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A PROOF OF NOETHER’S FORMULA FOR THE ARITHMETIC GENUS OF AN ALGEBRAIC SURFACE

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1. The proof

Let $X$ be a smooth, proper surface defined over an algebraically closed field $k$. Denote by $\chi(O_X) = \sum_{i=0}^2 (-1)^i \dim_k H^i(X, O_X)$ its Euler-Poincaré characteristic, by $c_i = c_i(O_X)$ the $i$th Chern class of its cotangent bundle, and by $f$ the degree of a zero-dimensional cycle in the Chow ring $A_X$. The above invariants of $X$ are related by the formula

$$12\chi(O_X) = f(c_1^2 + c_2),$$

due to Max Noether [9]. The formula is a special case of Hirzebruch’s Riemann–Roch theorem (however, it is not a special case of the original Riemann–Roch theorem for a surface (see [13]), which states that a certain inequality holds).

Here we give a proof of (1) more in the spirit of Noether’s original (see §2). First we realize $X$ as the normalization of a surface $X_0$ in $\mathbb{P}^3$, with ordinary singularities. Then we obtain expressions for $f c_1$, $f c_2$, and $\chi(O_X)$ in terms of numerical characters of $X_0$ and we verify that these expressions satisfy the relation (1).

By realizing $X$ as the normalization of a surface $X_0$ with ordinary singularities in $\mathbb{P}^3$ we mean the following. Let $X \hookrightarrow \mathbb{P}^N$ be any embedding of $X$. Replacing it by the embedding determined by hypersurface sections of degree $\geq 2$, we may assume that the projection $f : X \to \mathbb{P}^3$ of $X$ from any generically situated linear space of codimension 4 has the following properties [10, p. 206, theorem 3]:

(A) Put $X_0 = f(X)$. The map $f : X \to X_0$ is finite and birational (hence it is equal to the normalization map).

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(B) $X_0$ has only ordinary singularities: a double curve $\Gamma_0$, which has $t$ triple points (these being also triple for the surface) and no other singularities; a finite number of pinch points, these being the images of the points of ramification of $f$. The completion of the local ring of $X_0$ at a point $y$ of $\Gamma_0$ looks like

(a) $k[[t_1, t_2, t_3]]/(t_1t_2)$ for most points $y$ of $\Gamma_0$ and at such points $#f^{-1}(y) = 2$.

(b) $k[[t_1, t_2, t_3]]/(t_1t_2t_3)$ if $y$ is triple, and then $#f^{-1}(y) = 3$.

(c) $k[[t_1, t_2, t_3]]/(t_1^2 - t_2^2t_3)$ if $y$ is a pinch point and char $k \neq 2$ (otherwise the ring is $k[[t_1, t_1t_2, t_2^2 + t_3]]$) and $#f^{-1}(t) = 1$.

In order to compute the invariants of $X$ in terms of the numerical characters of $X_0$, we shall first make some observations concerning the scheme structure of the double curve $\Gamma_0$.

We let $\mathcal{C}_0 = \text{Hom}_{\mathcal{O}_{X_0}}(f^*\mathcal{O}_X, \mathcal{O}_{X_0})$ denote the conductor of $X$ in $X_0$ and put $\mathcal{C} = \mathcal{C}_0\mathcal{O}_X$. It follows that $f^*\mathcal{C} = \mathcal{C}_0$ holds. Moreover, using duality for the finite morphism $f$ [see 13, III, appendix by D. Mumford, p. 71; also 7, V. 7], we obtain a canonical isomorphism

$$\mathcal{C} \cong \Lambda^2 \Omega_X \otimes \mathcal{L}^{-n+4},$$

where $\mathcal{L} = f^*\mathcal{O}_{P^3}(1)$ is the pullback of the tautological line bundle on $P^3$ and $n$ is the degree of $X_0$ in $P^3$. In particular this shows that $\mathcal{C}$ is invertible.

Using (B) we see that the ideal $\mathcal{C}_0$ defines the reduced scheme structure on the double curve, call this scheme $\Gamma_0$ also. Now put $\Gamma = f^{-1}(\Gamma_0)$; thus $\Gamma$ is defined on $X$ by the ideal $\mathcal{C}$. This gives an equality in the Chow ring:

$$c_1 = c_1(\Omega_X) = (n - 4)c_1(\mathcal{L}) - [\Gamma].$$

The equality (2) allows us to compute $f c_1^2$. First, let us introduce the following numerical characters of $X_0$, in addition to its degree $n$, degree of $\Gamma_0 = m$,

$\#$ triple points of $\Gamma_0$ (or of $X_0$) = $t$,

grade (self-intersection) of $\Gamma$ on $X = \lambda$,

$\#$ (weighted) pinch points = $\nu_2$.

By definition $\nu_2$ is the degree of the ramification cycle of $f$ on $X$; this cycle is defined by the 0th Fitting ideal $F^0(\Omega_{X_0})$ of the relative differentials of $f$. (If char $k \neq 2$, $\nu_2$ is equal to the actual number of pinch points of $X_0$; if char $k = 2$, $\nu_2$ is twice the number of actual
pinch points [11, p. 163, prop. 6]. From (2) then we get the expression
\[ \int c_1^2 = (n - 4)^2 n - 4(n - 4)m + \lambda. \]

Here we used \( f \cdot C_1(F)^2 = n \) and \( f \cdot C_1(\mathcal{L})[\Gamma] = 2m \), which holds because the map \( f|_\Gamma : \Gamma \to \Gamma \) has degree 2.

For a surface with ordinary singularities in \( \mathbb{P}^3 \) there is the triple point formula:
\[ 3t = \lambda - mn + \nu_2, \]
due to Kleiman [7, I, 39]. Substituting the resulting value of \( \lambda \) in the above formula for \( \int c_1^2 \), we find
\[ (3) \quad \int c_1^2 = n(n - 4)^2 - (3n - 16)m + 3t - \nu_2. \]

Next we want to obtain an expression for \( \int c_2 \). Since there is an exact sequence
\[ f^* \Omega_{\mathbb{P}^3} \to \Omega_X \to \Omega_{\mathbb{P}^3} \to 0, \]
Porteous’ formula [6, p. 162, corollary 11] gives
\[ \nu_2 = \int (c_1^2 - c_2 + 4c_1 \cdot c_1(\mathcal{L}) + 6c_1(\mathcal{L})^2. \]

Using (2) and (3) we obtain
\[ (4) \quad \int c_2 = n(n^2 - 4n + 6) - (3n - 8)m + 3t - 2\nu_2. \]

The last invariant to be considered is \( \chi(\mathcal{O}_X) \). We claim that the arithmetic genus \( \chi(\mathcal{O}_X) - 1 \) satisfies the postulation formula (see §2),
\[ (5) \quad \chi(\mathcal{O}_X) - 1 = \binom{n - 1}{3} - (n - 4)m + 2t + g - 1, \]
where \( g \) denotes the (geometric) genus of \( \Gamma_0 \).

To prove (5) we consider the exact sequences \( 0 \to \mathcal{E} \to \mathcal{O}_X \to \mathcal{O}_\Gamma \to 0 \) and \( 0 \to \mathcal{E}_0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\Gamma_0} \to 0 \). Since \( f \) is finite, \( f^* \) is exact, and we have
seen that $f^*\mathcal{E} = \mathcal{O}_0$ holds. Therefore, by additivity of $\chi$, we obtain

$$
\chi(\mathcal{E}_0) = \chi(f^*\mathcal{O}_X) - \chi(f^*\mathcal{O}_U) = \chi(\mathcal{O}_{X_0}) - \chi(\mathcal{O}_{U_0}),
$$

hence

$$
\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X_0}) + \chi(\mathcal{O}_U) - \chi(\mathcal{O}_{U_0}).
$$

Moreover, since $X_0$ is a hypersurface of degree $n$ in $\mathbb{P}^3$, $\chi(\mathcal{O}_{X_0}) = \binom{n-1}{3} + 1$ holds. Since $\Gamma$ is a curve on a smooth surface, its arithmetic genus is given by the adjunction formula

$$
-\chi(\mathcal{O}_U) = \frac{1}{2} \int (\langle [\Gamma] + c_1 \rangle) \cdot [\Gamma],
$$

hence, using (2), we get

$$
\chi(\mathcal{O}_U) = -(n - 4)m.
$$

Finally, the equality

$$
\chi(\mathcal{O}_{U_0}) = 1 - g - 2t
$$

holds because the difference in arithmetic and geometric genus due to a triple point with linearly independent tangents is equal to 2. This is seen as follows. Consider the local ring $R$ of $U_0$ at a triple point, and let $R \to R'$ denote its normalization. By (B) the map on the completions looks like

$$
\hat{R} = k[[t_1, t_2, t_3]]/(t_1t_2, t_1t_3, t_2t_3) \to \hat{R}' = k[[t]]^3.
$$

The image of $\hat{R}$ in $\hat{R}'$ consists of triples $(\psi_1, \psi_2, \psi_3)$ such that $\psi_i(0) = \psi_i(0)$, the cokernel of $\hat{R} \to \hat{R}'$ is isomorphic to $k^2$, and the map $\hat{R}' \to k^2$ is given by

$$(\psi_1, \psi_2, \psi_3) \mapsto (\psi_1(0) - \psi_2(0), \psi_1(0) - \psi_3(0))$$

(Similar computations show that a triple point with coplanar tangents would diminish the genus by 3.) Thus we have proved (5).

Consider the curve $\Gamma'$; above each triple point of $\Gamma_0$ it has 3 ordinary double points. Hence the difference between its arithmetic and geometric genus is $3t$ (since $\Gamma$ has no other singularities). We have
observed that the map $f_1, -: F \to F_0$ has degree 2; since its ramification locus is equal to that of $f$, the Riemann-Hurwitz formula now gives a formula

$$2m(n - 4) - 6t = 2(2g - 2) + v_2.$$ 

Hence we can substitute for $g$ in (5) and multiply by 12 to get

$$12\chi(\mathcal{O}_X) = 2n(n^2 - 6n + 11) - 6(n - 4)m + 6t - 3v_2.$$ 

This equality, together with (3) and (4), now yields (1).

2. Historical note

Formula (1) was stated by Noether [9] as

$$(1') \quad \pi^{(1)} = 12(p + 1) - (p^{(1)} - 1).$$

He established it by considering a model of the surface in $\mathbb{P}^3$. Previously [8] he had found formulae for the arithmetic genus $p$ and the genus $p^{(1)}$ of a canonical curve in terms of the numerical characters of the model in $\mathbb{P}^3$. Now he showed that the expression he got for the difference $12(p + 1) - (p^{(1)} - 1)$ was equal to the expression for the invariant $\pi^{(1)}$ given by Zeuthen [14].

Clebsch [5] was the first to look for a class number of the birational class to which a surface belongs. He defined the genus of a surface as the number $p_g$ of independent everywhere finite double integrals. He showed that for a model $f(x, y, z) = 0$ of the surface in $\mathbb{P}^3$, of degree $n$, with only double and cuspidal curves, these integrals are of the form $\int \int \phi/f; dx dy$, where $\phi$ is a polynomial of degree $n - 4$ which vanishes on the singular curves of $f = 0$ (this result is attributed to Clebsch in [13, p. 157] but no reference is given). Noether [9] called the surfaces $\phi = 0$ adjoints to $f = 0$. He allowed more general singularities on $f = 0$. He proved that the number $p_g$ of independent adjoints is a birational invariant of the surface (this result was announced by Clebsch in [5]). In [9] Noether developed the theory of adjoints for higher dimensional varieties as well.

Let $S$ be a set of curves and points (with assigned multiplicities) in $\mathbb{P}^3$. Denote by $P(m, S)$ the number of conditions imposed on a surface of degree $m$ by requiring it to pass through $S$. The number $P(m, S)$ is called the postulation of $S$ with respect to surfaces of degree $m$. Cayley [3] was the first to consider $P(m, S)$ and give a formula for it,
under certain restrictions on the set $S$. The restrictions were relaxed by Noether [8].

The work of Clebsch [5] led Cayley [2] to derive a postulation formula for the genus (and again this was generalized by Noether [8]). According to this formula the genus is the postulated number $p_a$ of adjoints to a given model $f = 0$, hence equal to the number $\binom{n - 1}{3}$ of all surfaces of degree $n - 4$ minus the postulation $P(n - 4, S)$, where $S$ denotes the set of singular curves and points of $f = 0$. Zeuthen [14] uses Cayley's formula to show that $p_a$ is a birational invariant.

Both Cayley [4] and Noether [9] found that $p_a$ could be strictly less than the actual number, $p_p$, of adjoints. The breakthrough in understanding the difference $p_p - p_a$ was made by Enriques in 1896 [see 13, IV].

The next invariant $p^{(i)}$ that occurs in (1') is what Noether called the curve genus of the surface. He defined it, via a model $f = 0$, as the genus of the variable intersection curve of the surface $f = 0$ with a general adjoint $\phi = 0$, i.e. of a canonical curve. He showed, by what amounts to applying the adjunction formula, that $p^{(i)} - 1$ is equal to the self-intersection $\int c_1^2$ of a canonical curve.

Zeuthen [14] studied the behaviour of a surface under birational transformation by methods similar to those he had applied to curves. He considered enveloping cones of a model of the surface in $\mathbb{P}^3$ and looked for numbers of such a cone that were independent of the particular vertex and of the particular model. He discovered the invariant $\pi^{(i)}$ (equal to $\int c_2$), and found a formula for it in terms of characters of the model, including the class $n'$ (the class is the number of tangent planes that pass through a given point). Later Segre [12] studied pencils on a surface and found a formula for $\pi^{(i)} - 4$ in terms of characters of the pencil. The invariant $I = \pi^{(i)} - 4$ became known as the Zeuthen-Segre invariant of the surface, see also [1].

To deduce (1') Noether used his earlier formula [8] for the class $n'$ to eliminate $n'$ in Zeuthen's formula for $\pi^{(i)}$. He showed that the resulting expression for $\pi^{(i)}$ was equal to his expression for $12(p + 1) - (p^{(i)} - 1)$.

Added in proof

A proof of Noether's formula similar to the above has been given independently by P. Griffiths and J. Harris in their book "Principles of algebraic geometry" (Wiley Interscience, 1978).
REFERENCES


(Oblatum 23–III–1977)