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DOUBLE POINT RESOLUTIONS OF DEFORMATIONS OF RATIONAL SINGULARITIES

Joseph Lipman¹

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Let Y_0 be a normal algebraic surface over an algebraically closed field k , and let $f_0: Z_0 \rightarrow Y_0$ be the minimal resolution of singularities. There is a unique way to factor f_0 as $Z_0 \xrightarrow{g_0} X_0 \rightarrow Y_0$ with a proper birational g_0 and a normal surface X_0 , such that a reduced irreducible curve C on Z_0 which blows down to a point on Y_0 already blows down on X_0 iff C is non-singular and rational with self-intersection $C^2 = -2$ (cf. [1, p. 493, (2.7)]; or, for greater generality, [7, p. 275, (27.1)]). The singularities of X_0 are all **Rational Double Points**. X_0 , which is uniquely determined by Y_0 , will be called the **RDP-resolution** of Y_0 .

Suppose now that Y_0 is affine and has just one singularity y , y being *rational*. There is a conjecture of Wahl [9], and Burns-Rapoport [2, 7.4] to the effect that the Artin component in the (formal or henselian) versal deformation space of y is obtained from the deformation space of Z_0 by factoring out a certain Coxeter group. According to Wahl [10], this would imply that the Artin component is smooth, and is in fact formally identical with the deformation space of X_0 (at least after things are suitably localised). Wahl also shows how the conjecture reduces to the statement that under “blowing down”, the set of first order infinitesimal deformations of X_0 maps injectively into that of Y_0 . Our purpose here is to outline a proof that *this injectivity* (and so the conjecture) *does indeed hold*.

Wahl has previously established some special cases of the conjecture by showing that a certain cohomology group vanishes [10]. I am grateful to him for all the information and motivation he has provided.

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Let $g: Y \rightarrow S$ be a flat map of finite type of noetherian schemes, such that for each $s \in S$ the fibre Y_s is a normal surface having only rational singularities. We assume – for simplicity only – that every closed point of Y maps to a closed point of S whose residue field is algebraically closed. By an RDP-resolution of $Y \rightarrow S$ is meant a proper map $f: X \rightarrow Y$ such that $g \circ f: X \rightarrow S$ is flat, and for each s the map $f_s: X_s \rightarrow Y_s$ is the RDP-resolution of Y_s (see above).

THEOREM: *Given $Y \rightarrow S$ as above, there exists at most one (up to Y -isomorphism) RDP-resolution.*

(For $S = \text{Spec}(k[t]/t^2)$ we get the above-indicated injectivity statement.)

In fact we show the following:

Let $U \subseteq Y$ be the open set where g is smooth and let $i: U \rightarrow Y$ be the inclusion map. Set

$$\omega = i_* \Omega_{U/S}^2 \quad (\Omega^2 = \text{relative 2-differentials})$$

and for all $n \geq 0$, let ω^n be the image of the natural map $\omega^{\otimes n} \rightarrow i_((\Omega_{U/S}^2)^{\otimes n})$. Then, if an RDP-resolution $f: X \rightarrow Y$ exists, we must have*

$$X = \text{Proj}(\bigoplus_{n \geq 0} \omega^n)$$

(with f the canonical map). In other words X is the scheme-theoretic closure of U in the projective bundle $\mathbf{P}(\omega)$ [4, II, (4.1.1)].

EXAMPLE (Suggested by Riemenschneider). Let Y_0 be the cone over a non-singular rational curve of degree 4 in \mathbf{P}^4 (say over the complex numbers C). The versal deformation of the vertex has a one-dimensional *non-Artin* component, found by Pinkham: the corresponding deformation is $Y \rightarrow S = \text{Spec}(C[t])$, where $Y \subseteq C^5 \times S$ is the zero-set of the 2×2 minors of

$$\begin{pmatrix} x_1 & x_2 & x_3 + t \\ x_2 & x_3 & x_4 \\ x_3 + t & x_4 & x_5 \end{pmatrix}$$

Here, of course, no RDP-resolution can exist. One computes that ω^n is isomorphic to J^n , where J is the fractionary ideal generated by

$(1/x_2, 1/(x_3+t), 1/x_4)$; and $\text{Proj}(\bigoplus \omega^n)$ (\cong blow-up of J) is non-singular, but *not* an RDP-resolution of Y . (The fibre over $t = 0$ has two components!)

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After outlining the proof of some preliminary facts from duality theory (Lemma 1), we prove the Theorem in Lemmas 2 and 3 below.

Let $f: X \rightarrow Y$ be an RDP-resolution of $g: Y \rightarrow S$.

LEMMA 1: (cf. [5, p. 298] or 5', Prop. 9) (i) *Since $g: Y \rightarrow S$ is flat, with Cohen–Macaulay fibres, the relative dualizing complex $g^! \mathcal{O}_S$ has just one non-zero cohomology sheaf $\omega_{Y|S}$, and $\omega_{Y|S}$ is flat over S . Since $g \circ f$ is flat, with Gorenstein fibres, the similarly defined \mathcal{O}_X -module $\omega_{X|S}$ is invertible.*

(ii) *For any map $S' \rightarrow S$, with S' noetherian, if $Y' = Y \times_S S'$ and $\pi: Y' \rightarrow Y$ is the projection, then $\omega_{Y'|S'} = \pi^*(\omega_{Y|S})$.*

PROOF: We may assume $S = \text{Spec}(A)$, $Y = \text{Spec}(B)$, with B a homomorphic image of a polynomial ring $R = A[\xi_1, \dots, \xi_n]$ (cf. [5, p. 383, p. 388]). Then the first part of (i) states that

(i*): $\text{Ext}_R^i(B, R) = 0$ for $i \neq n - 2$, and $\text{Ext}_R^{n-2}(B, R)$ is A -flat;

and (ii) says that

(ii*): for any noetherian A -algebra A' , if $R' = R \otimes_A A'$ and $B' = B \otimes_A A'$, then the natural map

$$\text{Ext}_R^{n-2}(B, R) \otimes_A A' \rightarrow \text{Ext}_{R'}^{n-2}(B', R')$$

is an isomorphism.

(i*) and (ii*), together with Nakayama's lemma, reduce the proof of the last assertion in (i) to the well-known case where S is the spectrum of a field [5, p. 296, Prop. 9.3].

We first show that B has homological dimension $n - 2$ over R . For this we may replace R by its localization at an arbitrary maximal ideal \mathfrak{M} , and A by its localization at $\mathfrak{M} \cap A$ [8, p. 188, Thm. 11]. Let \mathfrak{m} be the maximal ideal of A , let $\bar{R} = R \otimes_A (A/\mathfrak{m})$ and $\bar{B} = B \otimes_A (A/\mathfrak{m})$, so that \bar{B} is a normal two-dimensional homomorphic image of the regular n -dimensional local ring \bar{R} , and so \bar{B} has homological dimen-

sion $n - 2$ over \bar{R} . (What matters here and subsequently is that \bar{B} is Cohen–Macaulay.) Let K be the residue field of R . Since R and B are A -flat, we have for any R -projective (hence A -flat) resolution P of B that the homology $H_j(P \otimes_A (A/\mathfrak{m})) = \text{Tor}_j^A(B, A/\mathfrak{m}) = 0$ for $j > 0$, i.e. $P \otimes_A (A/\mathfrak{m})$ is an \bar{R} -projective resolution of \bar{B} , whence, for all i ,

$$(\#) \quad \text{Tor}_i^R(B, K) = \text{Tor}_i^{\bar{R}}(\bar{B}, K);$$

and our assertion follows from [8, p. 193, Thm. 14].

Thus B has a finitely generated R -projective (hence A -flat) resolution

$$P: 0 \rightarrow P_{n-2} \rightarrow P_{n-3} \rightarrow \cdots \rightarrow P_0 \rightarrow 0.$$

Let Q be the complex with

$$Q_i = \text{Hom}_R(P_{n-2-i}, R) \quad (i \in \mathbb{Z}).$$

For any A -algebra A' and any i we have

$$(\#\#) \quad H_{n-2-i}(Q \otimes_A A') = \text{Ext}_R^i(B, R') = \text{Ext}_R^i(B', R').$$

(For the second equality cf. the proof of $(\#)$ above.)

Now (i^*) results from the following Lemma. (We assume again, as we may, that A and R are local, and let \mathfrak{m} be the maximal ideal of A .)

LEMMA 1a: *Let Q be an A -flat complex of finitely-generated R -modules, with $Q_i = 0$ for $i < 0$, and such that the homology $H_j(Q \otimes_A (A/\mathfrak{m})) = 0$ for all $j > 0$. Then $H_0(Q)$ is A -flat, and $H_j(Q) = 0$ for all $j > 0$.*

The proof is left to the reader.

Finally, applying [4, III', (7.3.1)(c) and (7.3.7)] to the homological functor T of A -modules M given by

$$T_p(M) = H_p(Q \otimes_A M) = \text{Ext}_R^{n-2-p}(B, R \otimes_A M) \quad (p \in \mathbb{Z})$$

(Q as above) we see that for every A -module M and every i , there is a natural isomorphism

$$\text{Ext}_R^i(B, R) \otimes_A M \xrightarrow{\sim} \text{Ext}_R^i(B, R \otimes_A M).$$

In view of $(\#\#)$, taking $M = A'$ we get (ii^*) . Q.E.D.

LEMMA 2: Let \mathcal{L} be the invertible \mathcal{O}_X -module $\omega_{X/S}$. Then:

- (i) \mathcal{L} is very ample for f .
- (ii) For every $n > 0$, the canonical map $(f_*\mathcal{L})^{\otimes n} \rightarrow f_*(\mathcal{L}^{\otimes n})$ is surjective.

PROOF: We first show that $R^1f_*(\mathcal{O}_X) = 0$: for any closed point y of Y , let \mathfrak{m}_y be the maximal ideal of $\mathcal{O}_{Y,y}$; then for all $r \geq 0$, $\mathfrak{m}_y^r \mathcal{O}_X / \mathfrak{m}_y^{r+1} \mathcal{O}_X$ is an $\mathcal{O}_{f^{-1}(y)}$ -module generated by its global sections; since Y_s ($s = g(y)$) has rational singularities, therefore $H^1(\mathcal{O}_{f^{-1}(y)}) = 0$, and since $f^{-1}(y)$ has dimension ≤ 1 , we get $H^1(\mathfrak{m}_y^r \mathcal{O}_X / \mathfrak{m}_y^{r+1} \mathcal{O}_X) = 0$; conclude with [4, III, (4.2.1)].

Now by [7, p. 220, (12.1) and p. 211, proof of (7.4)] it will be enough to show for any reduced irreducible curve $C \subseteq f^{-1}(y)$ that $(\mathcal{L}.C)$ (the degree of \mathcal{L} pulled back to C) is > 0 . Lemma 1 (ii) reduces us to the case $S = \text{Spec}(k)$, k an algebraically closed field. Let $p: Z \rightarrow X$ be a minimal resolution of singularities, and let \bar{C} be the component of $p^{-1}(C)$ which maps onto C . Then $(\mathcal{L}.C) = (p^{-1}\mathcal{L}.\bar{C})$. But $p^{-1}\mathcal{L}$ is a dualizing sheaf on Z [1, p. 493, (2.7)]; since (by definition of RDP-resolution) \bar{C} is not a nonsingular rational curve with $C^2 \geq -2$, therefore $(p^{-1}\mathcal{L}.\bar{C}) > 0$. Q.E.D.

Lemma 2(i) implies that $X = \text{Proj}(\bigoplus_{n \geq 0} f_*(\mathcal{L}^{\otimes n}))$, cf. [4, II, (4.6.2), (4.6.3), (5.4.4)] or [4, III, (2.3.4.1)]; so we need only show that $f_*(\mathcal{L}^{\otimes n}) = \omega^n$ ($n > 0$). This is given by (ii) of Lemma 2 and by:

LEMMA 3: Let $\mathcal{L} = \omega_{X/S}$. Then for all $n > 0$ there is a natural injective map

$$\theta_n : f_*(\mathcal{L}^{\otimes n}) \hookrightarrow i_*((\Omega_{U/S}^2)^{\otimes n});$$

and θ_1 is even bijective (i.e. $f_*\mathcal{L} = \omega$).

PROOF: Let $U \subseteq Y$ be as before, and let $j: f^{-1}(U) \rightarrow X$ be the inclusion map. f induces a proper map $f^{-1}(U) \rightarrow U$ which is fibrewise (over S) an isomorphism; hence $f^{-1}(U) \rightarrow U$ is an isomorphism. Now

$$\Omega_{U/S}^2 = \omega_{U/S} = j^*\mathcal{L},$$

and θ_n is obtained by applying f_* to the natural map $\mathcal{L}^{\otimes n} \rightarrow j_*j^*\mathcal{L}^{\otimes n}$. For the injectivity, it suffices that the $(X - f^{-1}(U))$ depth of \mathcal{L} be ≥ 1 [3, 1.9 and 3.8]. [4, IV, (11.3.8)] and (ii) of Lemma 1 above enable us to verify this fibrewise (over S); in other words, we need only check

the simple case where $S = \text{Spec}(k)$, k a field.

Similarly we see that the $(Y - U)$ -depth of $\omega_{Y/S}$ is ≥ 2 , and conclude that

$$\omega_{Y/S} = i_* i^* \omega_{Y/S} = \omega.$$

For the surjectivity of θ_1 , it suffices (by Nakayama's Lemma) that $f_* \mathcal{L} \rightarrow \omega_s = \omega_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_s}$ be surjective for all $s \in S$. Since $f^* f_* \mathcal{L} \rightarrow \mathcal{L}$ is surjective (Lemma 2), so is $f^* f_* \mathcal{K} \rightarrow \mathcal{K}$, where \mathcal{K} is the kernel of $\mathcal{L} \rightarrow \mathcal{L}_s = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_s}$; since $R^1 f_* \mathcal{O}_X = 0$ and f has fibres of dimension ≤ 1 , therefore $R^1 f_* \mathcal{K} = 0$, and $f_*(\mathcal{L}) \rightarrow f_*(\mathcal{L}_s)$ is surjective. But \mathcal{L}_s (resp. ω_s) is a dualizing sheaf on X_s (resp. Y_s), and since Y_s has rational singularities, therefore $f_*(\mathcal{L}_s) = \omega_s$ ([6, p. 606, 3.5] or, for greater generality, [7', §2]). Q.E.D.

REMARK: For $S = \text{Spec}(k[t]/t^2)$, Wahl has a proof of Lemma 3 which avoids duality theory [10, §2].

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