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## UPPER BOUNDS FOR AN IWASAWA INVARIANT

Frank Gerth III\*

### 1. Introduction

Let  $k_0$  be a finite extension of  $\mathbf{Q}$ , the field of rational numbers, and let  $K$  be a  $\mathbf{Z}_\ell$ -extension of  $k_0$  (that is,  $K$  is a Galois extension of  $k_0$ , and  $\text{Gal}(K/k_0)$  is topologically isomorphic to the additive group of the  $\ell$ -adic integers  $\mathbf{Z}_\ell$ ). Let the intermediate fields be denoted by  $k_0 \subset k_1 \subset k_2 \subset \cdots \subset k_n \subset \cdots \subset K$ , where  $k_n/k_0$  is a cyclic extension of degree  $\ell^n$ . Let  $\sigma$  be a topological generator of  $\text{Gal}(K/k_0)$ , and let  $\sigma_n = \sigma|_{k_n}$ . Then  $\text{Gal}(k_n/k_0)$  is generated by  $\sigma_n$ , and we shall sometimes denote  $\text{Gal}(k_n/k_0)$  by  $\langle \sigma_n \rangle$ . We define  $\tau_n = \sigma_n^{\ell^{n-1}}$ , and we note that  $\text{Gal}(k_n/k_{n-1}) = \langle \tau_n \rangle$ .

Next we let  $A_n$  denote the  $\ell$ -class group of  $k_n$  for all  $n$  (that is,  $A_n$  is the Sylow  $\ell$ -subgroup of the ideal class group of  $k_n$ ). For any finite group  $C$ , we let  $|C|$  denote the order of  $C$ . Then from the theory of  $\mathbf{Z}_\ell$ -extensions (see [4]),  $|A_n| = \ell^{e_n}$  with

$$(1) \quad e_n = \mu \ell^n + \lambda n + \nu$$

for  $n$  sufficiently large, where  $\mu, \lambda, \nu$  are integers (called the Iwasawa invariants of  $K/k_0$ ) with  $\mu \geq 0$  and  $\lambda \geq 0$ . In general it is difficult to compute  $\mu, \lambda, \nu$ , and we usually do not know how large  $n$  must be in order for equation 1 to be valid. Our goal in this paper is to specify an upper bound for  $\mu$  based on the number of ramified primes in  $K/k_0$  and on  $|A_{n_1}|$  for a particular  $n_1$ . Before stating our main theorem, we recall some facts about  $\mathbf{Z}_\ell$ -extensions (cf. [4]). It is known that only finitely many primes ramify in  $K/k_0$  (in fact, all the ramified primes are above the rational prime  $\ell$ ). If  $s_n$  denotes the number of primes of

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$k_n$  that are ramified over  $k_0$ , then  $s_{n+1} \geq s_n$  for all  $n$ , and there is an integer  $n_0$  such that  $s_n = s_{n_0}$  for all  $n \geq n_0$ . We let  $s$  denote  $s_{n_0}$ , and we observe that the  $s$  primes of  $k_{n_0}$  which are ramified over  $k_0$  are also totally ramified in  $K/k_{n_0}$ . We are now ready to state our main theorem.

**THEOREM 1.:** *Let  $K/k_0$  be a  $\mathbf{Z}_\ell$ -extension, where  $k_0$  is a finite extension of  $\mathbf{Q}$ . Let  $s$  be defined as above, and let  $n_1$  be chosen so that  $l^{n_1} > s - 1$ . If  $\mu$  is the Iwasawa invariant of  $K/k_0$ , then*

$$\mu \leq e_{n_1}/(l^{n_1} - s + 1),$$

where  $|A_{n_1}| = \ell^{e_{n_1}}$ , and  $A_{n_1}$  is the  $\ell$ -class group of  $k_{n_1}$ .

**REMARK:** In [3] Greenberg shows that  $\mu$  is ‘‘locally bounded’’ on certain  $\mathbf{Z}_\ell$ -extensions of  $k_0$ . Our Theorem 1 can be used to provide explicit local bounds for  $\mu$ .

## 2. Preliminary results

We let the notation be the same as in section 1. We shall use multiplicative notation in each  $A_n$ , and the action of  $\text{Gal}(k_n/k_0)$  shall be expressed by exponentiation. For  $j > i \geq 0$ , we define  $N_{i,j}: A_j \rightarrow A_i$  to be the map induced by the norm map from  $k_j$  to  $k_i$ , and we define  $J_{j,i}: A_i \rightarrow A_j$  to be the map induced by the inclusion map of  $k_i$  into  $k_j$ . For each  $n > 0$ , we define

$$T_n = 1 + \sigma_n^{\ell^{n-1}} + \cdots + \sigma_n^{(\ell-1)\ell^{n-1}}. \quad (2)$$

We note that  $a^{T_n} = J_{n,n-1}(N_{n-1,n}(a))$  for each  $a \in A_n$ . We now list several facts that are proved in [2].

**LEMMA 1:**  $T_n \equiv (1 - \sigma_n)^{(\ell-1)\ell^{n-1}} \pmod{\ell \mathbf{Z}_\ell[\langle \sigma_n \rangle]}$ .

**LEMMA 2:** *Let  $R = \mathbf{Z}_\ell[\langle \sigma_n \rangle]/(T_n)$ . Then  $R$  is a principal ideal domain. Next let  $A$  be a finite abelian  $\ell$ -group, and define  $\text{rank } A = \dim_{\mathbf{F}_\ell}(A/A^\ell)$ , where  $\mathbf{F}_\ell$  is the finite field with  $\ell$  elements. If  $A$  is also a cyclic  $R$  module, and  $\text{rank } A = r < (\ell - 1)\ell^{n-1}$ , then  $A$  is an elementary abelian  $\ell$ -group with  $\text{rank} = r$ .*

**LEMMA 3:** *For  $j > i \geq n_0$ ,  $N_{i,j}$  is surjective.*

The following lemma is a consequence of [5], Lemma 4.

LEMMA 4: Let  $A_n^{(\tau_n)} = \{a \in A_n \mid a^{\tau_n} = a\}$ . Then

$$|A_n^{(\tau_n)}| \leq |A_{n-1}| \cdot \ell^{s-1}.$$

### 3. Proof of Theorem 1

Let the notation be the same as in sections 1 and 2. Let  $V_n = A_n/A_n^\ell$ . We note that  $V_n$  may be viewed as a finite dimensional vector space over  $\mathbf{F}_\ell$ , and  $(1 - \sigma_n)$  is a nilpotent linear transformation on  $V_n$ . In fact

$$(1 - \sigma_n)^{\ell^n} \equiv 1 - \sigma_n^{\ell^n} \equiv 0 \pmod{(\ell \mathbf{Z}_\ell[\langle \sigma_n \rangle])}.$$

Then from linear algebra

$$V_n = \prod_{i=1}^{z_n} M_{n,i}$$

where  $\ell^n \geq \text{rank } M_{n,1} \geq \text{rank } M_{n,2} \geq \cdots \geq \text{rank } M_{n,z_n} \geq 1$ ;

$$z_n = \text{rank } V_n / V_n^{1-\sigma_n}; \text{ and}$$

$M_{n,i}$  is an elementary abelian  $\ell$ -group which is also a cyclic  $\mathbf{Z}_\ell[\langle \sigma_n \rangle]$  module.

If we let  $\alpha_{n,i} = \text{rank } M_{n,i}$ , then  $M_{n,i}^{(1-\sigma_n)^{\alpha_{n,i}}} = \{1\}$  but  $M_{n,i}^{(1-\sigma_n)^{\alpha_{n,i}-1}} \neq \{1\}$ . Also the integers  $\alpha_{n,1}, \dots, \alpha_{n,z_n}$  are uniquely determined by  $V_n$ .

Now we let  $\bar{N}_{i,j}: V_j \rightarrow V_i$  be the map induced by  $N_{i,j}$ , and we let  $\bar{J}_{j,i}: V_i \rightarrow V_j$  be the map induced by  $J_{j,i}$ . Since  $N_{i,j}$  is surjective for  $j > i \geq n_0$  by Lemma 3, then  $\bar{N}_{i,j}$  is surjective for  $j > i \geq n_0$ .

Unless otherwise noted, we shall assume  $n \geq n_2$ , where  $n_2 = \max(n_0, n_1)$ . (See Theorem 1 for the definition of  $n_1$ .) Let  $y_{0,n}$  denote the number of  $\alpha_{n,i}$  with  $\alpha_{n,i} = \ell^n$ . (Note: If  $\alpha_{n,1} < \ell^n$ , then  $y_{0,n} = 0$ .) We claim that  $y_{0,n+1} \leq y_{0,n}$ . In fact, if  $M_{n+1,i}$  is a cyclic  $\mathbf{Z}_\ell[\langle \sigma_{n+1} \rangle]$  module factor of  $V_{n+1}$  with rank  $= \ell^{n+1}$ , then letting  $M = \bar{N}_{n,n+1}(M_{n+1,i})$ , we see that  $\bar{J}_{n+1,n}(M) = \bar{J}_{n+1,n}(\bar{N}_{n,n+1}(M_{n+1,i})) = M_{n+1,i}^{T_{n+1}}$ . Since

$$T_{n+1} \equiv (1 - \sigma_{n+1})^{(\ell-1)\ell^n} \pmod{(\ell \mathbf{Z}_\ell[\langle \sigma_{n+1} \rangle])}$$

by Lemma 1, then  $\text{rank } \bar{J}_{n+1,n}(M) = \ell^{n+1} - (\ell-1)\ell^n = \ell^n$ . Then it is easy to see that  $M$  is a cyclic  $\mathbf{Z}_\ell[\langle \sigma_n \rangle]$  module factor of  $V_n$  with

rank =  $\ell^n$ . So for every cyclic module factor of  $V_{n+1}$  with rank =  $\ell^{n+1}$ , there is a corresponding cyclic module factor of  $V_n$  with rank =  $\ell^n$ . Then we see that  $y_{0,n+1} \leq y_{0,n}$ . Since each  $y_{0,n} \geq 0$ , there are integers  $\gamma_0 \geq 0$  and  $f_0 \geq n_2$  such that  $y_{0,n} = \gamma_0$  for all  $n \geq f_0$ .

Next let  $y_{1,n}$  denote the number of  $\alpha_{n,i}$  with  $\alpha_{n,i} = \ell^n - 1$ . By arguments similar to those above,  $y_{1,n+1} \leq y_{1,n}$  for all  $n \geq f_0$ , and there are integers  $\gamma_1 \geq 0$  and  $f_1 \geq f_0$  such that  $y_{1,n} = \gamma_1$  for all  $n \geq f_1$ . Similarly if  $y_{j,n}$  (for  $1 < j \leq s-1$ ) denotes the number of  $\alpha_{n,i}$  with  $\alpha_{n,i} = \ell^n - j$ , then there are nonnegative integers  $\gamma_2, \dots, \gamma_{s-1}$  and  $g = f_{s-1} \geq \dots \geq f_2 \geq f_1$  such that  $y_{j,n} = \gamma_j$  for all  $n \geq f_j$ . We observe that  $V_g = \prod_{i=1}^g M_{g,i}$  with

$$\alpha_{g,i} = \begin{cases} \ell^g & \text{for } i \leq \gamma_0 \\ \ell^g - 1 & \text{for } \gamma_0 < i \leq \gamma_0 + \gamma_1 \\ \vdots & \\ \ell^g - s + 1 & \text{for } \sum_{j=1}^{s-2} \gamma_j < i \leq \sum_{j=1}^{s-1} \gamma_j \end{cases}$$

and  $\alpha_{g,i} < \ell^g - s + 1$  for  $i > \sum_{j=1}^{s-1} \gamma_j$ . We let  $y$  denote  $\sum_{j=1}^{s-1} \gamma_j$ , and we let  $M_g$  denote the product of all  $M_{g,i}$  with  $\alpha_{g,i} < \ell^g - s + 1$ . So  $V_g = (\prod_{i=1}^y M_{g,i}) \times M_g$ . For each  $n > g$ , we define inductively  $M_{n,i}$  ( $1 \leq i \leq y$ ) and  $M_n$  such that  $\bar{N}_{n-1,n}(M_{n,i}) = M_{n-1,i}$ ;  $\bar{N}_{n-1,n}(M_n) = M_{n-1}$ ;  $V_n = (\prod_{i=1}^y M_{n,i}) \times M_n$ ; and  $\alpha_{n,i}$  satisfies the same equation as  $\alpha_{g,i}$  (for  $1 \leq i \leq y$ ) except with  $n$  replacing  $g$ . For  $n \geq g$ , we then define inductively  $\mathbf{Z}_\ell[\langle \sigma_n \rangle]$  modules  $B_n \supseteq A_n^\ell$  so that the image of  $B_n$  in  $V_n$  is  $M_n$  and  $N_{n,n+1}(B_{n+1}) = B_n$ . We note that  $|A_n/B_n| = |V_n/M_n| = \ell^{\beta_n}$  for  $n \geq g$ , where

$$\beta_n = \gamma_0 \ell^n + \gamma_1 (\ell^n - 1) + \dots + \gamma_{s-1} (\ell^n - s + 1).$$

Our next step is to estimate  $B_n^{(\tau_n)} = \{a \in B_n \mid a^{\tau_n} = a\} = B_n \cap A_n^{(\tau_n)}$ . First we note that for  $n > g$ ,  $J_{n,n-1}(A_{n-1}) \subseteq A_n^{(\tau_n)}$ , and

$$((J_{n,n-1}(A_{n-1})) \cdot B_n) / B_n = ((J_{n,n-1}(N_{n-1,n}(A_{n-1}))) \cdot B_n) / B_n = (A_n^{T_n} \cdot B_n) / B_n.$$

Since  $T_n \equiv (1 - \sigma_n)^{(\ell-1)\ell^{n-1}} \pmod{\ell \mathbf{Z}_\ell[\langle \sigma_n \rangle]}$  by Lemma 1, then

$$\begin{aligned} |(A_n^{T_n} \cdot B_n) / B_n| &= |A_n / B_n| \cdot |[A_n / (A_n^{T_n} \cdot B_n)]|^{-1} \\ &= \ell^{\beta_n} \cdot \ell^{-(\gamma_0 + \dots + \gamma_{s-1})(\ell-1)\ell^{n-1}} \\ &= \ell^{\beta_{n-1}}. \end{aligned}$$

Since  $A_n^{(\tau_n)}/(B_n \cap A_n^{(\tau_n)}) \cong (A_n^{(\tau_n)} \cdot B_n)/B_n$ , then  $|A_n^{(\tau_n)}/(B_n \cap A_n^{(\tau_n)})| = |(A_n^{(\tau_n)} \cdot B_n)/B_n| \geq \ell^{\beta_{n-1}}$ . Then

$$|B_n^{(\tau_n)}| = |B_n \cap A_n^{(\tau_n)}| \leq |A_n^{(\tau_n)}| \cdot \ell^{-\beta_{n-1}} \leq |A_{n-1}| \cdot \ell^{s-1} \cdot \ell^{-\beta_{n-1}}$$

by Lemma 4. Since  $|A_{n-1}/B_{n-1}| = \ell^{\beta_{n-1}}$ , then  $|A_{n-1}| \cdot \ell^{-\beta_{n-1}} = |B_{n-1}|$ . So  $|B_n^{(\tau_n)}| \leq |B_{n-1}| \cdot \ell^{s-1}$  for  $n > g$ .

Now we let  $W_n = B_n/B_n^\ell$ . We can apply the same procedures to the  $W_n$  that we applied to the  $V_n$ . So we can find submodules  $C_n$  of the  $B_n$  containing  $B_n^\ell$ , nonnegative integers  $\delta_0, \delta_1, \dots, \delta_{s-1}$ , and an integer  $r \geq g$  such that  $N_{n,n+1}(C_{n+1}) = C_n$  for  $n \geq r$ , and  $|B_n/C_n| = \ell^{\epsilon_n}$  for  $n \geq r$ , where

$$\epsilon_n = \delta_0 \ell^n + \delta_1(\ell^n - 1) + \dots + \delta_{s-1}(\ell^n - s + 1).$$

Also  $|C_n^{(\tau_n)}| \leq |C_{n-1}| \cdot \ell^{s-1}$  for  $n > r$ . We note that for  $n \geq r$ ,  $|A_n/C_n| = \ell^{\theta_n}$  with

$$\theta_n = (\gamma_0 + \delta_0)\ell^n + (\gamma_1 + \delta_1)(\ell^n - 1) + \dots + (\gamma_{s-1} + \delta_{s-1})(\ell^n - s + 1).$$

We can then consider  $C_n/C_n^\ell$  and repeat the above procedures. Eventually we obtain submodules  $H_n$  of the  $A_n$  and an integer  $n_3 \geq r$  such that  $N_{n,n+1}(H_{n+1}) = H_n$  for  $n \geq n_3$ , and  $H_n/H_n^\ell$  has no cyclic  $\mathbf{Z}_\ell[\langle \sigma_n \rangle]$  module factor with rank  $\geq \ell^n - s + 1$  for  $n \geq n_3$ . (Remark: Our procedure terminates after a finite number of steps because  $|A_{n_1}|$  is finite and  $|\bar{N}_{n_1,n}(M)| \geq \ell^{\ell^{n_1-s+1}} \geq \ell$  for each cyclic  $\mathbf{Z}_\ell[\langle \sigma_n \rangle]$  module factor  $M$  with rank  $\geq \ell^n - s + 1$ .) Also  $|A_n/H_n| = \ell^{\omega_n}$  for  $n \geq n_3$ , where

$$\omega_n = \psi_0 \ell^n + \psi_1(\ell^n - 1) + \dots + \psi_{s-1}(\ell^n - s + 1)$$

for some nonnegative integers  $\psi_0, \psi_1, \dots, \psi_{s-1}$ . Furthermore  $|H_n^{(\tau_n)}| \leq |H_{n-1}| \cdot \ell^{s-1}$  for  $n > n_3$ . We observe that

$$\omega_n = (\psi_0 + \psi_1 + \dots + \psi_{s-1})\ell^n - (\psi_1 + 2\psi_2 + \dots + (s-1)\psi_{s-1}).$$

We let  $\omega = \psi_0 + \psi_1 + \dots + \psi_{s-1}$  and  $\psi = \psi_1 + 2\psi_2 + \dots + (s-1)\psi_{s-1}$ . So  $\omega_n = \omega \ell^n - \psi$ . Now by equation 1,  $|A_n| = \ell^{e_n}$  with  $e_n = \mu \ell^n + \lambda n + \nu$  for  $n$  sufficiently large. Since  $|A_n/H_n| = \ell^{\omega_n}$ , then  $|H_n| = \ell^{e'_n}$  for  $n$  sufficiently large, where

$$(3) \quad e'_n = \mu' \ell^n + \lambda' n + \nu'$$

with  $\mu' = \mu - \omega$ ,  $\lambda' = \lambda$ , and  $\nu' = \nu + \psi$ . Also, from the structure of  $A_n/H_n$ , routine calculations show that

$$|A_{n_1}| \geq |N_{n_1, n_3}(A_{n_3})/N_{n_1, n_3}(H_{n_3})| = \ell^{\omega \ell^{n_1 - \psi}} \geq \ell^{\omega(\ell^{n_1 - s + 1})}.$$

So if  $|A_{n_1}| = \ell^{e_{n_1}}$ , then  $\omega \leq e_{n_1}/(\ell^{n_1} - s + 1)$ . To complete the proof of Theorem 1, it suffices to show that  $\mu' = 0$ . Then  $\mu = \omega \leq e_{n_1}/(\ell^{n_1} - s + 1)$ .

So we consider  $H_n$  with the following properties:  $H_n/H_n^\ell$  has no cyclic  $\mathbf{Z}_\ell[\langle \sigma_n \rangle]$  module factor with rank  $\geq \ell^n - s + 1$  for  $n \geq n_3$ ;  $N_{n, n+1}(H_{n+1}) = H_n$  for  $n \geq n_3$ ; and  $|H_n^{(\tau_n)}| \leq |H_{n-1}| \cdot \ell^{s-1}$  for  $n > n_3$ . We let  $P_n = H_n/H_n^\ell$ . Now

$$\begin{aligned} P_n/P_n^{1-\tau_n} &= (H_n/H_n^\ell)/(H_n/H_n^\ell)^{1-\tau_n} \cong H_n/(H_n^\ell \cdot H_n^{1-\tau_n}) \\ &\cong (H_n/H_n^{1-\tau_n})/(H_n/H_n^{1-\tau_n})^\ell. \end{aligned}$$

Since  $N_{n-1, n}(H_n) = H_{n-1}$  for  $n > n_3$ , then  $H_n/\ker N_{n-1, n} \cong H_{n-1}$ . Also it is easy to see that  $\ker N_{n-1, n} \supseteq H_n^{1-\tau_n}$  and  $|H_n/H_n^{1-\tau_n}| = |H_n^{(\tau_n)}|$ . It then follows that  $|\ker N_{n-1, n}/H_n^{1-\tau_n}| \leq \ell^{s-1}$ . Since  $H_{n-1}/H_{n-1}^\ell$  is the direct product of cyclic  $\mathbf{Z}_\ell[\langle \sigma_{n-1} \rangle]$  modules with ranks  $< \ell^{n-1} - s + 1$  for  $n > n_3$ , then  $(H_n/\ker N_{n-1, n})/(H_n/\ker N_{n-1, n})^\ell$  is the direct product of cyclic  $\mathbf{Z}_\ell[\langle \sigma_n \rangle]$  modules with ranks  $< \ell^{n-1} - s + 1$  for  $n > n_3$ . Then  $(H_n/H_n^{1-\tau_n})/(H_n/H_n^{1-\tau_n})^\ell$  and hence  $P_n/P_n^{1-\tau_n}$  are direct products of cyclic  $\mathbf{Z}_\ell[\langle \sigma_n \rangle]$  modules with ranks  $< \ell^{n-1}$  for  $n > n_3$ . Since  $1 - \tau_n = 1 - \sigma_n^{\ell^{n-1}} \equiv (1 - \sigma_n)^{\ell^{n-1}} \pmod{\ell \mathbf{Z}_\ell[\langle \sigma_n \rangle]}$ , we must have  $P_n^{1-\tau_n} = \{1\}$ . So

$$\begin{aligned} |P_n| &= |P_n/P_n^{1-\tau_n}| = |(H_n/H_n^{1-\tau_n})/(H_n/H_n^{1-\tau_n})^\ell| \\ &= \frac{|H_n/\ker N_{n-1, n}| \cdot |\ker N_{n-1, n}/H_n^{1-\tau_n}|}{|(H_n/H_n^{1-\tau_n})^\ell|} \\ &\leq \frac{|H_{n-1}| \cdot |\ker N_{n-1, n}/H_n^{1-\tau_n}|}{|H_{n-1}^\ell|} \\ &\leq |P_{n-1}| \cdot \ell^{s-1}. \end{aligned}$$

Next we note that  $H_n^{T_n} \subset H_n^\ell$  since

$$T_n \equiv (1 - \sigma_n)^{(\ell-1)\ell^{n-1}} \equiv (1 - \tau_n)^{\ell-1} \pmod{\ell \mathbf{Z}_\ell[\langle \sigma_n \rangle]}, \quad \text{and } P_n^{1-\tau_n} = \{1\}.$$

On the other hand,  $H_n/H_n^{T_n}$  is a finitely generated module over the principal ideal domain  $\mathbf{Z}_\ell[\langle \sigma_n \rangle]/(T_n)$ . Since

$$\begin{aligned}
(H_n/H_n^{T_n})/(H_n/H_n^{T_n})^\ell &\cong H_n/(H_n^\ell \cdot H_n^{T_n}) \\
&\cong (H_n/H_n^\ell)/(H_n/H_n^\ell)^{T_n} \\
&= P_n/P_n^{T_n} = P_n,
\end{aligned}$$

and  $P_n$  is the direct product of cyclic modules with ranks  $< \ell^{n-1} \leq (\ell-1)\ell^{n-1}$ , then  $H_n/H_n^{T_n}$  is the direct product of cyclic modules over  $\mathbf{Z}_\ell\langle\sigma_n\rangle/(T_n)$  with ranks  $< (\ell-1)\ell^{n-1}$ . So  $H_n/H_n^{T_n}$  is an elementary abelian  $\ell$ -group by Lemma 2. Hence  $H_n^{T_n} \supseteq H_n^\ell$ . Thus  $H_n^\ell = H_n^{T_n} = J_{n,n-1}(N_{n-1,n}(H_n)) = J_{n,n-1}(H_{n-1})$ . So  $|H_n^\ell| \leq |H_{n-1}|$ . Then

$$|H_n| = |H_n/H_n^\ell| \cdot |H_n^\ell| = |P_n| \cdot |H_n^\ell| \leq |P_{n-1}| \cdot \ell^{s-1} \cdot |H_{n-1}|.$$

Let  $t_{n-1} = \text{rank } P_{n-1}$ . Then  $|H_n| \leq |H_{n-1}| \cdot \ell^{t_{n-1}+s-1}$  for  $n > n_3$ . So  $|H_{n_3+1}| \leq |H_{n_3}| \cdot \ell^{t_{n_3}+s-1}$ ;

$$\begin{aligned}
|H_{n_3+2}| &\leq |H_{n_3+1}| \cdot \ell^{t_{n_3+1}+s-1} \leq (|H_{n_3}| \cdot \ell^{t_{n_3}+s-1}) \cdot \ell^{(t_{n_3}+s-1)+s-1} \\
&= |H_{n_3}| \cdot \ell^{2t_{n_3}+3(s-1)},
\end{aligned}$$

and by induction,  $|H_{n_3+i}| \leq |H_{n_3}| \cdot \ell^v$ , where  $v = it_{n_3} + (s-1)(i+1)(i)/2$  for  $i \geq 1$ . Now by equation 3,  $|H_{n_3+i}| = \ell^{e'_{n_3+i}}$  with  $e'_{n_3+i} = \mu' \ell^{n_3+i} + \lambda'(n_3+i) + \nu'$  for  $i$  sufficiently large. We note that  $\lim_{i \rightarrow \infty} (e'_{n_3+i}/\ell^{n_3+i}) = \mu'$ . From our above calculations,  $e'_{n_3+i} \leq e'_{n_3} + it_{n_3} + (s-1)(i+1)(i)/2$ . Then it is easy to see that  $\lim_{i \rightarrow \infty} (e'_{n_3+i}/\ell^{n_3+i}) = 0$ . So  $\mu' = 0$ . This is what we wanted to show, and hence the proof of Theorem 1 is complete.

#### 4. Some special cases

The following corollaries are immediate consequences of Theorem 1.

**COROLLARY 1:** *Let  $K/k_0$  be a  $\mathbf{Z}_\ell$ -extension of  $k_0$ , where  $k_0$  is a finite extension of  $\mathbf{Q}$ . Assume that there is only one ramified prime in  $K/k_0$ , and it is totally ramified. If  $|A_0| = \ell^{e_0}$ , where  $A_0$  is the  $\ell$ -class group of  $k_0$ , then the Iwasawa invariant  $\mu$  of  $K/k_0$  satisfies  $\mu \leq e_0$ .*

**REMARK:** Corollary 1 is also proved in [2].

**COROLLARY 2:** *Let  $K/k_0$  be a  $\mathbf{Z}_\ell$ -extension of  $k_0$ , where  $[k_0:\mathbf{Q}] = 2$ . If  $\ell$  splits in  $k_0$ , and both primes above  $\ell$  are totally ramified in  $K/k_0$ , then  $\mu \leq e_1/(\ell-1)$ .*

REMARK: Let  $K$  be the  $\mathbf{Z}_\ell$ -extension of  $k_0$  contained in the field obtained by adjoining to  $k_0$  all  $\ell$ -power roots of unity. Assume that  $k_0$  is an imaginary quadratic extension of  $\mathbf{Q}$ . In [1] Ferrero proves that  $\mu \leq e_0$  for this special type of  $\mathbf{Z}_\ell$ -extension. Also if  $k_0$  is any abelian extension of  $\mathbf{Q}$  and  $\ell = 2$  or  $3$ , Ferrero shows that  $\mu = 0$  for this special type of  $\mathbf{Z}_\ell$ -extension.

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