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## ON BIRCH AND SWINNERTON-DYER'S CONJECTURE FOR ELLIPTIC CURVES WITH COMPLEX MULTIPLICATION. I.

Nicole Arthaud

### Introduction

Let  $K$  be an imaginary quadratic field, and  $E$  an elliptic curve with complex multiplication by the ring of integers of  $K$ . Assume that  $E$  is defined over a finite extension  $F$  of  $K$ , and let  $L(E/F, s)$  be the Hasse-Weil zeta function of  $E$  over  $F$ . Deuring has proven that  $L(E/F, s)$  can be analytically continued over the whole complex plane, by identifying it with a product of Hecke  $L$ -series with Grössencharacters (see [7], Theorem 7.42). The conjecture of Birch and Swinnerton-Dyer asserts that  $L(E/F, s)$  has a zero at  $s = 1$  of order equal to  $g_F$ , the rank of the group  $E(F)$  of points of  $E$  with coordinates in  $F$ . Recently, Coates and Wiles [4] made some progress on a weak form of this conjecture. Namely, they showed that if  $K$  has class number 1 and  $F = K$ , then  $g_F \geq 1$  implies that  $L(E/F, s)$  does indeed vanish at  $s = 1$ . The aim of the present paper is to extend Coates and Wiles' proof to the case in which  $K$  has class number 1,  $E$  is still defined over  $K$ , but the base field  $F$  is now an arbitrary finite abelian extension of  $K$ .

**THEOREM 1:** *Let  $K$  be an imaginary quadratic field with class number 1, and  $E$  an elliptic curve defined over  $K$ , with complex multiplication by the ring of integers of  $K$ . If  $F$  is a finite abelian extension of  $K$  such that  $E$  has a point of infinite order with coordinates in  $F$ , then  $L(E/F, s)$  vanishes at  $s = 1$ .*

In a subsequent, but considerably more technical, paper [1] in preparation, we shall prove an analogous result when (i) no restriction is made on the class number of  $K$ , (ii) the base field  $F$  is again supposed to be an abelian extension of  $K$ , and finally (iii) the torsion

points of  $E$  are assumed to generate over  $F$  an abelian extension of  $K$  (see Theorem 7.44 of [7] for a necessary and sufficient condition for (iii) to be valid for  $E$ ). Since the methods of [4] depend crucially on the explicit knowledge of class field theory for abelian extensions of  $K$ , there seems to be little hope at present of proving results like Theorem 1 without hypotheses (ii) and (iii) above.

The broad outlines of the proof of Theorem 1 follow fairly closely the arguments in [4]. However, there are some significant and interesting innovations in dealing with an arbitrary finite abelian extension of  $K$  as base field. In particular, certain partial Hecke  $L$ -functions with Grössencharacters play a natural role in the proof. This is in striking analogy with the theory of cyclotomic  $\mathbb{Z}_p$ -extensions, where the values of partial  $L$ -functions formed with characters of finite order give the coefficients of Stickelberger ideals (see [2]). Also, we have simplified the proof of [4] in several cases (cf. the proof of Theorem 19).

In conclusion, I wish to thank John Coates for his guidance with this work.

### 1. Notation

To a large extent, we follow the notation of [4]. Thus  $K$  will denote an imaginary quadratic field with class number 1, lying inside the complex field  $\mathbb{C}$ , and  $\mathcal{O}$  the ring of integers of  $K$ . As in the Introduction,  $E$  will be an elliptic curve defined over  $K$ , whose ring of endomorphisms is isomorphic to  $\mathcal{O}$ . We fix a Weierstrass model for  $E$

$$(1) \quad y^2 = 4x^3 - g_2x - g_3,$$

where  $g_2, g_3$  belong to  $\mathcal{O}$ , and where the discriminant of (1) is divisible only by the primes of  $K$  where  $E$  has a bad reduction, and (possibly) by the primes of  $K$  above 2 and 3. Let  $\wp(z)$  be the associated Weierstrass function,  $L$  the period lattice of  $\wp(z)$ , and  $\xi(z) = (\wp(z), \wp'(z))$ . Choose  $\Omega \in L$  such that  $L = \Omega\mathcal{O}$ . We identify  $\mathcal{O}$  with the endomorphism ring of  $E$  in such a way that the endomorphism corresponding to  $\alpha \in \mathcal{O}$  is given by  $\xi(z) \mapsto \xi(\alpha z)$ . If  $\alpha \in \mathcal{O}$ , we write  $E_\alpha$  for the kernel of the endomorphism  $\alpha$  of  $E$ . Let  $\psi$  be the Grössencharacter of  $E$  over  $K$  as defined in [7], §7.8. We denote the conductor of  $\psi$  by  $\mathfrak{f}$ , and write  $f$  for some fixed generator of  $\mathfrak{f}$ .

Let  $F$  be an arbitrary finite abelian extension of  $K$ , which will be fixed for the rest of the paper. We write  $S$  for the finite set consisting of 2, 3, and all rational primes  $q$  which have a prime factor in  $K$ ,

which is either ramified in  $F$ , or at which  $E$  has a bad reduction. Henceforth,  $p$  will denote a rational prime, which splits in  $K$ , and which does not belong to the finite exceptional set  $S$ . We write  $\wp$  and  $\bar{\wp}$  for the factors of  $p$  in  $K$ , and put  $\pi = \psi(\wp)$ . Thus, by the definition of  $\psi$ ,  $\pi$  is a generator of the ideal  $\wp$ . Finally, let  $\mathfrak{g}$  denote the least common multiple of the conductor of  $\psi$  and the conductor of  $F/K$ .

## 2. Computation of conductors

We now compute the conductors of various abelian extensions of  $K$  which occur in the proof of Theorem 1. The arguments are similar to those in §2 of [4]. If  $\alpha \in \mathcal{O}$ , recall that  $E_\alpha$  is the group of  $\alpha$ -division points on  $E$ .

**LEMMA 2:** *Let  $\mathfrak{h} = (h)$  be any multiple of the conductor of  $\psi$ . Then  $K(E_h)$  is the ray class field of  $K$  modulo  $\mathfrak{h}$ .*

**PROOF:** By the classical theory of complex multiplication, the ray class field modulo  $\mathfrak{h}$  is contained in  $K(E_h)$ . To prove the converse, we use the notation and results of Shimura [7]. Let  $U(\mathfrak{h})$  be the subgroup of the idèle group of  $K$  as defined on p. 116 of [7], and let  $x$  be any element of  $U(\mathfrak{h})$  with  $x_\infty = 1$ . Since the conductor of  $\psi$  divides  $\mathfrak{h}$ , it follows from Shimura's reciprocity law (cf. the proof of Lemma 3 in [4]) that the Artin symbol  $[x, K]$  fixes  $E_h$ . Thus  $K(E_h)$  is contained in the ray class field modulo  $\mathfrak{h}$ , and the proof of the lemma is complete.

Recall that  $\mathfrak{g}$  is the least common multiple of the conductor of  $\psi$ , and the conductor of  $F/K$ . Also,  $p$  is any rational prime, not in  $S$ , which splits in  $K$ , say  $(p) = \wp\bar{\wp}$ .

**LEMMA 3:** *For each  $n \geq 0$ , the conductor of  $F_n = F(E_{\pi^{n+1}})$  over  $K$  is equal to  $\mathfrak{f}_n = \mathfrak{g}\wp^{n+1}$ . Moreover, if  $\mathcal{R}_n$  denotes the ray class field of  $K$  modulo  $\mathfrak{f}_n$ , then  $\mathcal{R}_n$  is the compositum of  $F_n$  and  $H = K(E_g)$ , and  $F_n \cap H = F$ .*

**PROOF:** Let  $\mathfrak{g}_n$  denote the conductor of  $F_n/K$ . Since  $F_n \subset K(E_{g\pi^{n+1}})$ , and the conductor of this latter field is  $\mathfrak{f}_n = \mathfrak{g}\wp^{n+1}$  by Lemma 2, we conclude that  $\mathfrak{g}_n$  divides  $\mathfrak{f}_n$ . On the other hand, it is clear that the conductor of  $F$  over  $K$  divides  $\mathfrak{g}_n$ . Also, as  $E$  has a good reduction everywhere over  $F_n$  (see Theorem 2 of [4]), the Grössencharacter of  $E$  over  $F_n$  must be unramified. As the Grössencharacter of  $E$  over  $F_n$  is the composition of the norm map from  $F_n$  to  $K$  with  $\psi$ , it follows

that the conductor  $f$  of  $\psi$  divides  $g_n$ . Combining these last two facts, we conclude that  $g$  divides  $g_n$ . But  $\wp^{n+1}$  divides  $g_n$  because  $F_n$  contains the ray class field modulo  $\wp^{n+1}$ . As  $(\wp, g) = 1$  by hypothesis, we deduce that  $g_n = f_n$ , as asserted. To prove the final statement of the lemma, we recall that  $\mathcal{R}_n = K(E_{g^{n+1}})$  by Lemma 2, and thus  $\mathcal{R}_n$  is certainly the compositum of  $F_n$  and  $H$ . Now  $\wp$  is totally ramified in  $K(E_{\pi^{n+1}})$  by the rudiments of Lubin-Tate theory. As  $\wp$  does not divide the conductor of  $F$  over  $K$ , it follows that each prime of  $F$  above  $\wp$  is totally ramified in  $F_n$ . Since  $\wp$  does not divide  $g$  by hypothesis, and  $H$  is the ray class field modulo  $g$  by Lemma 2, we deduce that  $F_n \cap H = F$ , as required.

### 3. $p$ -Adic logarithmic derivatives

We use the same notation as [4] for the formal groups  $\hat{E}$  and  $\mathcal{E}$ . Thus  $\hat{E}$  is the formal group giving the kernel of reduction modulo  $\wp$  on  $E$ , and  $\mathcal{E}$  is the Lubin-Tate formal group for which  $[\pi](w) = \pi w + w^p$ . By Lubin-Tate theory,  $\hat{E}$  and  $\mathcal{E}$  are isomorphic over the ring  $\mathcal{O}_\wp$  of integers of the completion  $K_\wp$  of  $K$  at  $\wp$ . For a fuller discussion, see §3 of [4].

Choose a fixed algebraic closure  $\bar{K}_\wp$  of  $K_\wp$ . We can assume that  $E_\pi$  lies in  $\bar{K}_\wp$ , and we define the extension  $\Phi$  of  $K_\wp$  by

$$\Phi = K_\wp(E_\pi) = K_\wp(\mathcal{E}_\pi).$$

Put  $G = G(\Phi/K_\wp)$ . Of course,  $G$  is endowed with the canonical character  $\chi$ , with values in  $\mathbb{Z}_p^\times$ , giving the action of  $G$  on  $E_\pi$ , or equivalently, on  $\mathcal{E}_\pi$ . Thus, if  $A$  is any  $\mathbb{Z}_p[G]$ -module, it has a canonical decomposition

$$(2) \quad A = \bigoplus_{k=1}^{p-1} A^{(k)},$$

where  $A^{(k)}$  is the submodule of  $A$  on which  $G$  acts via the  $k$ -th power of  $\chi$ .

Let  $u$  be a fixed generator for  $\mathcal{E}_\pi$ , so that  $u$  is a local parameter for  $\Phi$ . Let  $U$  be the group of units of  $\Phi$  which are  $\equiv 1 \pmod{u}$ . For  $1 \leq k \leq p - 2$ , we define homomorphisms

$$(3) \quad \varphi_k : U \rightarrow \mathcal{O}_\wp/\wp$$

as follows. If  $\alpha \in U$ , we choose any power series  $f(T) = \sum_{k=0}^{\infty} a_k T^k$ , with  $a_k \in \mathcal{O}_p$ , such that  $f(u) = \alpha$ . We then define  $\varphi_k(\alpha)$  to be the residue class in  $\mathcal{O}_p/\mathfrak{p}$  of the coefficient of  $T^k$  in the power series  $T(d/dT) \log f(T)$ . Since  $1 \leq k \leq p-2$  and the ramification index of  $\Phi$  over  $K_p$  is  $p-1$ , it is easy to see that  $\varphi_k(\alpha)$  is independent of the choice of  $f(T)$ , and so is well defined.

REMARK: In defining  $\varphi_k$  in [4], one insisted that the power series  $f(T)$  had  $a_0 = 1$ . It is more convenient for the arguments in §4 to work with power series whose constant term is not necessarily 1. Of course, the two definitions of  $\varphi_k$  are the same for  $1 \leq k \leq p-2$ . However, one cannot define  $\varphi_{p-1}$  by the present method.

In the proof of Theorem 1, we shall only be interested in the case in which  $\Phi$  contains no non-trivial  $p$ -power roots of unity. Recall that, by Lemma 12 of [4], if  $p > 5$ , then  $\Phi$  can contain a non-trivial  $p$ -th root of unity if and only if  $\pi + \bar{\pi} = 1$ . The next lemma is plain from Lemmas 9 and 10 of [4].

LEMMA 4: *Assume that  $\Phi$  contains no non-trivial  $p$ -th root of unity. Let  $k$  be an integer with  $1 \leq k \leq p-2$ . Then  $\varphi_k$  vanishes on  $U^{(j)}$  for  $j \not\equiv k \pmod{p-1}$ , and  $\varphi_k$  induces an isomorphism*

$$\tilde{\varphi}_k : U_0^{(k)} / (U_0^{(k)})^p \xrightarrow{\sim} \mathcal{O}_p/\mathfrak{p}.$$

Now consider our fixed finite abelian extension  $F$  of  $K$ , and  $F_0 = F(E_\pi)$ . Let  $\mathcal{S}$  be the set of primes of  $F_0$  above  $\mathfrak{p}$ . For each  $\mathfrak{q} \in \mathcal{S}$ , let  $F_{0,\mathfrak{q}}$  be the completion of  $F_0$  at  $\mathfrak{q}$ , and write  $U_{\mathfrak{q}}$  for the units in  $F_{0,\mathfrak{q}}$  which are  $\equiv 1 \pmod{\mathfrak{q}}$ . Put

$$(4) \quad \mathcal{U} = \prod_{\mathfrak{q} \in \mathcal{S}} U_{\mathfrak{q}}.$$

Now assume that  $\mathfrak{p}$  splits completely in  $F$ . Thus, for each  $\mathfrak{q} \in \mathcal{S}$ , there exists an isomorphism  $\tau_{\mathfrak{q}} : F_{0,\mathfrak{q}} \xrightarrow{\sim} \Phi$ , which preserves the valuations of both fields. Composing this isomorphism with the map  $\varphi_k$  given by (3), we obtain a homomorphism

$$(5) \quad \varphi_{\mathfrak{q},k} : U_{\mathfrak{q}} \rightarrow \mathcal{O}_p/\mathfrak{p} \quad (1 \leq k \leq p-2).$$

We define

$$(6) \quad \varphi_{F,k} : \mathcal{U} \rightarrow \prod_{\mathfrak{q} \in \mathcal{S}} (\mathcal{O}_p/\mathfrak{p})$$

to be the product of the homomorphisms (5) over all  $q \in \mathcal{S}$ . Plainly  $G = G(F_0/F) = G(\Phi/K_p)$  acts on (4), because it acts on each of the  $U_q$  in the natural way. The next lemma is now plain from Lemma 4.

**LEMMA 5:** *Assume that  $\Phi$  contains no non-trivial  $p$ -th root of unity, and that  $\wp$  splits completely in  $F$ . Let  $k$  be an integer with  $1 \leq k \leq p - 2$ . Then  $\varphi_{F,k}$  vanishes on  $\mathcal{U}^{(j)}$  for  $j \not\equiv k \pmod{p-1}$ , and  $\varphi_{F,k}$  induces an isomorphism*

$$\widetilde{\varphi}_{F,k}: \mathcal{U}^{(k)} / (\mathcal{U}^{(k)})^p \xrightarrow{\sim} \prod_{q \in \mathcal{S}} (\mathcal{O}_q / \wp).$$

Put  $d = [F:K]$ . In practice, we shall use the following immediate consequence of Lemma 5.

**COROLLARY 6:** *Under the same hypotheses as Lemma 5, let  $A$  be any  $Z_p[G]$ -submodule of  $\mathcal{U}$ . Then, for each integer  $k$  with  $1 \leq k \leq p - 2$ , the eigenspace  $(\mathcal{U}/A)^{(k)} \neq 0$  if and only if  $\varphi_{F,k}(A)$  has dimension less than  $d$  over the field  $\mathcal{O}_\wp / \wp$ .*

#### 4. Elliptic units

As in [4], a vital role in the proof of Theorem 1 is played by the elliptic units of Robert [6]. We begin by briefly recalling the definition of these elliptic units. Let  $\mathcal{S}$  be the set consisting of all pairs  $(A, \mathcal{N})$ , where  $A = \{\mathfrak{a}_j : j \in J\}$  and  $\mathcal{N} = \{n_j : j \in J\}$ , here  $J$  is an arbitrary finite index set, the  $\mathfrak{a}_j$  are integral ideals of  $K$  prime to  $S$  and  $p$ , and the  $n_j$  are rational integers satisfying  $\sum_{j \in J} n_j (N \mathfrak{a}_j - 1) = 0$ . Given such a pair  $(A, \mathcal{N})$ , we put

$$\Theta(z, A, \mathcal{N}) = \prod_{j \in J} \Theta(z, \mathfrak{a}_j)^{n_j},$$

where  $\Theta(z, \mathfrak{a}_j)$  is as defined at the beginning of §4 of [4]. Recall that  $\mathfrak{f}_n = \mathfrak{g}\wp^{n+1}$  is the conductor of  $F_n = F(E_{\pi^{n+1}})$  over  $K$ . As before, let  $\mathcal{R}_n$  be the ray class field of  $K$  modulo  $\mathfrak{f}_n$ . If  $\rho_n$  is an arbitrary primitive  $\mathfrak{f}_n$ -division point of  $L$ , Robert [6] has shown that  $\Theta(\rho_n, A, \mathcal{N})$  is a unit of the field  $\mathcal{R}_n$ . Moreover, as  $(A, \mathcal{N})$  ranges over  $\mathcal{S}$ , the  $\Theta(\rho_n, A, \mathcal{N})$  form a subgroup of the group of units of  $\mathcal{R}_n$ . We denote this subgroup by  $\mathcal{C}_n$ , and call it the group of elliptic units of  $\mathcal{R}_n$  (note that Robert's definition of the group of elliptic units is different from ours). A

similar argument to that given in the proof of Lemma 20 of [4] shows that  $\mathcal{C}_n$  is stable under the action of the Galois group of  $\mathcal{R}_n$  over  $K$ , and is independent of the choice of the particular primitive  $f_n$ -division point  $\rho_n$ . Finally, we define the elliptic units  $C_n$  of  $F_n = F(E_{\pi^{n+1}})$  to be the group consisting of the norms from  $\mathcal{R}_n$  to  $F_n$  of all units in  $\mathcal{C}_n$ . For simplicity, we often write  $C$  for  $C_0$ .

Let  $\rho = \Omega/g$ , where  $g = (g)$ . Here  $L = \Omega\mathcal{O}$  is the period lattice of  $\wp(z)$ . As above, let  $\mathcal{R}_0$  be the ray class field of  $K$  modulo  $f_0 = g\wp$ . Lemma 3 tells us that we have the diagram of fields

$$(7) \quad \begin{array}{ccc} & \mathcal{R}_0 = HF_0 & \\ & \swarrow \quad \searrow & \\ H = K(E_g) & & F_0 = F(E_\pi) \\ & \searrow \quad \swarrow & \\ & F = H \cap F_0 & \\ & | & \\ & K & \end{array}$$

If  $L$  is any finite abelian extension of  $K$ , and  $\mathfrak{c}$  is an integral ideal of  $K$  prime to the conductor of  $L/K$ , we write  $(\mathfrak{c}, L/K)$  for the Artin symbol of  $\mathfrak{c}$  for the extension  $L/K$ . We now choose and fix a set  $B$  of integral ideals of  $K$ , which are prime to  $f_0$ , and which are such that  $\{(\mathfrak{b}, \mathcal{R}_0/K) : \mathfrak{b} \in B\}$  is precisely the Galois group of  $\mathcal{R}_0/F_0$ . It is then plain from (7) that the restrictions of the  $(\mathfrak{b}, \mathcal{R}_0/K)$ ,  $\mathfrak{b} \in B$ , to  $H$  is precisely the Galois group of  $H/F$ .

If  $\mathfrak{a}$  is an arbitrary integral ideal of  $K$  prime to  $S$  and  $p$ , we define

$$\Lambda(z, \mathfrak{a}) = \prod_{\mathfrak{b} \in B} \Theta(z + \psi(\mathfrak{b})\rho, \mathfrak{a}).$$

LEMMA 7:  $\Lambda(z, \mathfrak{a})$  is a rational function of  $\wp(z)$  and  $\wp'(z)$  with coefficients in  $F$ .

PROOF: This is entirely similar to the first part of the proof of Lemma 21 of [4], and so we omit it.

It is now convenient to introduce some notation, which will be used repeatedly in this section. Let  $\mathcal{G}$  denote the Galois group of  $F$  over  $K$ . If  $\mathfrak{c}$  is an integral ideal of  $K$  prime to the conductor of  $F/K$ , we write  $\sigma_{\mathfrak{c}}$  for the Artin symbol  $(\mathfrak{c}, F/K)$ . Finally, if  $\sigma \in \mathcal{G}$  and  $R(z)$  is a rational function of  $\wp(z)$ ,  $\wp'(z)$  with coefficients in  $F$ , then  $R_{\sigma}(z)$  will denote the rational function of  $\wp(z)$ ,  $\wp'(z)$ , which is obtained by letting  $\sigma$  act on the coefficients of  $R(z)$ .

Let  $k$  be an integer  $\geq 1$ . Recall that  $\psi$  denotes the Grössencharacter of  $E$ . For each  $\sigma \in \mathcal{G}$ , we introduce the partial Hecke  $L$ -function

$$\zeta_F(\sigma, k; s) = \sum_{\substack{(\mathfrak{a}, \mathfrak{b})=1 \\ \sigma_{\mathfrak{a}}=\sigma}} \frac{\bar{\psi}^k(\mathfrak{a})}{(N\mathfrak{a})^s},$$

where the summation is over all integral ideals  $\mathfrak{a}$  of  $K$ , prime to  $\mathfrak{g}$ , such that the Artin symbol  $\sigma_{\mathfrak{a}}$  is equal to  $\sigma$ . It can be shown that  $\zeta_F(\sigma, k; s)$  can be analytically continued over the whole complex plane. Let  $\zeta_F(\sigma, k)$  denote the value of  $\zeta_F(\sigma, k; s)$  at  $s = k$ .

LEMMA 8: For each  $\sigma \in \mathcal{G}$ , we have

$$z \frac{d}{dz} \log \Lambda_{\sigma}(z, \mathfrak{a}) = \sum_{k=1}^{\infty} c_k(\mathfrak{a}, \sigma) z^k, \quad \text{where}$$

$$c_k(\mathfrak{a}, \sigma) = 12(-1)^{k-1} \rho^{-k} (N\mathfrak{a} \zeta_F(\sigma, k) - \psi^k(\mathfrak{a}) \zeta_F(\sigma\sigma_{\mathfrak{a}}, k)) \quad (k = 1, 2, \dots).$$

PROOF: Let  $\mathfrak{c}$  be an integral ideal of  $K$ , prime to  $\mathfrak{g}$ , such that  $\sigma = \sigma_{\mathfrak{c}}$ . By the definition of the Grössencharacter  $\psi$  in [7], we have

$$\xi(\psi(\mathfrak{b})\rho^{(\mathfrak{c}, H/K)}) = \xi(\psi(\mathfrak{bc})\rho).$$

It follows easily from the expression for  $\Theta(z + \psi(\mathfrak{b})\rho, \mathfrak{a})$  as a rational function of  $\wp(z), \wp'(z)$ , with coefficients in  $H$  (see (23) of [4]), that

$$\Lambda_{\sigma}(z, \mathfrak{a}) = \prod_{\mathfrak{b} \in B} \Theta(z + \psi(\mathfrak{bc})\rho, \mathfrak{a}).$$

If  $\mathcal{L}$  is any lattice in the complex plane, let  $\zeta(z, \mathcal{L})$  and  $\wp(z, \mathcal{L})$  be the Weierstrass zeta and  $\wp$ -functions of  $\mathcal{L}$ . Define

$$\Omega(z, \mathcal{L}) = z \frac{d}{dz} \log \left( \prod_{\mathfrak{b} \in B} \theta(z + \psi(\mathfrak{bc})\rho, \mathcal{L}) \right).$$

Then (cf. the proof of Lemma 21 of [4])  $\Omega(z, \mathcal{L})$  has the power series expansion  $\sum_{k=1}^{\infty} d_k(\mathcal{L})z^k$ , where  $\eta = \psi(\mathfrak{c})\rho$  and

$$(8) \quad d_1(\mathcal{L}) = 12 \sum_{\mathfrak{b} \in B} (\zeta(\psi(\mathfrak{b})\eta, \mathcal{L}) - s_2(\mathcal{L})\psi(\mathfrak{b})\eta),$$

$$(9) \quad d_2(\mathcal{L}) = -12 \sum_{\mathfrak{b} \in B} (\wp(\psi(\mathfrak{b})\eta, \mathcal{L}) + s_2(\mathcal{L})),$$

$$(10) \quad d_k(\mathcal{L}) = -12 \sum_{\mathfrak{b} \in B} \wp^{(k-2)}(\psi(\mathfrak{b})\eta, \mathcal{L}) / (k-1)! \quad (k \geq 3).$$

Thus we must show that  $c_k(\mathfrak{a}, \sigma)$ , as defined in Lemma 8, satisfies

$$(11) \quad c_k(\mathfrak{a}, \sigma) = N \mathfrak{a} d_k(L) - d_k(\mathfrak{a}^{-1}L) \quad (k \geq 1).$$

As in [4], we put  $\lambda_k = 12(-1)^{k-1} \rho^{-k}$ . We write  $\mathcal{B}$  for a fixed set of generators of the ideals in  $B$ . Also, we let  $\gamma$  denote a fixed generator of the ideal  $\mathfrak{a}$ , and  $c$  a fixed generator of  $\mathfrak{c}$ . The argument now breaks up into three cases. Much of the reasoning is similar to that in the proof of Lemma 21 of [4], so that we refer there for details from time to time.

*Case 1.* We suppose that  $k \geq 3$ . Since

$$\wp^{(k-2)}(z, \mathcal{L}) = (-1)^k (k-1)! \sum_{\omega \in \mathcal{L}} (z - \omega)^{-k} \quad (k \geq 3),$$

we conclude easily from (10) that

$$d_k(L) = \lambda_k \sum_{b \in B} \sum_{\alpha \in \mathfrak{a}} (\psi(bc) - \alpha)^{-k}.$$

We now write  $\psi(bc) = \epsilon(bc)bc$ , where  $b$  is the generator of  $\mathfrak{b}$  in  $\mathcal{B}$ , and  $\epsilon(bc)$  is a root of unity in  $K$ , and argue in exactly the same way as in Case 1 of the proof of Lemma 21 in [4]. In this way, it follows that

$$d_k(L) = \lambda_k \sum_{b \in \mathcal{B}} \sum_{\alpha \in \mathfrak{a}} \bar{\psi}^k((bc - \alpha)) N(bc - \alpha)^{-k},$$

where  $N$  denotes the norm from  $K$  to  $\mathbb{Q}$ . Let  $W$  denote the group of roots of unity of  $K$ . Since the Grössencharacter  $\psi$  is defined modulo  $\mathfrak{g}$ , the natural map of  $W$  into  $(\mathcal{O}/\mathfrak{g})^\times$  is plainly injective. Now, as  $H$  is the ray class field modulo  $\mathfrak{g}$  by Lemma 2, we can identify the Galois group of  $H$  over  $K$  with  $(\mathcal{O}/\mathfrak{g})^\times/W$  via the Artin map. Since the Artin symbol of  $\mathfrak{c} = (c)$  for  $F/K$  is equal to  $\sigma$ , it is therefore clear that  $\{\mu bc : \mu \in W, b \in \mathcal{B}\}$  is a complete set of representatives of those elements in  $(\mathcal{O}/\mathfrak{g})^\times$ , whose Artin symbol has restriction to  $F$  equal to  $\sigma$ . In other words,

$$\{\mu bc - \alpha : \mu \in W, b \in \mathcal{B}, \alpha \in \mathfrak{g}\}$$

is the set of all algebraic integers in  $K$ , prime to  $\mathfrak{g}$ , such that the Artin symbol for  $F/K$  of the associated principal ideal is equal to  $\sigma$ . Since

we can plainly rewrite the above expression for  $d_k(L)$  as

$$d_k(L) = \frac{\lambda_k}{w_k} \sum_{\mu \in W} \sum_{b \in \mathfrak{B}} \sum_{\alpha \in \mathfrak{g}} \bar{\psi}^k((\mu bc - \alpha)) N(\mu bc - \alpha)^{-k},$$

where  $w_k$  denotes the number of roots of unity in  $K$ , it follows that

$$d_k(L) = \lambda_k \zeta_F(\sigma, k).$$

Now consider  $d_k(\mathfrak{a}^{-1}L)$ . Recalling that  $\mathfrak{a} = (\gamma)$ , it follows from (10) that

$$d_k(\mathfrak{a}^{-1}L) = \lambda_k \gamma^k \sum_{b \in \mathfrak{B}} \sum_{\alpha \in \mathfrak{g}} (\gamma \psi(bc) - \alpha)^{-k}.$$

Substitute  $\gamma = \psi(\mathfrak{a})\epsilon^{-1}(\gamma)$  for the first occurrence of  $\gamma$  on the right hand side of this equation. Again arguing in the same way as in Case 1 of the proof of Lemma 21 in [4], we obtain

$$d_k(\mathfrak{a}^{-1}L) = \lambda_k \psi^k(\mathfrak{a}) \sum_{b \in \mathfrak{B}} \sum_{\alpha \in \mathfrak{g}} \bar{\psi}^k((\gamma bc - \alpha)) N(\gamma bc - \alpha)^{-k}.$$

Now

$$\{\mu \gamma bc - \alpha : \mu \in W, b \in \mathfrak{B}, \alpha \in \mathfrak{g}\}$$

is the set of all algebraic integers in  $K$ , prime to  $\mathfrak{g}$ , such that the Artin symbol for  $F/K$  of the associated principal ideal is equal to  $\sigma\sigma_{\mathfrak{a}}$ . Thus

$$d_k(\mathfrak{a}^{-1}L) = \lambda_k \psi^k(\mathfrak{a}) \zeta_F(\sigma\sigma_{\mathfrak{a}}, k).$$

We have therefore proven (11) in this case.

*Case 2.* We assume that  $k = 2$ . Now, for any lattice  $\mathcal{L}$ ,

$$\wp(z, \mathcal{L}) = \lim_{\substack{s \rightarrow 0 \\ s > 0}} \sum_{\omega \in \mathcal{L}} (z - \omega)^{-2} |z - \omega|^{-2s} - s_2(\mathcal{L}),$$

where  $s_2(\mathcal{L})$  is as defined at the beginning of §4 of [4]. Taking  $\mathcal{L} = L$ , we deduce from (9) that

$$d_2(L) = \lambda_2 \lim_{\substack{s \rightarrow 0 \\ s > 0}} \sum_{b \in \mathfrak{B}} \sum_{\alpha \in \mathfrak{g}} (\psi(bc) - \alpha)^{-2} |\psi(bc) - \alpha|^{-2s}.$$

Arguing as in the previous case, we obtain  $d_2(L) = \lambda_2 \zeta_F(\sigma, 2)$ . Similarly,  $d_2(\mathfrak{a}^{-1}L) = \lambda_2 \psi^2(\mathfrak{a}) \zeta_F(\sigma \sigma_{\mathfrak{a}}, 2)$ , and so we obtain (11) in this case.

*Case 3.* We assume that  $k = 1$ . If  $\mathcal{L}$  is any lattice, let  $H(s, z, \mathcal{L})$  denote the analytic continuation in  $s$  of the series

$$\sum_{\omega \in \mathcal{L}} (\bar{z} + \bar{\omega}) |z + \omega|^{-2s}$$

(this series converges for  $R(s) > 3/2$ ). Then, as is shown in case 3 of the proof of Lemma 21 of [4], we have

$$\zeta(z, \mathcal{L}) - z s_2(\mathcal{L}) = H(1, z, \mathcal{L}) + \bar{z} g(\mathcal{L}),$$

where  $g(\mathcal{L})$  is defined in the same proof. First take  $\mathcal{L} = L$ . It follows from (8) that

$$d_1(L) = \lambda_1 \lim_{s \rightarrow 1} \sum_{\mathfrak{b} \in B} \sum_{\alpha \in \mathfrak{a}} \frac{\bar{\psi}(\mathfrak{b}\mathfrak{c}) + \bar{\alpha}}{|\psi(\mathfrak{b}\mathfrak{c}) + \alpha|^{2s}} + r g(L),$$

where  $r = \sum_{\mathfrak{b} \in B} (\bar{\psi}(\mathfrak{b}\mathfrak{c}) \bar{\rho})$  (here, by the limit as  $s \rightarrow 1$ , we mean the value of the analytic continuation at  $s = 1$ ). As before, we deduce easily that

$$d_1(L) = \lambda_1 \zeta_F(\sigma, 1) + r g(L).$$

Next take  $\mathcal{L} = \gamma^{-1}L$ . Then

$$d_1(\mathfrak{a}^{-1}L) = \lambda_1 \lim_{s \rightarrow 1} \sum_{\mathfrak{b} \in B} \sum_{\alpha \in \gamma^{-1}\mathfrak{a}} \frac{\bar{\psi}(\mathfrak{b}\mathfrak{c}) + \bar{\alpha}}{|\psi(\mathfrak{b}\mathfrak{c}) + \alpha|^{2s}} + r g(\gamma^{-1}L).$$

Taking the factor  $\gamma^{-1}$  out of each  $\alpha$ , and recalling that  $g(\gamma^{-1}L) = N \mathfrak{a} g(L)$ , we conclude that

$$d_1(\mathfrak{a}^{-1}L) = \lambda_1 \gamma \lim_{s \rightarrow 1} \sum_{\mathfrak{b} \in B} \sum_{\alpha \in \mathfrak{a}} \frac{\bar{\gamma} \bar{\psi}(\mathfrak{b}\mathfrak{c}) + \bar{\alpha}}{|\gamma \psi(\mathfrak{b}\mathfrak{c}) + \alpha|^{2s}} + r N \mathfrak{a} g(L).$$

We now argue in the same way as in case 1 to deduce that

$$d_1(\mathfrak{a}^{-1}L) = \lambda_1 \psi(\mathfrak{a}) \zeta_F(\sigma \sigma_{\mathfrak{a}}, 1) + r N \mathfrak{a} g(L).$$

Combining these two expressions for  $d_1(L)$  and  $d_1(\mathfrak{a}^{-1}L)$ , we see that (11) is true for  $k = 1$ . This completes the proof of Lemma 8.

COROLLARY 9: *For each integer  $k \geq 1$ , and each  $\sigma \in \mathcal{G}$ ,  $\Omega^{-k}\zeta_F(\sigma, k)$  belongs to  $F$ . Moreover, if  $\tau \in \mathcal{G}$ , then  $(\Omega^{-k}\zeta_F(\sigma, k))^\tau = \Omega^{-k}\zeta_F(\tau\sigma, k)$ .*

PROOF: The first assertion is plain from Lemmas 7 and 8, on taking  $\mathfrak{a} \neq 1$  to be an integral ideal of  $K$ , prime to  $S$  and  $p$ , such that  $\sigma_{\mathfrak{a}} = 1$ . The second assertion follows similarly, on noting that  $c_k(\mathfrak{a}, \sigma)^\tau = c_k(\mathfrak{a}, \tau\sigma)$  for all  $k \geq 1$  because  $\Lambda_\sigma(z, \mathfrak{a})^\tau = \Lambda_{\tau\sigma}(z, \mathfrak{a})$ . Here  $\Lambda_\sigma(z, \mathfrak{a})^\tau$  denotes the rational function of  $\wp(z)$  and  $\wp'(z)$ , with coefficients in  $F$ , which is obtained by letting  $\tau$  act on the coefficients of  $\Lambda_\sigma(z, \mathfrak{a})$ .

Let  $\psi_F$  denote the Grössencharacter of  $F$ , which is obtained by composing  $\psi$  with the norm map from  $F$  to  $K$ . Plainly  $\psi_F$  is unramified outside  $\mathfrak{g}$ . Thus, for each integer  $k \geq 1$ , we can define

$$L_F(\bar{\psi}_F^k, s) = \prod_{(\mathfrak{P}, \mathfrak{g})=1} (1 - \bar{\psi}_F^k(\mathfrak{P})(N\mathfrak{P})^{-s})^{-1},$$

the product being taken over all primes  $\mathfrak{P}$  of  $F$  which do not divide  $\mathfrak{g}$ . Of course,  $L_F(\bar{\psi}_F^k, s)$  will not, in general, be a primitive Hecke  $L$ -function, but this will not be important in the proof of Theorem 1. Let  $\hat{\mathcal{G}}$  denote the group of all homomorphisms from  $\mathcal{G}$  into the group of non-zero complex numbers. If  $\theta \in \hat{\mathcal{G}}$ , we associate with it the complex  $L$ -function

$$L_F(\bar{\psi}^k\theta, s) = \sum_{\sigma \in \mathcal{G}} \theta(\sigma)\zeta_F(\sigma, k; s).$$

One verifies immediately that we have the product decomposition

$$(12) \quad L_F(\bar{\psi}_F^k, s) = \prod_{\theta \in \hat{\mathcal{G}}} L_F(\bar{\psi}^k\theta, s).$$

The next lemma gives the basic rationality properties of the value of  $L_F(\bar{\psi}_F^k, s)$  at  $s = k$ .

LEMMA 10: *For each integer  $k \geq 1$ ,  $\Omega^{-kd}L_F(\bar{\psi}_F^k, k)$  belongs to  $F$ , and the ideal that it generates is fixed by the action of  $\mathcal{G}$ .*

PROOF: By (12) and the first assertion of Corollary 9, we see that  $\nu_k = \Omega^{-kd}L_F(\bar{\psi}_F^k, k)$  belongs to  $M$ , where  $M$  is the field obtained by adjoining to  $F$  the values of all  $\theta \in \hat{\mathcal{G}}$ . But, again by (12), it is clear that  $\nu_k$  is fixed by the Galois group of  $M$  over  $F$ , and so belongs to  $F$ . Now take  $\tau$  to be any element of  $\mathcal{G}$ , and let  $\tau_1$  be an element of  $G(M/K)$  whose restriction to  $F$  is  $\tau$ . The second assertion of Corol-

lary 9 implies that

$$(13) \quad \Omega^{-k} L_F(\bar{\psi}^k \theta, k)^{\tau_1} = \theta^{\tau_1(\tau^{-1})} \Omega^{-k} L_F(\bar{\psi}^k \theta^{\tau_1}, k),$$

whence it is plain from (12) that the ideal in  $F$  generated by  $\nu_k$  is fixed by  $\mathcal{G}$ .

REMARK: If  $\mathcal{G}$  has no quadratic characters, (12) and (13) show that  $\Omega^{-kd} L_F(\bar{\psi}_F^k, k)$  is actually fixed by  $\mathcal{G}$ , and so belongs to  $K$ .

We now investigate the integrality properties of the numbers in Corollary 9 and Lemma 10. Let  $\mathfrak{P}$  be any prime of  $F$  lying above  $\wp$ ,  $F_{\mathfrak{P}}$  the completion of  $F$  at  $\mathfrak{P}$ , and  $\mathcal{O}_{\mathfrak{P}}$  the ring of integers of  $F_{\mathfrak{P}}$ . We can view  $\Lambda_{\sigma}(z, \mathfrak{a})$  as being a rational function of  $\wp(z)$  and  $\wp'(z)$  with coefficients in  $F_{\mathfrak{P}}$ , via the canonical inclusion of  $F$  in  $F_{\mathfrak{P}}$ . Hence we can expand  $\Lambda_{\sigma}(z, \mathfrak{a})$  in terms of the parameter  $t = -2\wp(z)/\wp'(z)$  of the formal group  $\hat{E}$ .

LEMMA 11: *Let  $\mathfrak{P}$  be any prime of  $F$  above  $\wp$ . In terms of the parameter  $t = -2\wp(z)/\wp'(z)$ ,  $\Lambda_{\sigma}(z, \mathfrak{a})$  has an expansion*

$$\Lambda_{\sigma}(z, \mathfrak{a}) = \sum_{k=0}^{\infty} h_{k,\sigma}(\mathfrak{a}, \mathfrak{P}) t^k,$$

whose coefficients all belong to  $\mathcal{O}_{\mathfrak{P}}$ , and where  $h_{0,\sigma}(\mathfrak{a}, \mathfrak{P})$  is a unit in  $\mathcal{O}_{\mathfrak{P}}$ .

PROOF: This is the same as the proof of Lemma 23 of [4] (on recalling that  $(\mathfrak{g}, \wp) = 1$  by hypothesis), and so we omit the details.

LEMMA 12: *Let  $k$  be an integer with  $1 \leq k \leq p - 1$ . Then (i) for  $\sigma \in \mathcal{G}$ ,  $\Omega^{-k} \zeta_F(\sigma, k)$  is integral at each prime of  $F$  above  $\wp$ , and (ii)  $\Omega^{-kd} L_F(\bar{\psi}_F^k, k)$  is integral at each prime of  $F$  above  $\wp$ .*

PROOF: In view of (12), it is plain that (ii) is a consequence of (i). We now proceed to deduce (i) from the previous lemma. Let  $w$  be the parameter of the Lubin-Tate formal group  $\mathcal{E}$  such that  $[\pi](w) = \pi w + w^p$  (cf. §3 of [4]). Fix a prime  $\mathfrak{P}$  of  $F$  above  $\wp$ . For the moment, take  $\mathfrak{a}$  to be an arbitrary integral ideal of  $K$ , prime to  $S$  and  $p$ . Since  $t$  can be written as a power series in  $w$  with coefficients in  $\mathcal{O}_{\mathfrak{P}}$ , it follows from Lemma 11 that  $\Lambda_{\sigma}(z, \mathfrak{a})$  can be expanded as a power series in  $w$ , say  $f(w)$ , with coefficients in  $\mathcal{O}_{\mathfrak{P}}$ , and whose constant term  $f(0)$  is a unit in  $\mathcal{O}_{\mathfrak{P}}$ . Moreover, since  $z = w + \sum_{i=2}^{\infty} a_i w^i$ , where  $a_i = 0$  unless

$i \equiv 1 \pmod{p-1}$  (cf. Lemma 7 of [4]), the coefficients of  $z^k$  and  $w^k$  ( $0 \leq k \leq p-1$ ) in the  $z$ -expansion of  $\Lambda_\sigma(z, \mathfrak{a})$  and in  $f(w)$  are plainly equal. It follows that the coefficients of  $z^k$  and  $w^k$  ( $1 \leq k \leq p-1$ ) in the  $z$ -expansion of  $z(d/dz) \log \Lambda_\sigma(z, \mathfrak{a})$  and in  $w(d/dw) \log f(w)$  are also equal. But the coefficients of this latter series lie in  $\mathcal{O}_{\mathfrak{P}}$ , because the constant term  $f(0)$  of  $f(w)$  is a unit in  $\mathcal{O}_{\mathfrak{P}}$ . We conclude from Lemma 8 that

$$(14) \quad \Omega^{-k}(N\mathfrak{a}\zeta_F(\sigma, k) - \psi^k(\mathfrak{a})\zeta_F(\sigma\sigma_{\mathfrak{a}}, k))$$

is integral at  $\mathfrak{P}$  for  $1 \leq k \leq p-1$ . We now make a special choice of the ideal  $\mathfrak{a}$ . Let  $e$  denote a generator of the ideal  $(12g) \cap Z$ . Choose  $n$  to be a rational integer, prime to  $p$ , such that  $1 + ne\pi$  is not divisible by  $\bar{\rho}$ , and take  $\mathfrak{a} = (1 + ne\pi)$ . Then  $N\mathfrak{a} \equiv 1 \pmod{\rho}$ . Also  $\sigma_{\mathfrak{a}} = 1$  because the conductor of  $F/K$  divides  $e$ , and  $\psi^k(\mathfrak{a}) = (1 + en\pi)^k \equiv 1 \pmod{\rho}$ , because the conductor of  $\psi$  divides  $e$ . Thus  $N\mathfrak{a} - \psi^k(\mathfrak{a})$  is a unit at  $\rho$ , and so assertion (i) follows from (14). This completes the proof of Lemma 12.

We now prove a technical lemma, which establishes the existence of  $d$  pairs  $(A, \mathcal{N})$  in  $\mathcal{J}$ , with properties which will be needed later in this section. To simplify the statement of the lemma, we choose a fixed numbering of the elements of  $\mathcal{G}$ , say  $\sigma_1, \dots, \sigma_d$ , with  $\sigma_1 = 1$ .

LEMMA 13: *Let  $k$  be an integer with  $1 \leq k \leq p-2$ . Then there exist  $d$  pairs  $(A^{(h)}, \mathcal{N}^{(h)}) \in \mathcal{J}$ , where*

$$A^{(h)} = \{\mathfrak{a}_1^{(h)}, \mathfrak{a}_2^{(h)}\}, \quad \mathcal{N}^{(h)} = \{n_1^{(h)}, n_2^{(h)}\} \quad (1 \leq h \leq d),$$

*with the following properties. Firstly,  $\psi^k(\mathfrak{a}_2^{(1)}) \not\equiv 1 \pmod{\rho}$ . Secondly, for  $1 \leq h \leq d$ , we have (i)  $\psi^k(\mathfrak{a}_1^{(h)}) \equiv 1 \pmod{\rho}$ , (ii)  $\sigma_{\mathfrak{a}_2^{(h)}} = 1$ , (iii)  $\sigma_{\mathfrak{a}_1^{(h)}} = \sigma_h^{-1}$ , and (iv)  $n_2^{(h)}$  is prime to  $p$ .*

PROOF: Let  $e$  denote a generator of the ideal  $(12g) \cap Z$ , and let  $\beta \pmod{\rho}$  be a generator of  $(\mathcal{O}/\rho)^\times$ . First consider the case  $h = 1$ . Let  $n$  be a rational integer, prime to  $p$ , such that  $1 + ne\pi$  is prime to  $\bar{\rho}$ , and take  $\mathfrak{a}_1^{(1)} = (1 + en\pi)$ . Choose  $\mathfrak{a}_2^{(1)} = (\alpha_2^{(1)})$ , where  $\alpha_2^{(1)}$  is an algebraic integer in  $K$  satisfying  $\alpha_2^{(1)} \equiv 1 \pmod{e\bar{\pi}}$ , and  $\alpha_2^{(1)} \equiv \beta \pmod{\pi}$ . Let  $n_1^{(1)} = N\mathfrak{a}_2^{(1)} - 1$  and  $n_2^{(1)} = -(N\mathfrak{a}_1^{(1)} - 1)$ , so that  $n_2^{(1)}$  is prime to  $p$  because  $(p, ne) = 1$ . Moreover, as the conductor of  $\psi$  divides  $e$ , we have  $\psi^k(\mathfrak{a}_1^{(1)}) \equiv 1 \pmod{\rho}$ , and  $\psi^k(\mathfrak{a}_2^{(1)}) \equiv \beta^k \not\equiv 1 \pmod{\rho}$ . Finally, both ideals are prime to  $S$  and  $p$  by construction, and  $\sigma_{\mathfrak{a}_1^{(1)}} = \sigma_{\mathfrak{a}_2^{(1)}} = 1$  because the conductor of  $F$  over  $K$  also divides  $e$ . This completes the case  $h = 1$ .

For  $h > 1$ , again choose  $\mathfrak{a}_1^{(h)} = (1 + n\pi)$  and  $n_2^{(h)} = -(N\mathfrak{a}_1^{(h)} - 1)$ . Take  $\mathfrak{a}_2^{(h)}$  to be an integral ideal of  $K$ , prime to  $S$  and  $p$ , such that  $\sigma_{\mathfrak{a}_2^{(h)}} = \sigma_h^{-1}$ , and let  $n_1^{(h)} = N\mathfrak{a}_2^{(h)} - 1$ . The proof of the lemma is now complete.

So far in this section, we have made no hypothesis on the decomposition of  $\wp$  in the extension  $F/K$ , other than requiring that  $\wp$  does not ramify in  $F/K$ . We now suppose, until further notice, that  $\wp$  splits completely in  $F$ . We use the notation of the last part of §I3. Thus  $\mathcal{S}$  will denote the set of prime of  $F_0 = F(E_\pi)$  above  $\wp$ , and  $\mathcal{U}$  will again be given by (4). Let

$$(15) \quad i : F_0 \rightarrow \prod_{\mathfrak{q} \in \mathcal{S}} F_{0,\mathfrak{q}}$$

be the canonical embedding of  $F_0$  in the product of its completions at the primes  $\mathfrak{q}$  in  $\mathcal{S}$ . Recall that  $C$  denotes the group of elliptic units of  $F_0$ , as defined at the beginning of this section. We write  $\mathcal{C}$  for the subgroup of  $C$  consisting of all elements which are  $\equiv 1 \pmod{\mathfrak{q}}$  for each  $\mathfrak{q} \in \mathcal{S}$ . Let  $\overline{i(\mathcal{C})}$  be the closure of  $i(\mathcal{C})$  in the  $\wp$ -adic topology. Our aim is to compute, for  $1 \leq k \leq p - 2$ , the image of  $\overline{i(\mathcal{C})}$  under the homomorphism  $\varphi_{F,k}$  given by (6).

Recall that  $\Phi$  is the field  $K_\wp(E_\pi)$ , which lies inside our fixed algebraic closure of  $K_\wp$ . Since  $\wp$  splits completely in  $F$  by hypothesis, the completion of  $F_0$  at each  $\mathfrak{q}$  in  $\mathcal{S}$  is plainly topologically isomorphic to  $\Phi$ . To simplify notation, we adopt the following convention. We fix one embedding of  $F_0$  in  $\Phi$ , and view this embedding as simply being an inclusion. This amounts to choosing one fixed prime in  $\mathcal{S}$ , which we denote by  $\mathfrak{q}$ . Let  $\Omega$  denote the Galois group of  $F_0$  over  $K(E_\pi)$ . Since  $\wp$  is totally ramified in  $K(E_\pi)$ , and splits completely in  $F_0/K(E_\pi)$ , the other primes in  $\mathcal{S}$  are given precisely by the  $\mathfrak{q}^\sigma$  for  $\sigma \in \Omega$ , and the embedding of  $F_0$  in  $\Phi$  corresponding to  $\mathfrak{q}^\sigma$  is given by  $\sigma$  itself. With this convention, the map (15) is simply given by

$$(16) \quad i(x) = (x^\sigma)_{\sigma \in \Omega}.$$

Now take  $x$  to be any elliptic unit in  $\mathcal{C}$ . More explicitly, let  $\xi(\tau)$  be the point of  $E_\pi$  corresponding to our chosen generator  $u$  of  $\mathcal{E}_\pi$  under our fixed isomorphism from  $\hat{E}$  to  $\mathcal{E}$ . Then, by definition,  $x$  will be of the form

$$(17) \quad x = \prod_{j \in J} \Lambda(\tau, \mathfrak{a}_j)^{n_j}$$

for some pair  $(A, \mathcal{N})$  belonging to  $\mathcal{J}$ . Now  $\Omega = G(F_0/K(E_\pi))$  is canonically isomorphic to  $\mathcal{G} = G(F/K)$  under the restriction map, and we shall identify these two Galois groups in this way when there is no danger of confusion. Since  $\Omega$  fixes  $E_\pi$ , it is then plain that

$$x^\sigma = \prod_{j \in J} \Lambda_\sigma(\tau, a_j)^{n_j} \quad \text{for } \sigma \in \Omega,$$

where  $\Lambda_\sigma(z, a_j)$  is as defined just after Lemma 7

LEMMA 14: *Let  $x$  be the elliptic unit in  $\mathfrak{C}$  given by (17). Then, for each integer  $k$  with  $1 \leq k \leq p - 2$ , we have*

$$\varphi_{F,k}(i(x)) = \left( \lambda_k \sum_{j \in J} n_j (N a_j \zeta_F(\sigma, k) - \psi^k(a_j) \zeta_F(\sigma \sigma_{a_j}, k)) \bmod \mathfrak{q}^\sigma \right)_{\sigma \in \Omega},$$

where  $\lambda_k = 12(-1)^{k-1} \rho^{-k}$ .

PROOF: We can obtain a power series  $f_\sigma(w)$ , with coefficients in  $\mathcal{O}_\rho$ , such that  $f_\sigma(u) = x^\sigma$  in the following manner. Let  $w$  be the parameter of the Lubin-Tate formal group  $\mathcal{E}$ , and expand the rational function of  $\wp(z)$  and  $\wp'(z)$ , with coefficients in  $F$ , given by

$$(18) \quad \prod_{j \in J} \Lambda_\sigma(z, a_j)^{n_j}$$

as a formal power series in  $w$ . Denote the power series obtained in this way by  $f_\sigma(w)$ . By lemma 11 and the fact that  $t$  can be written as a power series in  $w$  with coefficients in  $\mathcal{O}_\rho$ , we conclude that  $f_\sigma(w)$  does indeed have coefficients in  $\mathcal{O}_\rho$ . It is then plain that  $x^\sigma = f_\sigma(u)$ . Moreover, as  $z = w + \sum_{i=2}^\infty a_i w^i$ , where  $a_i = 0$  unless  $i \equiv 1 \pmod{p-1}$  (cf. Lemma 7 of [4]), we see that the coefficients of  $z^k$  and  $w^k$  ( $0 \leq k \leq p-1$ ) in the series expansions of (18) in terms of  $z$  and  $w$  must be equal. Thus the conclusion of the lemma is now clear from Lemma 8 and the definition of  $\varphi_{F,k}$ .

We now come to the first main result of this section. Since the elliptic units of  $F_0$  are stable under the action of the Galois group of  $F_0$  over  $K$  (cf. Lemma 20 of [4]), it follows, in particular, that  $\bar{i}(\mathfrak{C})$  is a  $\mathbb{Z}_p[G]$ -submodule of  $\mathcal{U}$ , where  $G = G(F_0/F)$ . We can therefore take the canonical decomposition (2) of  $\mathcal{U}/\bar{i}(\mathfrak{C})$ . We follow the terminology of [4] and say that  $p$  is anomalous for  $E$  if  $\pi + \bar{\pi} = 1$ .

**THEOREM 14:** *Assume that  $p$  is a prime number  $>5$  satisfying (i)  $p$  does not belong to the finite exceptional set  $S$ , (ii)  $p$  splits in  $K$ , say  $(p) = \wp \bar{\wp}$ , (iii)  $\wp$  splits completely in  $F|K$ , and (iv)  $p$  is not anomalous for  $E$ . Let  $\mathcal{C}$  be the group of elliptic units of  $F_0 = F(E_\pi)$ , which are  $\equiv 1 \pmod{\mathfrak{q}}$  for each  $\mathfrak{q} \in \mathcal{S}$ . Then, for each integer  $k$  with  $1 \leq k \leq p - 2$ , the eigenspace  $(\mathcal{U}/i(\mathcal{C}))^{(k)}$  is non-trivial if and only if  $\Omega^{-kd} L_F(\bar{\psi}_F^k, k) \equiv 0 \pmod{\mathfrak{q}}$  for each  $\mathfrak{q} \in \mathcal{S}$ .*

**REMARK:** By Lemma 10,  $\Omega^{-kd} L_F(\bar{\psi}_F^k, k) \equiv 0 \pmod{\mathfrak{q}}$  for one prime  $\mathfrak{q}$  in  $\mathcal{S}$  if and only if the same congruence is valid for all  $\mathfrak{q}$  in  $\mathcal{S}$ .

**PROOF:** We adopt the same convention as before, in which we have fixed one prime  $\mathfrak{q}$  in  $\mathcal{S}$ , and view  $F_0$  as being contained in  $\Phi$ . We make use of the following formal identity in the group ring  $F[\mathcal{G}]$ , which is very reminiscent of computations with Stickelberger elements in cyclotomic fields. For each  $\sigma \in \mathcal{G}$ , put

$$\zeta_{\mathbb{F}}^*(\sigma, k) = \lambda_k \zeta_F(\sigma, k).$$

By Corollary 9,  $\zeta_{\mathbb{F}}^*(\sigma, k)$  belongs to  $F$ . Write

$$(19) \quad \alpha = \sum_{\sigma \in \mathcal{G}} \zeta_{\mathbb{F}}^*(\sigma, k) \sigma^{-1}.$$

Then, for each integral ideal  $\mathfrak{a}$  of  $K$  which is prime to  $\mathfrak{g}$ , we plainly have

$$(20) \quad (N\mathfrak{a} - \psi^k(\mathfrak{a})\sigma_{\mathfrak{a}})\alpha = \sum_{\sigma \in \mathcal{G}} \delta_k(\sigma, \mathfrak{a}) \sigma^{-1},$$

where

$$(21) \quad \delta_k(\sigma, \mathfrak{a}) = N\mathfrak{a} \zeta_{\mathbb{F}}^*(\sigma, k) - \psi^k(\mathfrak{a}) \zeta_{\mathbb{F}}^*(\sigma\sigma_{\mathfrak{a}}, k).$$

By Corollary 6, the eigenspace  $(\mathcal{U}/i(\mathcal{C}))^{(k)}$  will be trivial if and only if  $\varphi_{F,k}(i(\mathcal{C}))$  has dimension  $d$  over the finite field  $F_p$  with  $p$  elements. This suggests that we study the image under  $\varphi_{F,k}$  of any  $d$  elements of  $i(\mathcal{C})$ . Suppose therefore that  $(A^{(h)}, \mathcal{N}^{(h)})$  ( $1 \leq h \leq d$ ) are any  $d$  elements of  $\mathcal{S}$ . Let  $x_h$ , given by (17), be the elliptic unit corresponding to  $(A^{(h)}, \mathcal{N}^{(h)})$ . We assume that  $x_1, \dots, x_d$  belong to  $\mathcal{C}$ . Write

$$A^{(h)} = \{\mathfrak{a}_j^{(h)} : j \in J_h\}, \quad \mathcal{N}^{(h)} = \{n_j^{(h)} : j \in J_h\},$$

and

$$\gamma_h = \sum_{j \in J_h} n_j^{(h)} (N \mathbf{a}_j^{(h)} - \psi^k(\mathbf{a}_j^{(h)}) \sigma_{\mathbf{a}_j^{(h)}}).$$

For  $\sigma \in \mathcal{G}$  and  $1 \leq h \leq d$ , we define

$$b_{h\sigma} = \sum_{j \in J_h} n_j^{(h)} \delta_k(\sigma, \mathbf{a}_j^{(h)}),$$

where  $\delta_j(\sigma, \mathbf{a}_j^{(h)})$  is given by (21). It is then plain from (20) that we have the identity

$$(22) \quad \gamma_h \alpha = \sum_{\sigma \in \mathcal{G}} b_{h\sigma} \sigma^{-1} \quad (1 \leq h \leq d).$$

We let  $\Xi$  denote the  $d \times d$ -determinant form from the  $b_{h\sigma}$  ( $h = 1, \dots, d, \sigma \in \mathcal{G}$ ).

By Lemma 14, the determinant of the  $d$  vectors

$$\varphi_{F,k}(i(x_h)) \quad (1 \leq h \leq d)$$

is equal to  $\Xi \bmod \mathfrak{q}$ . We now proceed to compute  $\Xi$ . To this end, let  $\hat{\mathcal{G}}$  be the group of homomorphisms from  $\mathcal{G}$  to the multiplicative group of non-zero complex numbers. Let  $\sigma_1 = 1, \sigma_2, \dots, \sigma_d$  denote the distinct elements of  $\mathcal{G}$ , and  $\chi_1 = 1, \chi_2, \dots, \chi_d$  the distinct elements of  $\hat{\mathcal{G}}$ . Write  $\Gamma$  and  $\Sigma$  for the  $d \times d$ -determinants formed from the  $\chi_i(\gamma_h), \chi_i(\sigma_h^{-1})$  ( $1 \leq i, h \leq d$ ), respectively. Applying each of the  $\chi_i$  to the equation (22), we conclude that

$$(23) \quad \left( \prod_{i=1}^d \chi_i(\alpha) \right) \Gamma = \Sigma \Xi.$$

We now make two observations. Put  $L_F^*(\bar{\psi}_F^k, k) = \lambda_k^d L_F(\bar{\psi}_F^k, k)$ . Then it is plain from (12) and (19) that

$$(24) \quad \prod_{i=1}^d \chi_i(\alpha) = L_F^*(\bar{\psi}_F^k, k).$$

Secondly,  $\Sigma \neq 0$  and  $\Gamma/\Sigma$  is an algebraic integer in  $K$ . The former assertion is clear. To prove the latter one, we note that we can write

$$(25) \quad \gamma_h = \sum_{\sigma \in \mathcal{G}} e_{h\sigma} \sigma^{-1},$$

where the  $e_{h\sigma}$  are algebraic integers in  $K$ . Applying each of the  $\chi_i$  to (25), it follows that  $\Gamma = \Lambda\Sigma$ , where  $\Lambda$  is the  $d \times d$ -determinant formed from the  $e_{h\sigma}$ . Since  $\Sigma$  is obviously an algebraic integer in  $K$ , it follows that the same is true for  $\Sigma = \Gamma/\Lambda$ .

We can now complete the proof of Theorem 14. Suppose first that  $L_{\mathbb{F}}^*(\bar{\psi}_{\mathbb{F}}^k, k) \equiv 0 \pmod{\mathfrak{q}}$ . Then we conclude from (23), (24) and the above remarks that  $\Xi \equiv 0 \pmod{\mathfrak{q}}$  for all choices of the  $d$  pairs  $(A^{(h)}, \mathcal{N}^{(h)})$  in  $\mathcal{J}$ . Thus  $\varphi_{F,k}(i(\mathbb{C}))$  has dimension strictly less than  $d$  over  $F_p$ , and hence  $(\mathcal{U}/i(\mathbb{C}))^{(k)} \neq 0$ . Conversely, assume that  $L_{\mathbb{F}}^*(\bar{\psi}_{\mathbb{F}}^k, k) \not\equiv 0 \pmod{\mathfrak{q}}$ . Then it follows from (23) and (24) that  $\Xi \not\equiv 0 \pmod{\mathfrak{q}}$  only if we can choose the  $d$  pairs  $(A^{(h)}, \mathcal{N}^{(h)})$  such that the determinant  $\Lambda$  defined above is not congruent to 0 modulo  $\wp$ . But this is always possible. Indeed, make the choice of the  $d$  pairs  $(A^{(h)}, \mathcal{N}^{(h)})$  specified in Lemma 13. Note that, by multiplying each of the  $n_1^{(h)}, n_2^{(h)}$  ( $1 \leq h \leq d$ ) by  $p - 1$  (which changes none of the other conditions in Lemma 13), we can certainly assume that the corresponding elliptic units lie in  $\mathbb{C}$ . Using the relation  $\sum_{j=1}^2 n_j^{(h)}(Na_j^{(h)} - 1) = 0$  and the fact that  $\psi^k(a_1^{(h)}) \equiv 1 \pmod{\wp}$ , we conclude that

$$\gamma_h \equiv n_2^{(h)} - n_2^{(h)}\psi^k(a_2^{(h)})\sigma_h^{-1} \pmod{\wp} \quad (1 \leq h \leq d);$$

here the congruence mod  $\wp$  means that we have taken the coefficients in the group ring mod  $\wp$ . It is now trivial to verify from the other conditions of Lemma 13 that  $\Lambda \not\equiv 0 \pmod{\wp}$ . This completes the proof of Theorem 14.

LEMMA 15: *There are infinitely many rational primes  $p$  satisfying conditions (i), (ii), (iii), and (iv) of Theorem 14.*

PROOF: As before, let  $H = K(E_g)$ . Applying Chebotarev's density theorem to a Galois extension of  $\mathbb{Q}$  containing  $H$ , we conclude that there are infinitely many rational primes  $p$  which split completely in  $H$ . We claim that any rational prime  $p$ , not in  $S$ , which splits completely in  $H$ , satisfies (i), (ii), (iii) and (iv). The only part which is not obvious is that such a  $p$  satisfies (iv). Take such a  $p$ , and let  $(p) = \wp\bar{\wp}$  be its factorization in  $K$ . Since  $\wp$  splits completely in  $H$ , the Artin symbol  $(\wp, H/K)$  fixes  $E_g$ . On the other hand, as  $\psi(\wp) = \pi$ , Shimura's reciprocity law gives  $\xi(\rho)^{(\wp, H/K)} = \xi(\pi\rho)$  for each  $\rho \in E_g$ . Thus we must have  $\pi \equiv 1 \pmod{g}$ . Now, if  $p$  were anomalous, it would follow that  $\pi\bar{\pi} = (\pi - 1)(\bar{\pi} - 1)$ , and this is clearly impossible because  $p$  was prime to  $g$  by hypothesis. This completes the proof.

We now begin the proof of the second main result of this section.

As before, let  $F_n = F(E_{\pi^{n+1}})$ . Since  $\wp$  is totally ramified in  $K(E_{\pi^{n+1}})$ , it is clear that each prime of  $F$  above  $\wp$  is totally ramified in  $F_n$ . Write  $\mathcal{S}_n$  for the set of primes of  $F_n$  above  $\wp$ . Let  $C_n$  be the group of elliptic units of  $F_n$ , as defined at the beginning of this section, and let  $\mathfrak{C}_n$  be the subgroup of  $C_n$  consisting of all elements which are  $\equiv 1 \pmod{\mathfrak{q}}$  for each  $\mathfrak{q} \in \mathcal{S}_n$ . If  $m \geq n$ , we write  $N_{m,n}$  for the norm map from  $F_m$  to  $F_n$ . The next lemma, which is, in essence, one of the main results of [6], is valid without any hypothesis on the decomposition of  $\wp$  in  $F$ .

LEMMA 16: *For each  $m \geq n \geq 0$ , we have  $N_{m,n}(\mathfrak{C}_m) = \mathfrak{C}_n$ .*

PROOF: Recall that  $\mathfrak{f}_n = \mathfrak{q}\wp^{n+1}$  is the conductor of  $F_n$  over  $K$ , by Lemma 3. Let  $f_n$  denote a generator of the ideal  $\mathfrak{f}_n \cap \mathbb{Z}$ , and let  $g_n$  be the largest divisor of  $f_n$  such that the  $g_n$ -th roots of unity lie in  $F_n$ . We claim that  $g_n = g_0$  for all  $n \geq 0$ , and that  $g_0$  is prime to  $p$ . Indeed,  $F_n$  can contain no non-trivial  $p$ -power roots of unity, because  $\bar{\wp}$  does not divide the conductor of  $F_n/K$ . Moreover, since  $F_n/F_0$  is totally ramified at the primes above  $\wp$ , it follows that  $F_n$  and  $F_0$  have the same group of roots of unity for all  $n \geq 0$ . Let  $D$  be the group of  $g_0$ -th roots of unity in  $F_0$ . Robert (cf. [6], p. 43) has defined  $\Omega_{F_n}$  to be the group  $DC_n$ . Moreover, since  $\mathfrak{f}_0$  divides  $\mathfrak{f}_n$  and  $\mathfrak{f}_n$  are divisible by the same primes, it is shown in [6] (cf. Proposition 17, p. 43) that  $N_{m,n}(\Omega_{F_m})D = \Omega_{F_n}$ . Since the order of  $D$  is prime to  $p$  (and hence no element of  $D$  is  $\equiv 1 \pmod{\mathfrak{q}}$  for  $\mathfrak{q} \in \mathcal{S}_n$ ), it follows immediately that  $N_{m,n}(\mathfrak{C}_m) = \mathfrak{C}_n$ . This completes the proof.

For each integer  $n \geq 0$ , let  $\Phi_n = K_{\wp}(E_{\pi^{n+1}})$ , and let  $\wp_n$  be the maximal ideal of  $\Phi_n$ . Write  $U_n$  for the units of  $\Phi_n$  which are  $\equiv 1 \pmod{\wp_n}$ , and  $U'_n$  for the subgroup of  $U_n$  consisting of all elements with norm 1 to  $K_{\wp}$ . Plainly

$$(26) \quad (U'_n)^{(k)} = U_n^{(k)} \quad \text{for} \quad k \not\equiv 0 \pmod{p-1}.$$

If  $m > n$ , we also write  $N_{m,n}$  for the norm map from  $\Phi_m$  to  $\Phi_n$ .

LEMMA 17: *Suppose that  $k \not\equiv 0 \pmod{p-1}$ . If  $m \geq n$ , then the norm map from  $U_m^{(k)}$  to  $U_n^{(k)}$  is surjective, and its kernel is equal to  $(U_m^{(k)})^{1-\tau}$ , where  $\tau$  is a generator of  $G(\Phi_m/\Phi_n)$ .*

PROOF: The norm map from  $U'_m$  to  $U'_n$  is surjective, because  $U'_n$  consists of those elements of  $U_n$  which are norms from  $\Phi_m$  for all  $m \geq n$  (cf. Lemma 8 of [4]). Thus the first assertion is plain from (26). As for the second, let  $V_m$  denote the kernel of the norm map from  $U_m$

to  $U_n$ . Since  $\Phi_m/\Phi_n$  is a totally ramified cyclic extension of degree  $p^{m-n}$ , a standard computation (cf. [5], p. 188) shows that

$$[V_m : U_m^{1-\tau}] = [V_m^{(0)} : U_m^{(0)(1-\tau)}] = p^{m-n}.$$

Hence  $[V_m^{(k)} : U_m^{(k)(1-\tau)}] = 1$  for all  $k \not\equiv 0 \pmod{p-1}$ , as required.

The following elementary lemma is certainly well known, but we have been unable to find a suitable reference.

LEMMA 18: *Let  $\Lambda$  be a cyclic group of prime order  $p \neq 2$ , operating on a finitely generated  $\mathbb{Z}_p$ -module  $M$ . Let  $\tau$  be a generator of  $\Lambda$ . If  $M = (\tau - 1)M$ , then  $M = 0$ .*

PROOF: Since  $\tau^p = 1$  and  $p$  is odd, it is clear that

$$(27) \quad (\tau - 1)^p \in pZ[\Lambda],$$

where  $Z[\Lambda]$  is the group ring of  $\Lambda$  with coefficients in  $\mathbb{Z}$ . Let  $N$  be the torsion submodule of  $M$ , so that  $M/N$  is a free  $\mathbb{Z}_p$ -module of finite rank with  $(\tau - 1)(M/N) = (M/N)$ . But this shows that  $(\tau - 1)^p$  is surjective on  $M/N$ , and this is impossible by (27) unless  $M/N = 0$ . Hence we can suppose that  $M$  is a finite abelian  $p$ -group. But again (27) implies that  $M = 0$  if  $(\tau - 1)M = M$ . This completes the proof.

For each  $q \in \mathcal{S}_n$ , let  $F_{n,q}$  be the completion of  $F_n$  at  $q$ , and again let  $i$  be the canonical inclusion of  $F_n$  in  $\prod_{q \in \mathcal{S}_n} F_{n,q}$ . Write  $U_{n,q}$  for the units in  $F_{n,q}$  which are  $\equiv 1 \pmod{q}$ , and put

$$(28) \quad \mathcal{U}_n = \prod_{q \in \mathcal{S}_n} U_{n,q}.$$

Thus, in terms of our earlier notation,  $\mathcal{U}_0 = \mathcal{U}$  and  $\mathcal{C}_0 = \mathcal{C}$ .

THEOREM 19: *Let  $p$  be a prime number satisfying (i)  $p$  does not belong to  $S$ , (ii)  $p$  splits in  $K$ ,  $(p) = \wp, \bar{\wp}$ , and (iii)  $\wp$  splits completely in  $F$ . Let  $k$  be an integer with  $1 \leq k \leq p - 2$ . Let  $m, n$  be any two integers  $\geq 0$ , with  $m > n$ . Then  $(\mathcal{U}_m / i(\mathcal{C}_m))^{(k)} \neq 0$  if and only if  $(\mathcal{U}_n / i(\mathcal{C}_n))^{(k)} \neq 0$ .*

PROOF: Since  $\wp$  splits completely in  $F$ , we can identify  $F_{n,q}$ , for each  $q \in \mathcal{S}_n$ , with the field  $\Phi_n$ , and  $U_{n,q}$  with  $U_n$ . Let  $N_{m,n} : \mathcal{U}_m \rightarrow \mathcal{U}_n$  be the map given by the product of the local norms from  $\Phi_m$  to  $\Phi_n$  at each  $q \in \mathcal{S}_n$ . Suppose now that  $1 \leq k \leq p - 2$ . Put  $A_n = \mathcal{U}_n^{(k)} / i(\mathcal{C}_n)^{(k)}$ . It

follows from the first part of Lemma 17 that the norm map from  $\mathcal{U}_m^{(k)}$  to  $\mathcal{U}_n^{(k)}$  is surjective, whence the induced map from  $A_m^{(k)}$  to  $A_n^{(k)}$  is also surjective. Thus it is clear that  $A_m^{(k)} = 0$  implies that  $A_n^{(k)} = 0$ . To prove the converse, we note that Lemmas 16 and 17 together imply that the kernel of the norm map from  $A_m^{(k)}$  to  $A_n^{(k)}$  is  $(A_m^{(k)})^{1-\tau}$ , where  $\tau$  is a generator of the Galois group of  $F_m$  over  $F_n$ . Suppose now that  $A_n^{(k)} = 0$ . Since  $A_{n+1}^{(k)}$  is a finitely generated  $\mathbb{Z}_p$ -module, we conclude from Lemma 18 that  $A_{n+1}^{(k)} = 0$ . Repeating the argument a finite number of times, it follows that  $A_m^{(k)} = 0$  for all  $m \geq n$ . This completes the proof.

### 5. Proof of Theorem 1

We can now complete the proof of Theorem 1 in an entirely similar fashion to the proof of Theorem 1 in [4]. If  $N$  is an abelian extension of  $F_n$ , which is Galois over  $F$ , then  $G_n = G(F_n/F)$  operates on  $X = G(N/F_n)$  via inner automorphisms in the usual way. In particular,  $G = G(F_0/F)$  operates on  $X$ , because we can identify  $G$  with a subgroup of  $G_n$ . Thus, if  $N$  is a  $p$ -extension of  $F_n$ , we can take the canonical decomposition (2) of  $X$  into eigenspaces for the action of  $G$ .

As before, let  $\mathcal{S}_n$  be the set of primes of  $F_n$  over  $\wp$ . Let  $M_n$  denote the maximal abelian  $p$ -extension of  $F_n$ , which is unramified outside  $\mathcal{S}_n$ , and let  $L_n$  be the  $p$ -Hilbert class field of  $F_n$ . Let  $\mathcal{U}_n$  be defined by (28), that is,  $\mathcal{U}_n$  is the product of the local units  $\equiv 1$  in the completions of  $F_n$  at the primes  $\mathfrak{q} \in \mathcal{S}_n$ . Write  $N_{F_n/K} : \mathcal{U}_n \rightarrow K_\wp$  for the map given by the product of the local norms at all  $\mathfrak{q} \in \mathcal{S}_n$ . We denote the kernel of  $N_{F_n/K}$  by  $\mathcal{U}'_n$ . Plainly

$$(29) \quad \mathcal{U}_n^{(k)} = (\mathcal{U}'_n)^{(k)} \quad \text{whenever} \quad k \not\equiv 0 \pmod{p-1}.$$

As is explained in detail in [3], global class field theory gives the following explicit description of  $G(M_n/L_n F_\infty)$  as a  $G_n$ -module, where  $F_\infty = \bigcup_{n \geq 0} F_n$ . Let  $E_n$  be the group of all global units of  $F_n$  which are  $\equiv 1 \pmod{\mathfrak{q}}$  for each  $\mathfrak{q} \in \mathcal{S}_n$ . Let  $\overline{i(E_n)}$  be the closure of  $i(E_n)$  in  $\mathcal{U}_n$  in the  $\wp$ -adic topology.

**THEOREM 20:** *For each  $n \geq 0$ ,  $\mathcal{U}'_n / \overline{i(E_n)}$  is isomorphic as a  $G_n$ -module, via the Artin map, to  $G(M_n/L_n F_\infty)$ .*

Suppose now that there does exist a point  $P$  in  $E(F)$  of infinite

order. Take  $p$  to be a rational prime satisfying (i)  $p$  does not belong to  $S$ , (ii)  $p$  splits in  $K$ ,  $(p) = \wp \bar{\wp}$ , and (iii)  $\wp$  splits completely in  $F$ . As before, let  $\pi = \psi(\wp)$ . For each  $n \geq 0$ , choose  $Q_n$  in  $E(\bar{F})$  such that  $\pi^{n+1}Q_n = P$ , and form the extension  $H_n = F_n(Q_n)$ . Thus  $H_n/F_n$  is a cyclic extension of degree dividing  $p^{n+1}$ , and as  $P$  lies in  $E(F)$ , one verifies easily that

$$(30) \quad x^\sigma = \chi(\sigma)x \quad \text{for all } x \in G(H_n/F_n) \quad \text{and} \quad \sigma \in G.$$

An entirely similar argument to that given in Lemma 33 of [4] shows that  $H_n/F_n$  is unramified outside  $\mathcal{S}_n$ . Finally, as  $\wp$  splits completely in  $\bar{F}$ , the local arguments in Theorem 11 and Lemma 35 of [4] again show that the extension  $H_n F_\infty/F_\infty$  is non-trivial and ramified for all sufficiently large  $n$ .

Assume now that  $n$  is so large that  $H_n F_\infty/F_\infty$  is non-trivial and ramified. Hence the extension  $H_n L_n F_\infty/L_n F_\infty$  is non-trivial. As this extension lies inside  $M_n$ , we conclude from (29), (30) and Theorem 20 that

$$(31) \quad (\mathcal{U}_n / i(\overline{E_n}))^{(1)} \neq 0.$$

As before, let  $\mathcal{C}_n$  be the group of elliptic units of  $F_n$ , which are  $\equiv 1 \pmod{\mathfrak{q}}$  for each  $\mathfrak{q} \in \mathcal{S}_n$ . As  $\mathcal{C}_n \subset E_n$ , it follows that  $(\mathcal{U}_n / i(\overline{\mathcal{C}_n}))^{(1)} \neq 0$ . Therefore, by Theorem 19,  $(\mathcal{U}_0 / i(\overline{\mathcal{C}_0}))^{(1)} \neq 0$ . Assume, in addition, that  $p > 5$  and is not anomalous for  $E$ . Theorem 14 then implies that

$$\Omega^{-d} L_F(\bar{\psi}_F, 1) \equiv 0 \pmod{\mathfrak{q}} \quad \text{for each } \mathfrak{q} \in \mathcal{S}_n.$$

But, by Lemma 15, there certainly are infinitely many rational primes  $p$  satisfying the conditions we have imposed on  $p$ . Thus  $\Omega^{-d} L_F(\bar{\psi}_F, 1)$  is divisible by infinitely many distinct prime ideals of  $F$ , and so must be equal to 0. Since the Hasse-Weil zeta function of  $E$  over  $F$  is equal to  $L_F(\psi_F, s) L_F(\bar{\psi}_F, s)$ , up to finitely many Euler factors which do not vanish at  $s = 1$  (cf. Theorem 7.42 of [7]), this completes the proof of Theorem 1.

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