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SIMULTANEOUS RESOLUTION OF RATIONAL SINGULARITIES

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Abstract

Let Spec R be a rational surface singularity over $\mathbb{C}$. Generalizing work of Brieskorn, Artin, and others, we prove there is a smooth irreducible component A of the moduli space of Spec R, consisting of deformations which resolve simultaneously in a family after Galois base change. Further, the group is a direct product of Weyl groups associated to -2 configurations in the graph of R. We also prove that for a determinantal singularity, A consists of the determinantal deformations.

0. Introduction

Let R be a two-dimensional normal local ring over $\mathbb{C}$ with a rational singularity at the closed point, and $X \to \operatorname{Spec} R$ the minimal resolution. The simplest examples are those of embedding dimension $e = 3$, the rational double points (or RDP's). These are the Kleinian singularities $\mathbb{C}^2/G$, where $G \subset SL(2, \mathbb{C})$ is a finite subgroup; they are called A_n (G cyclic), D_n (binary dihedral), E_6 (binary tetrahedral), E_7 (binary octahedral), and E_8 (binary icosahedral). The exceptional fibre E in X is a configuration of non-singular rational curves, of self-intersection -2, whose (weighted) dual graph is the Dynkin diagram of the corresponding simple Lie algebra.

Brieskorn discovered ([5], [6], [7]) a relationship between the deformation theory of such an R and the Weyl group W of the Lie algebra. We say a deformation $\mathcal{V} \to T$ *resolves simultaneously* if there

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is a smooth map $\mathscr{X} \to T$, factoring through $\mathscr{V}$, such that for each $t \in T$, $\mathscr{X}_t \to \mathscr{V}_t$ is a (minimal) resolution of singularities. Atiyah observed in 1958 [4] that the versal (analytic) deformation of A_1 resolves simultaneously after a $\mathbb{Z}/2$-base change (i.e., $x^2 + y^2 + z^2 = t^2$ resolves simultaneously).

THEOREM (Brieskorn [7]): *The versal deformation of a rational double point resolves simultaneously after a Galois base change, with group W.*

This was proved independently by Tjurina [19] and (for A_n) Kas [13], using Brieskorn's earlier work.

There is a more precise picture of how the simple algebraic groups $G = \mathrm{SL}(n)$, $\mathrm{Sp}(n)$, etc., themselves come into play, and not just the Weyl groups [1], [7]. The idea (due to Grothendieck) is to study the subregular elements of G; in particular, one looks at the singular locus of $G \to T/W$ (T = maximal torus), sending an element of G to the conjugacy class of its semi-simple part. (If $G = \mathrm{SL}(n)$, this sends a matrix to its characteristic polynomial).

Artin and Schlessinger [2] generalized part of Brieskorn's result (and also a result of Huikeshoven [12]) to rational singularities of higher multiplicity, and made it more algebraic; however, one must work in a suitably localized algebraic category (e.g., algebraic spaces, or local henselian schemes).

THEOREM [2]: *There is a smooth space* Res *parametrizing deformations of* Spec *R with simultaneous resolution, and a finite map* $\Phi : \mathrm{Res} \to \mathrm{Def}$ *into the deformation space, whose image is an irreducible component A of* Def. (*A = Artin component*).

When $e = 3$ or 4, then Def is smooth, hence Φ is surjective. However, Pinkham [16] showed that for the cone over $\mathbf{P}^1 \to \mathbf{P}^4$ ($e = 5$), Def has one- and three-dimensional components; every deformation is a smoothing, but simultaneous resolution takes place on only the second component.

The main purpose of this paper is to prove

THEOREM 1: $\Phi : \mathrm{Res} \to \mathrm{Def}$ *is Galois onto A, with group* $W = \Pi W_i$, *the product of the Weyl groups associated to the maximal connected* -2 *configurations in the graph of R. In particular, A is smooth.*

This had been conjectured by Burns–Rapoport [8] and Wahl [21].

The first authors had noticed that each -2 curve gives an automorphism of Res $\to$ Def (an "elementary transformation"). We proved that the dimension of the kernel of the tangent space map of Res $\to$ Def is the number of -2 curves; in particular, if there are no -2's, then Res $\xrightarrow{\sim} A$ [20]. It is recent work of J. Lipman [15] which completes the proof.

The idea of the proof is rather simple. First, interpret Res as the deformation space of X ([2], 4.6). Next, blow down the -2 configurations in X to rational double points, obtaining $X \to V \to \operatorname{Spec} R$. This gives blowing-down maps

$$\text{Res} = \text{Def}(X) \to \text{Def}(V) \to \text{Def}(R) = \text{Def}.$$

Third, using Brieskorn's rational double point theorem and Burns–Wahl [9] on the relation of local to global deformations, one deduces $\text{Def}(X) \to \text{Def}(V)$ is Galois and surjective, with group W. Therefore, it remains only (!) to show Def V injects into Def(R) (i.e., Def V is the Artin component). In an earlier version of this paper (cf. [24]), we used a cohomological argument to prove Theorem 1 in case the fundamental cycle has multiplicity 1 at the -3 curves, e.g., for determinantal or quotient singularities. Lipman proves the injectivity directly; the point is that V is "canonically" obtained from Spec R, even after deformation of each.

There is one case where the result is more concrete.

THEOREM 2: *Let R be determinantal, of multiplicity d, hence defined by the 2×2 minors of a $2 \times d$ matrix. Then the Artin component is the versal determinantal deformation.*

That is, the deformations of R corresponding to perturbations of the entries of the defining matrix form an irreducible component of Def, equal to A. First, we observe that determinantal deformations yield deformations of V, owing to the simple construction of V in this case [23]. Then, we recognize the determinantal nature of R from a morphism $X \to \mathbb{P}^1 \times \mathbb{P}^{d-1}$; standard obstruction theory shows this map lifts to each deformation $\bar{X}$ of X, whence $\Gamma(\mathcal{O}_{\bar{X}})$ is also determinantal.

In a forthcoming paper [25], we study the finer structure of Res $\to$ A, especially the irreducible components of the discriminant locus, and the fact that the monodromy group over A is W.

In §1, we define the action of W on Res; our treatment there is influenced by a letter from E. Horikawa. We outline a proof of

Lipman's result in §2, while §3 discusses determinantal rational singularities.

§1. The action of W on Res

(1.1) We assume known the basic facts about rational singularities ([3], [14]). Moduli spaces are minimally versal deformation spaces. Spec R, and moduli spaces like Res and Def, are assumed local henselian or local analytic spaces, in order to avoid the non-separatedness of Res as an algebraic space.

(1.2) Interpreting Res as Def X, there is a blowing-down map Def $X \to$ Def V arising from $X \to V$ ([17], [9]). Denote by $p_1, \ldots, p_r$ the RDP's on V, and by $S_1, \ldots, S_r$ their moduli spaces. The composition Def $X \to$ Def $V \to \Pi S_i$ factors via ΠZ_i, where $Z_i \to S_i$ is the Res $\to$ Def map for the RDP p_i [9]. (Thinking of Z_i as the deformation space of some neighborhood U_i of the exceptional fibre of p_i in X, one has simply that deformations of X give deformations of each U_i).

THEOREM 1.3: *The diagram*

$$\begin{array}{ccc} \mathrm{Def}\, X & \to & \Pi Z_i \\ \downarrow & & \downarrow \\ \mathrm{Def}\, V & \to & \Pi S_i \end{array}$$

is cartesian, all spaces are smooth, the horizontal maps are smooth, and the vertical maps are Galois (and surjective), with group $W = \Pi W_i$.

PROOF: The cartesian property is [9], 2.6 (it is assumed there that V is projective, but this is not needed for the proof). All spaces are obviously smooth (all global H^2's and local T^2's vanish). The top map is smooth by [9], 2.14; the bottom, because it is surjective on the tangent spaces:

$$\mathrm{Ext}^1_V(\Omega^1_V, \mathcal{O}_V) \to H^0(T^1_V) \to H^2(\theta_V) = 0.$$

Finally, Brieskorn's RDP theorem gives the Galois property of the right-hand map; since the diagram is cartesian, these automorphisms (and the Galois property) pull back to Def $X \to$ Def V.

(1.4) Thus, $\mathrm{Res}/W = \mathrm{Def}\, X/W \xrightarrow{\sim} \mathrm{Def}\, V$. Now, $W = \Pi W_i$ is generated by reflections; we give a more geometric picture of the action of a reflection σ corresponding to E_1, a -2 curve on X. This is the "elementary operation" of [8], §7, or [11], Appendix B.

First, let $Z \to S$ be Res $\to$ Def for a single -2 curve (A_1-singularity), with $\mathscr{X} \to Z$ the total family. Here, $\dim Z = \dim S = 1$, and $\dim \mathscr{X} = 3$. Then the exceptional curve $E \subset X \subset \mathscr{X}$ has normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ [11]. Let $p : \mathscr{Y} \to \mathscr{X}$ be the blow-up of E; then $p^{-1}(E) \to E$ is $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 = E$, and the normal bundle of $p^{-1}(E)$ in $\mathscr{Y}$ is (- diagonal). By [10], $p^{-1}(E)$ may be blown down in the direction of the other ruling, yielding $\mathscr{Y} \to \mathscr{X}' \to Z$. By functoriality, $\mathscr{X}' \to Z$ is obtained from $\mathscr{X} \to Z$ by an automorphism $\sigma : Z \to Z$, of order 2 by construction.

For a general rational singularity, let V_1 be the space obtained from X by contracting E_1, and consider the (cartesian) diagram

$$\begin{array}{ccc} \text{Def } X & \to & Z \\ \downarrow & & \downarrow \\ \text{Def } V_1 & \to & S. \end{array}$$

Let $R_1 \subset \text{Def } X$ be the fibre over the origin of Z; R_1 is a smooth, codimension 1 subvariety corresponding to deformations of X to which E_1 lifts. If $\mathscr{X} \to \text{Def } X$ is the total space, and $\mathscr{X}_1 \to R_1$ is the induced deformation, one has a relative effective Cartier divisor $\mathscr{E} \subset \mathscr{X}_1$, which lifts E_1. Blow up $\mathscr{E} \subset \mathscr{X}$, then blow down in another direction as before, yielding again the reflection σ.

(1.5) Note that $\sigma(R_1) = R_1$. In fact, for an RDP, the R_i's correspond to hyperplanes left fixed by a basis of the positive roots when one views W as acting on the usual complex vector space as a reflection group. To see the other hyperplanes as subvarieties of Res, and for the generalization to all rational singularities, see [25].

(1.6) Curiously, the Weyl group of E_8 cannot appear unless R is the RDP E_8. That is, an E_8-configuration in the graph of a rational singularity is necessarily the entire graph. In light of Lemma 1.7 below, this is because the fundamental cycle of E_8 has multiplicity ≥ 2 at every component.

Lemma 1.7: *Inside a rational configuration, suppose L is a reduced, connected curve intersecting an irreducible E_1 in one point; let E_2 be the curve in L with $E_1 \cdot E_2 = 1$. Then the multiplicity of E_2 in Z_L, the fundamental cycle of L, is* 1.

Proof: We must have $(Z_L + E_1) \cdot (Z_L + E_1 + K) \leq -2$. Since

$Z_L \cdot (Z_L + K) = E_1 \cdot (E_1 + K) = -2$, we deduce $Z_L \cdot E_1 \leq 1$. This implies the result.

§2. Lipman's theorem

(2.1) As mentioned in the introduction, the main theorem will follow from the injectivity of the blowing-down map Def $V \to$ Def R. The usual functorial argument shows that it suffices to prove injectivity on the tangent spaces, since $H^0(\theta_V) \xrightarrow{\sim} \theta_R$ ([22], 1.12). We will sketch (a slight variant of) Lipman's proof.

THEOREM 2.2 (Lipman [15]): Def V *injects into* Def R.

PROOF: We show first that $V = \text{Proj} \bigoplus H^0(X, \omega_X^{\otimes n})$ (as schemes over Spec R). Letting $f: X \to V$, we have $f_* \omega_X = \omega_V$ (dualizing differentials on V), so $H^0(X, \omega_X^{\otimes n}) = H^0(V, \omega_V^{\otimes n})$. We show ω_V is very ample for $V \to \text{Spec}\, R$. By [14], 12.1, it suffices to show that $(\omega_V \cdot F_1) > 0$, for each exceptional curve F_1 in V. Since $f^* \omega_V = \omega_X$ (V has only RDP's), this intersection number is $(\omega_X \cdot E_1)$, where E_1 is a non-2 curve in X, hence is positive. Using the surjectivity $\Gamma(\omega_X^{\otimes m}) \otimes \Gamma(\omega_X^{\otimes n}) \to \Gamma(\omega_X^{\otimes (m+n)})$ ([14], 7.3), the claim now follows.

Next, if $U = X - E = \text{Spec}\, R - \{m\}$, we have

$$H^0(X, \omega_X^{\otimes n}) = H^0(U, \omega_U^{\otimes n}), \quad n = 0, 1 \tag{2.2.1}$$

$$H^0(X, \omega_X^{\otimes n}) = \text{Im}(\phi_n : H^0(U, \omega_U)^{\otimes n} \to H^0(U, \omega_U^{\otimes n})), \quad n \geq 1. \tag{2.2.2}$$

(2.2.1) follows from the exact sequence of local cohomology, since $H_E^1(\mathcal{O}_X) = 0$ (Grauert–Riemenschneider – see [20], Theorem A), and $H_E^1(\omega_X) = 0$ (dual to $H^1(\mathcal{O}_X) = 0$). For (2.2.2), we use

$$\begin{array}{ccc} H^0(X, \omega_X)^{\otimes n} & \longrightarrow & H^0(X, \omega_X^{\otimes n}) \\ \downarrow & & \downarrow \\ H^0(U, \omega_U)^{\otimes n} & \xrightarrow{\phi_n} & H^0(U, \omega_U^{\otimes n}). \end{array} \tag{2.2.3}$$

The top row is surjective as above, and the right map is injective, whence (2.2.2). Putting everything together gives that V is computable canonically from U, viz.

$$V = \text{Proj}(\bigoplus \text{Im}\, \phi_n) \tag{2.2.4}$$

Let $\bar{V}$ be a deformation of V over $D = \text{Spec}\, \mathbb{C}[\epsilon]/\epsilon^2$, and $\omega_{\bar{V}}$ the relative canonical sheaf of $\bar{V}$ over D (this is the unique lifting of the line bundle ω_V to $\bar{V}$). We claim $\bar{V} \xrightarrow{\sim} \mathrm{Proj} \bigoplus H^0(\bar{V}, \omega_{\bar{V}}^{\otimes n})$; since a map between deformations is automatically an isomorphism, it suffices to show $H^0(\bar{V}, \omega_{\bar{V}}^{\otimes n})$ is D-flat, all $n \geq 0$. But $R^1 f_*(\omega_X^{\otimes n}) = 0$ and $H^1(X, \omega_X^{\otimes n}) = 0$ ([14], 7.3), so $H^1(V, \omega_V^{\otimes n}) = 0$. Therefore, $H^0(\bar{V}, \omega_{\bar{V}}^{\otimes n})$ is D-flat, by [22], 0.4.4.

If $\bar{V}$ blows down to a trivial deformation, then, as in [22], §1, the induced deformation $\bar{U}$ of U is trivial. Let $\omega_{\bar{U}} = \Omega^2_{\bar{U}/D}$. The barred analogues of (2.2.1)–(2.2.3) are still true (use $H^0(\bar{V}, \omega_{\bar{V}}^{\otimes n})$ instead of $H^0(X, \omega_X^{\otimes n})$), again using [22], Theorem 0.4. Therefore, $\bar{V} \xrightarrow{\sim} \mathrm{Proj}(\bigoplus \mathrm{Im}\, \bar{\phi}_n)$. But since $\bar{U}$ is a trivial deformation, $\bar{\phi}_n$ is a product, hence $\bar{V}$ is trivial. This completes the proof.

(2.3) One can identify directly the kernel of the tangent space map of $\mathrm{Def}\, V \to \mathrm{Def}\, R$ as $\mathrm{Ext}^1_V(\Omega^1_{V/R}, \mathcal{O}_V)$. If $f: X \to V$, this can be recomputed as $\mathrm{Ext}^1_X(f^*\Omega^1_{V/R}, \mathcal{O}_X)$, and a (non-obvious) reduction equates injectivity with $\mathrm{Hom}_X(f^*\Omega^1_V, \mathcal{O}_Z(Z)) = 0$, where Z is the fundamental cycle. This should be viewed as a vanishing theorem analogous to those of [20]. After computing $f^*\Omega^1_V$ near the fibre of each RDP, we identified "easy cases" (cf. [21]) in which the theorem was true – determinantal singularities, and those with no -3 curves [24]. A more careful analysis of bad cycles gives the result if Z has multiplicity 1 at the -3 curves (e.g., for quotient singularities). Fortunately, Lipman's theorem proves injectivity in complete generality, and without our long and complicated method.

§3. Determinantal deformations

(3.1) A determinantal rational singularity R has equations given by the 2×2 minors of a $2 \times d$ matrix, d = multiplicity of R (see [23], §3 for a full discussion). There is a smooth subvariety (or subfunctor) Det of Def consisting of determinantal deformations; merely perturb arbitrarily the entries of the given matrix defining R. We will show $\mathrm{Det} = A$, obtaining another proof of Theorem 2.2 in this case. Thus, Det is independent of the matrix used. Assume $d \geq 3$.

THEOREM 3.2: *For a determinantal rational singularity,* $\mathrm{Det} = A$; *i.e., the determinantal deformations are exactly those which, after base change, simultaneously resolve in a family.*

PROOF: Let $X \to V \to \mathrm{Spec}\, R$ be as usual. We show first that the inclusion $\mathrm{Det} \subset \mathrm{Def}$ factors via $\mathrm{Def}\, V$, hence via A. Then we prove that Def $X \to A$, which is surjective (as a map of spaces, not functors), factors via Det.

Recall V has a simple construction [23]. Assume R is defined (formally, say) by

$$rk\begin{pmatrix} f_1 \cdots f_d \\ g_1 \cdots g_d \end{pmatrix} \leq 1, \tag{3.2.1}$$

where f_i, g_i are in a power series ring of $d+1$ variables. Then V is the closure of the graph of the rational map Spec $R \to \mathbb{P}^1$ defined by the columns. In fact, $V \subset \operatorname{Spec} R \times \mathbb{P}^1$ is defined by $sf_i = tg_i$, where s, t are homogeneous coordinates on $\mathbb{P}^1$. If now

$$rk\begin{pmatrix} F_1 \cdots F_d \\ G_1 \cdots G_d \end{pmatrix} \leq 1$$

defines a determinantal deformation Spec $\bar{R}$, we may use $sF_i = tG_i$ to define a deformation $\bar{V}$ of V. The verification that $\bar{V}$ is flat is done by using, e.g., that at least $d-1$ of the f_i's have linearly independent leading forms ([23], 3.4). We omit the details. This shows Det $\subset A$.

On X, denote by E_0 the $-d$ curve, and by E_i $(i>0)$ the other -2 curves; recall E_0 has multiplicity 1 in the fundamental cycle Z. Pulling back $\mathcal{O}(1)$ from $X \to V \to \mathbb{P}^1$ gives an invertible sheaf $\mathcal{L}$ on X, with $\mathcal{L} \cdot E_0 = 1$, $\mathcal{L} \cdot E_i = 0$ $(i>0)$, and $\omega_X = \mathcal{L}^{\otimes(d-2)}$. Note that $h^1(\mathcal{L}) = 0$, $h^0(\mathcal{L} \otimes \mathcal{O}_Z) = 2$ (use Riemann–Roch).

Suppose that $Z \cdot E_0 < 0$; this means that the entries of the matrix (3.2.1) generate the maximal ideal of R, or the strict tangent cone is 0. Therefore, the rational map Spec $R \to \mathbb{P}^{d-1}$ (defined by the rows) is well-defined after one blow-up; in particular, there is a map $X \to \mathbb{P}^{d-1}$. Denote by $\mathcal{M}$ by pull-back of $\mathcal{O}(1)$. Since $\mathcal{M}$ is generated by its global sections, $h^1(\mathcal{M}) = 0$; also, $h^0(\mathcal{M} \otimes \mathcal{O}_Z) = d$. Now, the projectivized tangent cone Proj$\bigoplus H^0(Z, \mathcal{O}_Z(-nZ))$ embeds in $\mathbb{P}^1 \times \mathbb{P}^{d-1}$ (since $Z \cdot E_0 < 0$), so $\mathcal{O}_Z(-Z) \cong \mathcal{L} \otimes \mathcal{M} \otimes \mathcal{O}_Z$, whence $\mathcal{O}_X(-Z) \simeq \mathcal{L} \otimes \mathcal{M}$ (recall that numerically equivalent line bundles are isomorphic). Thus, the map $\pi : X \to \mathbb{P}^1 \times \mathbb{P}^{d-1}$ has $\pi^*(\mathcal{O}(1) \otimes \mathcal{O}(1)) = \mathcal{O}(-Z)$. The maps

$$\Gamma(\mathbb{P}^1 \times \mathbb{P}^{d-1}, \mathcal{O}(n) \otimes \mathcal{O}(n)) \to \Gamma(X, \mathcal{O}_X(-nZ)) \subset \Gamma(X, \mathcal{O}_X)$$

give

$$C = \bigoplus_{n=0}^{\infty} \Gamma(\mathbb{P}^1 \times \mathbb{P}^{d-1}, \mathcal{O}(n) \otimes \mathcal{O}(n)) \to \Gamma(X, \mathcal{O}_X). \tag{3.2.2}$$

We claim (3.2.2) is surjective on completions, by showing the map of gr's is surjective. The map of n^{th} graded pieces is

$$\Gamma(\mathbf{P}^1 \times \mathbf{P}^{d-1}, \mathcal{O}(n) \otimes \mathcal{O}(n)) \to \Gamma(Z, \mathcal{O}_Z(-nZ)).$$

But consider

$$\begin{array}{ccc}
(\overset{n}{\otimes} \Gamma(\mathbf{P}^1, \mathcal{O}(1))) \otimes (\overset{n}{\otimes} \Gamma(\mathbf{P}^{d-1}, \mathcal{O}(1))) & \to & \overset{n}{\otimes} \Gamma(\mathcal{O}_Z(-Z)) \\
\wr\!\wr & & \Big| \\
(\overset{n}{\otimes} \Gamma(\mathcal{L} \otimes \mathcal{O}_Z)) \otimes (\overset{n}{\otimes} \Gamma(\mathcal{M} \otimes \mathcal{O}_Z)) & & \Big| \\
\downarrow & & \downarrow \\
\Gamma(\mathbf{P}^1, \mathcal{O}(n)) \otimes \Gamma(\mathbf{P}^{d-1}, \mathcal{O}(n)) & \longrightarrow & \Gamma(\mathcal{O}_Z(-nZ)).
\end{array}$$

The top and right maps are surjective, by [14], 7.3, whence so is the bottom. This proves the claim. Note C is the generic $2 \times d$ determinantal singularity, and (3.2.2) shows how to write the matrix (3.2.1) from $\pi: X \to \mathbf{P}^1 \times \mathbf{P}^{d-1}$.

But $\mathcal{L}$ and $\mathcal{M}$ lift uniquely to any deformation of X (since $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$), and the sections of $H^0(\mathcal{L})$ and $H^0(\mathcal{M})$ lift as well (since their H^1's are 0). Thus, the map $X \to \mathbf{P}^1 \times \mathbf{P}^{d-1}$ lifts, and (3.2.2) lifts after deformation (of course, C is rigid). This shows how to perturb the entries of the matrix (3.2.1) after deformation of X. (We use implicitly that a map between deformations is an isomorphism). Thus, Def X maps into Det.

Next, suppose $Z \cdot E_0 = 0$; it is no longer true that the projectivized tangent cone embeds in $\mathbf{P}^1 \times \mathbf{P}^{d-1}$. Define inductively cycles L_j, B_j, where $L_0 = E$, $B_0 = Z$, and

(i) L_{j+1} = connected component of $\{E_i \subset L_j \mid B_j \cdot E_i = 0\}$ containing E_0

(ii) B_{j+1} = fundamental cycle of L_{j+1}.

Eventually, $B_k \cdot E_0 < 0$, some k. Let $Z_1 = B_0 + \cdots + B_k$. Then $Z_1 \cdot E_0 < 0$, $B_i \cdot B_j = 0$, $i \neq j$, so $h^0(\mathcal{O}_{Z_1}) = k + 1$, and $h^1(\mathcal{O}(-Z_1)) = 0$. (Use Lemma 1.7 to show $Z_1 \cdot E_i \leq 0$, all i; e.g., end curves of L_{j+1} have multiplicity 1 in B_{j+1}). By construction, $k + 1$ = multiplicity of E_0 in Z_1 = number of blow-ups of Spec R needed to drop the multiplicity (cf. [18]). Also, $H^0(\mathcal{O}(-Z_1)) = I$ is the complete ideal, of colength $k + 1$, generated by the entries of the matrix (3.2.1).

The rational map Spec $R \to \mathbf{P}^{d-1}$ is well-defined after $k + 1$ blow-ups of Spec R (following the point of multiplicity d), hence there is a map $X \to \mathbf{P}^{d-1}$. Let $\mathcal{M}$ be the pull-back of $\mathcal{O}(1)$. Then $\mathcal{L} \otimes \mathcal{M}$ is the pull-back of $\mathcal{O}(1) \otimes \mathcal{O}(1)$ from $X \to \mathbf{P}^1 \times \mathbf{P}^{d-1}$; but by construction, this is $I\mathcal{O}_X$, where I is the ideal generated by the entries of the matrix. Thus, $\mathscr{L}\otimes\mathscr{M}\simeq\mathcal{O}(-Z_1)$. The map $Z\to\mathbb{P}^1\times\mathbb{P}^{d-1}$ again expresses the determinantal nature of the projectivized tangent cone. There is a map

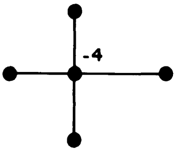
$$\bigoplus\Gamma(\mathbb{P}^1\times\mathbb{P}^{d-1},\mathcal{O}(n)\otimes\mathcal{O}(n))\to\Gamma(X,\mathcal{O}_X) \tag{3.2.4}$$

so that the map on the n^{th} piece of the associated graded is

$$\Gamma(\mathscr{L}^n\otimes\mathcal{O}_Z)\otimes\Gamma(\mathscr{M}^n\otimes\mathcal{O}_Z)\to\Gamma(\mathcal{O}_Z(-nZ_1))\simeq I^n/mI^n$$

(compare to the preceding). In fact, the completion of (3.2.4) maps onto $\mathbb{C}+I$, a subring of finite colength in R. Nonetheless, we proceed as before. Deformations of X carry the map into $\mathbb{P}^1\times\mathbb{P}^{d-1}$ ($H^1(\mathscr{L})=H^1(\mathscr{M})=0$), hence (3.2.4) deforms, and one again knows how to perturb the entries of the matrix defining R. This completes the proof of Theorem 3.2.

EXAMPLE (3.3) (See [23], 5.5): A particular rational singularity of multiplicity 4 with graph



may be written determinantally via the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_1+x_5^2 \end{pmatrix}$$

One computes that $\dim T_R^1=10$. The versal determinantal deformation, of dimension 8, is given by

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2+t_1+t_6x_5 & x_3+t_2+t_7x_5 & x_4+t_3+t_8x_5 & x_1+x_5^2+t_4+t_{10}x_3 \end{pmatrix}. \tag{3.3.1}$$

(Note t_{10} is the "equisingular" parameter). Another four-dimensional family (not obviously an irreducible component) can be read off the 2×2 minors of the symmetric 3×3 matrix:

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3+t_5+t_9x_5 & x_4 \\ x_3 & x_4 & x_1+x_5^2+t_4+t_{10}x_3 \end{pmatrix}. \tag{3.3.2}$$

Now, the one-parameter subfamily of (3.3.2) given by $t_4 = s^2$, $t_5 = s^3$, $t_9 = s$, $t_{10} = 0$, has a family of -4 singularities along the section $x_1 = x_2 = x_3 = x_4 = 0$, $x_5 = s$; also, it intersects (3.3.1) only at $s = 0$. But a check shows that (3.3.2) acts as the non-Artin component along the s-curve. By local versality of Def, and Pinkham's description [16] of the moduli space for -4, it follows that there is another component, of dimension ≥ 6, acting as the Artin component along the s-curve. In fact, a computation shows Def has 4, 6, and 8-dimensional components; the 6-dimensional one is singular, with smooth normalization.

REMARK (3.4): For a general determinantal singularity, Det is not an irreducible component of the moduli space; in fact, it will depend on which matrix representation is used. For instance, the moduli space of 4 lines through the origin in $\mathbb{C}^4$ is a cone over $\mathbb{P}^1 \times \mathbb{P}^3$ (hence irreducible, but not smooth). Note, incidentally, that this singularity is the affine form of the projectivized tangent cone in (3.3) above (i.e., set $x_5 = 0$ in the equation).

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