(d) *If a vanishing set strictly contains a set of the form* $\{ab\}$ *then it contains at least* 18 *coordinates and, moreover, contains a standard* 9-*set* $\{abc\} \cup \{fgh\}$.

The proofs depend on choosing the correct equations to work with. For example, in (a) use equations of type (3) and for (b) use equations of type (2).

In order to find the structure of the larger vanishing sets, we have to recode the coordinates. This is because the number of different possibilities becomes far too large to cope with in the present notation. This "super-notation" and its uses were discovered by J.H. Conway.

Pick out the following nine coordinates and relabel them as follows:

$$\infty = [9];\quad \frac{1238}{4567} = [8];\quad \frac{1237}{4568} = [7];\quad \frac{1236}{4578} = [6];$$

$$\frac{1235}{4678} = [5];\quad \frac{1234}{5678} = [4];\quad \frac{1278}{3456} = [3];$$

$$\frac{1378}{2456} = [2];\quad \frac{1456}{2378} = [1].$$

Note that these nine coordinates actually form a 9-set; namely $\{123\} \cup \{456\}$. Now split the numbers $1, \ldots, 9$ into three blocks:

$$1, 2, 3/4, 5, 6/7, 8, 9.$$

The remaining 27 coordinates are coded by a choice of one number from each of the three blocks. If we choose 9 (from the third block) then we define:

$$[ij9] = \frac{ij78}{\text{complement}} \quad \mathrm{i} \in (1, 2, 3), \quad \mathrm{j} \in (4, 5, 6)$$

On the other hand, we define:

$$[ij8] = \frac{ij'j''8}{i'i''j7} \quad \text{and} \quad [ij7] = \frac{ij'j''7}{i'i''j8}$$

where i' and i'' form the complement of i in $(1, 2, 3)$ and j', j'' the complement of j in $(4, 5, 6)$.

We can now think of the residual 27 coordinates as points of a cube in 3-space, by letting $(1, 2, 3)$; $(4, 5, 6)$; $(7, 8, 9)$ be coordinates on the three coordinate axes. Before we can make general geometric statements about vanishing sets vis à vis the cube, we will need the following result.

PROPOSITION 3.1: *The full group of automorphisms of the cube can be realised by elements of $S_6(2)$ which fix the set* $([1], [2], \ldots, [9])$.

The proof of this proposition is straightforward.

We are now in a position to prove the following lemma.

LEMMA 3.2: *Suppose that, in the new notation, the coordinates* [1], . . ., [9] *all vanish. Then, on the cube, if any two collinear points vanish, so does the third point on that line. Moreover, if any point vanishes then two of the three lines passing through it also vanish.*

PROOF: Because of the proposition, we only have to prove these results once on the cube. For the collinearity result, take the line [148], [248], [348] which coordinates in the old language are $\frac{1568}{2347}$, $\frac{2568}{1347}$, $\frac{3568}{1247}$ respectively. Now, ex hypothesi, $4568 = 7568 = 0$ and $\infty = 0$. Then the linear quartic coded by {568} gives the result.

For the second part, suppose $1248 = 0 = [347]$. The three lines through it are: [347], [247], [147]; [347], [357], [367]; [347], [348], [349] which in old coordinates are respectively: 1248, 1348, 2348; 1248, 1258, 1268; 1248, 3568, 3478. Now consider the biquadratic equations (without their square powers):

$$\infty\, 1678 \pm 1248\ \ 1358 \pm 1258\ \ 1348 \pm 1458\ \ 1238 = 0$$

$$\infty\, 1236 \pm 1248\ \ 1257 \pm 1258\ \ 1247 \pm 1245\ \ 1278 = 0$$

$$\infty\, 1456 \pm 1437\ \ 1428 \pm 1427\ \ 1438 \pm 1487\ \ 1432 = 0$$

Using the known vanishings we get

$$[357] \text{ or } [247] = 0$$

$$\text{and } [357] \text{ or } [348] = 0$$

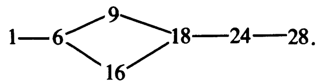
$$\text{and } [348] \text{ or } [247] = 0$$

Now combine this with the collinearity result and we have proved the original claim. Q.E.D.

We can now make simple geometric observations about the cube to determine all vanishing sets containing [1], . . ., [9] and hence all vanishing sets.

Suppose [1], . . ., [9] all vanish along with one point of the cube, which we may take to be a corner point. Then, repeatedly using lemma 3.2, we see that only three cases occur: either one whole face vanishes, or two adjacent faces vanish, or three faces which meet in a point vanish. Coupling this with lemma 3.1, we are able to enunciate the following.

THEOREM 3.1: *The only possible vanishing sets have the following sizes*:



To finish off this section we shall describe one technical result essential in the sequel. Not surprisingly, the equations defining V_3 give more information about the fourth powers of coordinates than the coordinates themselves. That is, if we denote by θ^4 the regular map from $\mathbf{P}^{35}$ to itself which raises each coordinate to its fourth power, then we will be able to work more easily with the variety $\theta^4(V_3)$. Fortunately, it turns out that the quotient group $\Gamma(2)/\pm\Gamma(4,8)$ is transitive on the fibres of θ^4 when restricted to V_3. Hence in order to compare V_3 and S_3 we may look at their images under θ^4. The proof of this transitivity depends upon splitting into two cases: multiplication of coordinates by ± 1 (governed by equations of type (3)) and multiplication by i. The equations of types 2 and 3 greatly restrict the possible configurations of multiplications of coordinates by fourth roots of unity. It remains to be shown that the possibilities are cut down to precisely those generated by the group $\Gamma(2)/\pm\Gamma(4,8)$. For example, $\Gamma(2,4)/\pm\Gamma(4,8)$ is the group which multiplies coordinates by ± 1, and has order 2^{20}. An element of this group acting on an equation of type three either fixes each term, or multiplies all three by -1. Hence the product of any two terms' sign changes must be identically 1. This can all be translated to the vector space $\mathbf{F}^{36}$ where the group generates a subspace and this subspace is cut out by hyperplanes of the form: a sum of 8 coordinates equals zero (corresponding to a pair of terms of an equation). See [7] Chapter 3 for details. Of course this proof only works for points on V_3 with no zero coordinates; for as soon as coordinates vanish, the groups involved get smaller; but then the equations change and, in fact, everything works out as expected.

4. Classification of the points of V_3

In this final section our aim is to prove that $V_3 = S_3$ and, in the process, we shall classify the points of S_3 vis à vis the points of $\mathcal{H}_3$.

The analysis proceeds as follows: first we look at points with no zero coordinates, then points with one zero coordinate and, finally, points with six or more zero coordinates.

Our first concern is with non-hyperelliptic curves of genus 3 and these are all non-singular plane quartics. The background to our work derives from the papers of Riemann [4, 5] and their sequel by Weber [6]. In these papers they describe the relationship between the 28 bitangents of a plane quartic and the 28 odd reduced characteristics. After some long calculations they arrive at a sextet of complex numbers which serve as "moduli" for plane quartics. In the sequel a certain fixed pairing of the coordinates x_i in $\mathbb{P}^{35}$ with the 36 even reduced characteristics is used (we shall not bother to write it down explicitly). Following Weber ([6] p. 108) we define the following.

Let $(x_1, \ldots, x_{36})$ be a point of V_3 with the property that none of the coordinates is zero. Now define nine complex numbers:

$$(4.1)\qquad \begin{aligned} \alpha_1 &= i\,\frac{x_{21}x_6}{x_{35}x_{11}}; & \alpha_1' &= i\,\frac{x_6x_{28}}{x_{19}x_{35}}; & \alpha_1'' &= -\frac{x_{28}x_{21}}{x_{11}x_{19}} \\ \alpha_2 &= i\,\frac{x_{16}x_{27}}{x_{17}x_{31}}; & \alpha_2' &= i\,\frac{x_{27}x_3}{x_{25}x_{17}}; & \alpha_2'' &= \frac{x_3x_{16}}{x_{31}x_{25}} \\ \alpha_3 &= i\,\frac{x_8x_{23}}{x_{12}x_{36}}; & \alpha_3' &= i\,\frac{x_{23}x_{13}}{x_{30}x_{12}}; & \alpha_3'' &= \frac{x_{13}x_8}{x_{36}x_{30}} \end{aligned}$$

We note that if $\tau \in \mathcal{H}_3$ represents a non-hyperelliptic curve then the corresponding point $(\theta[^{000}_{000}](\tau), \ldots)$ of V_3 has no vanishing coordinates (by a combination of the Riemann vanishing theorem and Riemann-Roch).

The proof of the following theorem involves writing down hordes of skilfully chosen rational functions of the α, making use of the equations defining V_3, and multiplying lots of expressions together.

THEOREM 4.1: *Given* $(x_1, \ldots, x_{36})$ *in* V_3 *with all* $x_i \neq 0$, *every ratio* x_j^4/x_k^4 *is a rational function of the nine constants* $\alpha_i, \alpha_i', \alpha_i''$ *defined by* (4.1).

In fact, there are formulae which express the α_i'' as functions of the α_i, α_i' (see [6] p. 108 et. seq.) Thus it is the α_i, α_i' which serve as "the moduli". These formulae only determine the α_i'' up to multiplication of all three of them by -1 but, because the rational functions in the above theorem are invariant under this, it does us no harm.

In Weber's development of the theory, a non-hyperelliptic curve is put into a normal form namely, the rationalisation of

$$(4.2)\qquad (t_1\xi_1)^{1/2} + (t_2\xi_2)^{1/2} + (t_3\xi_3)^{1/2} = 0$$

where ξ_1, ξ_2, ξ_3 are linear functions of the coordinates t_1, t_2, t_3 in $\mathbb{P}^2$. They are defined by the equations:

$$(4.3)\qquad \begin{aligned} &\xi_1+\xi_2+\xi_3+t_1+t_2+t_3=0\\ &\frac{1}{\alpha_1}\xi_1+\frac{1}{\alpha_2}\xi_2+\frac{1}{\alpha_3}\xi_3+\alpha_1t_1+\alpha_2t_2+\alpha_3t_3=0\\ &\frac{1}{\alpha_1'}\xi_1+\frac{1}{\alpha_2'}\xi_2+\frac{1}{\alpha_3'}\xi_3+\alpha_1't_1+\alpha_2't_2+\alpha_3't_3=0 \end{aligned}$$

The strategy is, given P on V_3 with no zero coordinates, to define the α by (4.1) and reverse Weber's steps to give a curve. This is almost possible, in the sense that all the necessary conditions are open so that at worst we come out with a limit of curves. Let us now give the details. Define a rational map

$$\psi:\mathbb{C}^6\to\mathbb{P}^{14}\times\mathbb{C}^9 \text{ by}$$

$$(4.4)\qquad \begin{aligned} &\psi(\alpha_1,\alpha_2,\alpha_3,\alpha_1',\alpha_2',\alpha_3')=\\ &\qquad\left(\sum_{i=1}^{3}(t_i\xi_i)^{1/2};\alpha_1,\alpha_2,\alpha_3,\alpha_1',\alpha_2',\alpha_3',\alpha_1''^2,\alpha_2''^2,\alpha_3''^2\right) \end{aligned}$$

where $\Sigma_{i=1}^3(t_i\xi_i)^{1/2}$ stands for the rationalised version of that curve (a quartic); the ξ_i are defined by (4.3); and, as mentioned previously, the α_i, α_i' determine the $\alpha_i''^2$.

Set $X=\psi(\mathbb{C}^6)$. It is clear that the Zariski closure $\bar{X}$ of X is an irreducible variety. Now define a Zariski open subset U of $\bar{X}$ by the following open conditions:

(1) the curve $\Sigma_{i=1}^3(t_i\xi_i)^{1/2}$ is non-singular. Call this quartic C.

(2) the seven lines $t_1=0$, $t_2=0$, $t_3=0$, $t_1+t_2+t_3=0$, $\alpha_1t_1+\alpha_2t_2+\alpha_3t_3=0$, $\alpha_1't_1+\alpha_2't_2+\alpha_3't_3=0$, $\alpha_1''t_1+\alpha_2''t_2+\alpha_3''t_3=0$ and all bitangents to C.

(3) each subset of three lines from the seven in (2) has the property: the points of intersection with C do not lie on a single conic.

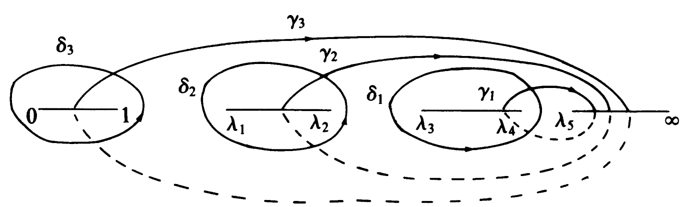
The last condition is the only one that isn't obviously open. This condition is important because geometrically it means that the other 21 bitangents can be constructed out of these seven. To verify the openness, reduce first to the case where all triples of lines are linearly independent-being an open condition this is acceptable. Thus it remains to prove that: if a quartic in $\mathbb{P}^2$ has $t_1=0$, $t_2=0$, $t_3=0$ as bitangents then the condition that their points of intersection lie on a conic is a closed one. The proof of this is straightforward. (See [7] Chap. 4 for details).

Finally, take P on V_3 with no zero coordinates, form the α_i, α_i', α_i'' by (4.1) and thus a point of $\bar{X}$ by (4.4). Now U is strongly dense in $\bar{X}$

hence we obtain a sequence of non-hyperelliptic curves such that their α_i, α'_i, α''_i converge strongly to those for P (the α''_i needing a scintilla of analysis). Invoking theorem 4.1, and the transitivity described at the end of section 3, we have proved

THEOREM 4.2: *Any point P of V_3 with no zero coordinates is the strong limit in V_3 of a sequence of points corresponding to non-hyperelliptic curves. In particular, P is in S_3.*

Hyperelliptic curves of genus three always have precisely one vanishing theta constant and, as we shall see now, the reverse is also true. In the hyperelliptic case there is a very obvious choice for the moduli namely, the five un-normalised branch-points. By choosing a special canonical homology basis, these five branch points can be expressed as rational functions of even theta constants in that homology basis. More precisely, given five complex numbers $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ distinct from 0, 1 and each other, define a hyperelliptic Riemann surface of genus 3 as a two-sheeted branched cover of $\mathbf{P}^1$ in the standard way. On it there is the following canonical homology basis:



the dashes indicating that the path is in the lower sheet. It is then a simple matter to compute the values (being half-periods) of the Abel-Jacobi map ϕ at the branch points. All theta functions considered in this paragraph are evaluated at the $\tau \in \mathcal{H}_3$ defined by the above canonical homology basis. Now examine the pullbacks of the theta functions $\theta[^{\epsilon}_{\epsilon'}](\phi(P), \tau)$. Knowing the values of ϕ at branch points as we do we can work out at which branch points any particular theta vanishes. Look for pairs of thetas whose quotient has a double zero at 0 and a double pole at ∞ (i.e. the branch points above $0, \infty$). These are constant multiples of the covering map z and, plugging in values like $P = z^{-1}(\lambda_1)$ we get, for example

$$\lambda_1 = \frac{\theta\begin{bmatrix}000\\000\end{bmatrix}(0,\tau)\,\theta\begin{bmatrix}000\\010\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}000\\001\end{bmatrix}(0,\tau)\,\theta\begin{bmatrix}000\\011\end{bmatrix}(0,\tau)}$$

In the book [3] p. 186 it is shown that for the above canonical homology basis $\theta\left[\begin{smallmatrix}101\\111\end{smallmatrix}\right](0,\tau)=0$. Given a point P on V_3 with one vanishing coordinate (the one corresponding to $\theta\left[\begin{smallmatrix}101\\111\end{smallmatrix}\right]$, and called x_{25} in our fixed ordering) define 5 constants, using the formulae for the λ_i described above:

$$\lambda_1=\frac{x_1x_3}{x_2x_4},\quad \lambda_2=\frac{x_{22}x_{26}}{x_{23}x_{27}},\quad \lambda_3=\frac{x_5x_{13}}{x_6x_{14}}$$

$$\lambda_4=\frac{x_{15}x_{20}}{x_{16}x_{21}},\quad \lambda_5=\frac{x_7x_{28}}{x_8x_{29}} \tag{4.5}$$

Using the equations for V_3 with $\theta\left[\begin{smallmatrix}101\\111\end{smallmatrix}\right]=0$ we can show that: for the point P, every ratio x_i^4/x_j^4 $(j\neq 25)$ can be expressed as a rational function of the λ_n. This is just like theorem 4.1 and is proved in the same way. Now the λ_i in (4.5) define a hyperelliptic curve and hence its theta constants for our special canonical homology basis. Thus $\theta^4(P)=\theta^4$ (point from hyperelliptic curve) and by the transitivity result from the end of section 3 we have proved that:

THEOREM 4.3: *Any point P of V_3 with precisely one vanishing coordinate is a point of S_3* corresponding to a hyperelliptic curve.

It remains for us to consider the points of V_3 with several zero coordinates (vide theorem 3.1). Because all vanishing 6-sets are equivalent under the group, we may concentrate on the six coordinates $\mathcal{S}=\{\theta\left[\begin{smallmatrix}101\\101\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}101\\111\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}111\\101\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}011\\011\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}011\\111\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}111\\011\end{smallmatrix}\right])$. Recall the formula:

$$\theta\begin{bmatrix}\epsilon_1\epsilon_2\epsilon_3\\ \epsilon_1'\epsilon_2'\epsilon_3'\end{bmatrix}\left(0,\begin{pmatrix}\pi_2 & \begin{smallmatrix}0\\0\end{smallmatrix}\\ 00 & \tau\end{pmatrix}\right)=\theta\begin{bmatrix}\epsilon_1\epsilon_2\\ \epsilon_1'\epsilon_2'\end{bmatrix}(0,\pi_2)\,\theta\begin{bmatrix}\epsilon_3\\ \epsilon_3'\end{bmatrix}(0,\tau)$$

where $\pi_2\in\mathcal{H}_2$, $\tau\in\mathcal{H}_1$. It motivates the definition of the composite map:

$$\phi:\mathbb{P}^9\times\mathbb{P}^2\to\mathbb{P}^{29}\to\mathbb{P}^{35}$$

where the first map is the Segre embedding with coordinates ordered so that the split on the diagonal (mentioned above) is respected; and the second map is the identity map where we set the six coordinates of $\mathcal{S}$ equal to zero. The theta constants of genera one and two define varieties in $\mathbb{P}^2$ and $\mathbb{P}^9$ of respective dimensions one and three. Call them S_1 and S_2 respectively. In each case the equations defining them are easy to write down: S_1 is just the curve $x^4-y^4-z^4=0$: whereas

S_2 is defined by 30 quartics. These 30 quartics are generated by a procedure quoted in the book [3] p. 43.

Take a point P of V_3 with the coordinates of $\mathcal{S}$ equal to zero. Letting θ^2 be the regular map from $\mathbb{P}^{35}$ to itself which squares each coordinate, I claim.

PROPOSITION 4.1: *There is a point Q in $S_2 \times S_1$ such that $\theta^2(P) = \theta^2(\phi(Q))$. Hence by transitivity on the fibres of θ^2 (see end of section* 3) *we have that P belongs to the subvariety $\phi(S_2 \times S_1)$ of S_3, or to one of its translates under $\Gamma(1)$.*

The proof of this proposition is not difficult. Once we have Q in $\mathbb{P}^9 \times \mathbb{P}^2$ with the required property it is a simple matter to find equations implying that Q is in $S_2 \times S_1$. The rest of the proof involves a counting argument, namely, to show that there are enough biquadratic equations (those of type 2), with the coordinates of $\mathcal{S}$ equal to zero, to imply all the "interchanges" required by the Segre embedding (at the θ^2-level which is sufficient).

The higher vanishing sets all give points which translate under the group action into special subvarieties of $\phi(S_2 \times S_1)$. These subvarieties are the obvious ones obtained by killing coordinates in S_2 and S_1.

This completes our aim: to prove that $V_3 = S_3$ and, at the same time classify points of S_3.

Added in proof

I now believe that Theorem 4.2 can be improved upon in the following sense. Namely, that to a point with no vanishing coordinates in V_3 there actually corresponds a non-hyperelliptic curve. The choice of the said curve being obvious, the difficulty lies in showing that it is non-singular. I am indebted to Mr. Patrick Du Val for showing me this proof. A paper containing the result and its proof has been accepted for publication.

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