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A NOTE ON A 3-DIMENSIONAL HOMOGENEOUS SPACE

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We give a calculation of the homeomorphism type of the homogeneous space $SL(2, R)/SL(2, Z)$ using only elementary properties of Seifert manifolds (cf. p. 84 of Milnor's book, Introduction to algebraic K-theory, for a different proof). As a corollary, the well-known presentation of $SL(2, Z)$ follows.

THEOREM: *The (right) coset space of $SL(2, Z)$ in $SL(2, R)$ is homeomorpic to the complement of the trefoil knot in S^3.*

PROOF: Since $SL(2, Z)$ is a discrete subgroup of $SL(2, R)$, the coset space which we denote by M is a 3-manifold without boundary. Right multiplication of $SO(2, R)$ on $SL(2, R)$ commutes with left multiplication by $SL(2, Z)$. Therefore there is an induced action of $SO(2, R)$ on M. We show that M is a Seifert manifold with two exceptional fibres of multiplicity 2 and 3 and with orbit surface equal to an open disk. This implies that M and the complement of the trefoil knot in S^3 have the same Seifert invariants and so are homeomorphic (see [2] or [3] for the properties of Seifert spaces).

Let $A \in SO(2, R)$. Then A is in the stabilizer of some point of M exactly when the equation $\Sigma S = SA$ can be solved for $\Sigma \in SL(2, Z)$ and $S \in SL(2, R)$. In this case $\Sigma = SAS^{-1}$ and $\operatorname{tr} A = \operatorname{tr} \Sigma$ is an integer. If λ, γ are the eigenvalues of A, then $|\lambda| = |\gamma| = 1$ since A is orthogonal, and $\lambda\gamma = 1$ as $\det A = 1$. Therefore $\lambda = \bar{\gamma}$ and because $\lambda + \gamma$ is an integer we see that λ is an n^{th} root of unity for $n = 1, 2, 3, 4$ or 6. Consequently $A^n = I$. Also $-I$ acts trivially on M and so we conclude that the stabilizer of every point of M is Z_2, Z_4 or Z_6.

Consider the case when A is of order 4, i.e. when $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

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Observe that $S^t = AS^{-1}A^{-1}$, so that $SS^t = SAS^{-1}A^{-1} = \Sigma A^{-1} \in SL(2, Z)$. Let $SS^t = \begin{pmatrix} m & p \\ p & n \end{pmatrix}$ and let $B = \begin{pmatrix} 0 & 1 \\ -1 & \pm 1 \end{pmatrix}$ or $\begin{pmatrix} \pm & 1 \\ -1 & 0 \end{pmatrix}$. Then BSS^tB^t has its off-diagonal entry equal to $\pm n - p$ or $\pm m - p$ respectively. Now $mn - p^2 = 1$ and $m > 0$, $n > 0$. Consequently either $m \leq |p|$ or $n \leq |p|$, and by appropriate choice of B we get that the off-diagonal term of BSS^tB^t has absolute value strictly less than $|p|$. So by a sequence of such transformations, a matrix $\Sigma_1 \in SL(2, Z)$ can be found for which $\Sigma_1 SS^t \Sigma_1^t = I$. This implies that $\Sigma_1 S \in SO(2, R)$ and the coset of S is in the same orbit of $SO(2, R)$ as the coset of I. Therefore there is only one orbit of $SO(2, R)$ in M with stabilizer Z_4.

For the case when the stabilizer is Z_6, let A be the matrix of rotation by $\pi/3$. Then $SAS^{-1} \in SL(2, Z)$ if and only if $\sqrt{3}\, SS^t$ is a matrix with integer entries, determinant equal to 3, odd integers on the diagonal and even integers off the diagonal. By the same argument as above, we can find a matrix $\Sigma_1 \in SL(2, Z)$ so that $\Sigma_1 \sqrt{3}\, SS^t \Sigma_1^t = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore $\Sigma_1 SS^t \Sigma_1^t = S_1 S_1^t$ where $S_1 = \begin{pmatrix} \sqrt[4]{3} & 0 \\ 0 & -\sqrt[4]{3} \end{pmatrix}$. This implies $S_1^{-1} \Sigma_1 S \in SO(2, R)$ and the coset of S is in the same orbit as the coset of S_1. Consequently, there is just one orbit with stabilizer Z_6.

To complete the proof we have to determine the orbit surface F of M. Every matrix in $SL(2, R)$ can be written uniquely as the product of a real lower triangular matrix and a matrix in $SO(2, R)$. So there is a homeomorphism of $SL(2, R)$ onto $R^2 \times S^1$ and the inclusion of $SO(2, R)$ in $SL(2, R)$ induces an isomorphism of fundamental groups. Since $M = SL(2, R)/SL(2, Z)$, there is an exact sequence $1 \to Z \to \pi_1(M) \to SL(2, Z) \to 1$, and $SL(2, Z)$ is the quotient of $\pi_1(M)$ by a cyclic normal subgroup with generator α. Note that α is the image of a generator of $\pi_1(SO(2, R))$ under the mapping $SO(2, R) \to SL(2, R) \to M$. Let h be the homotopy class of an orbit in M of the $SO(2, R)$ action, with stabilizer Z_2 (i.e. an ordinary fibre in the terminology of [3]). Without loss of generality $\alpha = h^2$ and $SL(2, Z)$ is the quotient of $\pi_1(M)$ by the cyclic normal subgroup generated by h^2.

Consequently there is an epimorphism from $SL(2, Z)$ to $\pi_1(F)$ and also from $H_1(SL(2, Z))$ to $H_1(F)$. Now it is well-known that $SL(2, Z)$ is generated by $X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$. So since $X^4 = 1$, $Y^6 = 1$ and $X^2 = Y^3 = -I$, we see that $H_1(SL(2, Z))$ has order at most 12. Therefore F must be an open disk, a 2-sphere or a projective plane.

If $F = S^2$ then M is a lens space and so cannot be covered by $SL(2, R)$. Suppose next that $F = RP^2$. As M is orientable, the relation matrix for $H_1(M)$ (see [2] or [3]) shows that h^2 is null homologous and $H_1(M)$ has order at least 24. But $h^2 \sim 0$ implies $H_1(M) = H_1(SL(2, Z))$, which gives a contradiction. This completes the proof.

COROLLARY: $SL(2, Z) = Z_4 *_{Z_2} Z_6$.

PROOF: $\pi_1(M)$ has generators h, x, y and relations $[h, x] = [h, y] = 1$, $x^2h = y^3h = 1$ (see [2] or [3]). Then $SL(2, Z)$ is given by adding the relation $h^2 = 1$ to $\pi_1(M)$, and so has the presentation $\{x, y \mid x^4 = 1, x^2 = y^3\}$.

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