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SOME RESULTS ON BANACH SPACES WITHOUT LOCAL UNCONDITIONAL STRUCTURE

Gilles Pisier

Introduction

In the first part of this paper, we consider the following question: When does the Banach space c_p – the Schatten p-class of operators on Hilbert space – imbed (for $1 < p \neq 2 < \infty$) in a space with an unconditional basis? Kwapién and Pełczyński proved that c_∞ or c_1 does not embed isomorphically into a space with an unconditional basis ([10] theorem 2.3). Gordon and Lewis proved in [5] (see also [6], [11]) that the spaces c_p fail to have an unconditional basis (or even merely l.u.st.) unless $p = 2$. The question whether or not c_p embeds in a space with an unconditional basis for $1 < p \neq 2 < \infty$ was answered affirmatively by Lindenstrauss (1974 unpublished), who even showed that c_p embeds in a *reflexive* space with unconditional basis (the reflexivity follows from theorem 3.1 in [3]). He asked whether one could embed c_p $(1 < p \neq 2 < \infty)$ in a space with unconditional basis not containing l_∞^n's uniformly. We will answer this question in the negative. We also show below that, except for the trivial case $p = 2$, c_p is not isomorphic to a quotient of a subspace of a uniformly convex Banach lattice (see th. 2.1).

Stated in finite dimensional terms, our results provide a complete answer to the question of McCarthy in [14] (see theorem 2.2), yielding the correct order of growth of the shortest distance of $c_p(l_2^n)$ to a subspace of L_p. The results are derived in a simple way from an approach of Gordon and Lewis [5], together with general results of Maurey [12] on spaces which do not contain l_∞^n's uniformly. (We could have deduced them from the results of [2] or [7] which both use [12], we have chosen a more direct route).

One can remark that all the known examples of spaces which do not have l.u.st. ([5], [21], [13], [18]) also do not embed into a space with l.u.st. not containing l_∞^n's uniformly. This is the motivation for our section 3. We show there the existence of subspaces of $L_p(T, m)$ ((T, m) is the circle with Lebesgue measure) for each $p > 4$ which do not have l.u.st. The proof is based on a deep result of Rudin concerning $\Lambda(p)$ sets. Moreover, we slightly generalise a result of Varopoulos characterising Sidon sets in terms of the isomorphic structure of the invariant subspace which they generate in the space of continuous functions (on a compact abelian group).

§1. Preliminaries

We first recall the definition of several ideals of operators which we will use: If $0 < p < \infty$, an operator $A: E \to F$ is called p-absolutely summing if there exists a scalar λ such that

$$\Sigma \|A(x_n)\|^p \leq \lambda^p \max\{\Sigma |\xi(x_n)|^p | \xi \in E^*, \quad \|\xi\| \leq 1\}$$

for all finite sequences (x_n) in E. The smallest of such λ is denoted $\pi_p(A)$.

If $1 \leq p \leq \infty$, an operator $A: E \to F$ is said to 'factor' through L_p if for some L_p space there exist $A_1: E \to L_p$ and $A_2: L_p \to F^{**}$ such that

$$A_2 A_1 = JA$$

where J denotes the canonical injection of F into F^{**}; the number $\gamma_p(A)$ is defined as the infinum of $\|A_1\| \cdot \|A_2\|$ over all such factorizations of A.

The distance between two Banach spaces E, F is defined as:

$$d(E, F) = \inf\{\|T\| \, \|T^{-1}\|\}$$

where the infimum runs over all isomorphisms T from E onto F (with the convention: $\inf \phi = \infty$). By definition, a Banach space E contains l_∞^n's uniformly if, for any $\epsilon > 0$ and any integer n there is a subspace $E_n \subset E$ such that

$$d(E_n, l_\infty^n) \leq 1 + \epsilon.$$

l_∞^n denotes as usual the n dimensional space with the norm:

$$(\alpha_1, \ldots, \alpha_n) \to \sup |\alpha_i|.$$

We will also need the following:

PROPOSITION 1.1: *Let (Ω, P) be a probability space; consider elements $\varphi_1, \ldots, \varphi_N$ in $L_p(\Omega, P)$ $(1 \le p \le \infty)$ and elements $x_1, \ldots, x_N$ of a Banach space X, and define the operator $u : X^* \to L_p(\Omega, P)$ by: $\forall \xi \in X^*\ u(\xi) = \Sigma_1^N \varphi_i(\cdot)\xi(x_i)$. Then the following inequality holds:*

$$\pi_p(u) \le \left(\int \|\Sigma_1^N \varphi_i x_i\|^p \, \mathrm{d}P \right)^{1/p} \le \pi_p({}^t u). \tag{1.1}$$

The left part of (1.1) *is obvious; The right one is a particular case of a general statement concerning radonifying maps* (*see* [8] *th.* 2).

Next, we recall some notions of unconditionality. A finite or infinite family $(e_n)_{n \in I}$ in a Banach space is called unconditionally basic if there exists a constant K such that

$$\left\| \sum_n \epsilon_n \alpha_n e_n \right\| \le K \left\| \sum \alpha_n e_n \right\|$$

for any sequence of signs $\epsilon_n = \pm 1$ and any sequence of scalars (α_n) with only finitely many non zero terms. The smallest constant K with this property will be denoted $\chi\{e_n\}$. When it is total the family $\{e_n\}$ is called an unconditional basis of the Banach space under consideration.

We will work with the notion of local unconditional structure introduced by Gordon and Lewis in [5].

DEFINITION 1.1: *A Banach space E is said to have local unconditional structure (in short l.u.st.) if there exists a scalar λ with the following property: Given any finite dimensional subspace $F \subset E$, there exists a space U with an unconditional basis $\{u_n\}$ and operators A from F to U and B from U to E such that BA is the identity on F and $\|A\| \cdot \|B\| \cdot \chi\{u_n\} \le \lambda$. Moreover, we will denote $\chi_u(E)$ the smallest scalar λ with the above property.*

REMARK 1.1: In [3] (remark 2.3) it is shown that a space E has l.u.st. in the above sense if and only if its bidual E^{**} is isomorphic to a complemented subspace of a Banach lattice. As an immediate consequence, it follows that E has l.u.st. in the above sense if and only if the same is true for its dual E^*. Moreover, the proof in [3] shows that $\chi_u(E) = \chi_u(E^*)$.

In [5], Gordon and Lewis distinguish spaces with l.u.st. by the property that every 1-absolutely summing operator on the space factors through L_1. The preceding remarks show that we can replace E by E^* in Lemma 3.3 in [5] and we immediately get

LEMMA 1.1: *Let A be a 1-absolutely summing operator on E^* with values in an arbitrary Banach space, then $\gamma_1(A) \leq \chi_u(E)\pi_1(A)$.*

The preceding lemma can also be derived from Lemma 3.3 in [5] using the theory of duality for ideals of operators.

The next theorem (and its reformulations) was essentially observed by Bernard Maurey as a corollary of his results in [12]. It was verbally communicated to the author in 1974.

THEOREM 1.1: *Assume that the space E has l.u.st.*

(i) *If E does not contain l_∞^n's uniformly, then there exist q, $2 \leq q < \infty$, and a constant C such that:*

(I) *Any E-valued operator A satisfies:*

$$\pi_q(A) \leq C\pi_1({}^tA).$$

(ii) *If neither E nor E^* contain l_∞^n's uniformly, then there exist q with $2 \leq q < \infty$, p with $1 < p \leq 2$ and a constant C such that:*

(II) *Any E-valued operator A satisfies:*

$$\pi_q(A) \leq C\pi_p({}^tA).$$

PROOF OF (i): As was proved by Maurey ([12] or [20] théorème 1.2.a) we have: If $E \not\supset l_\infty^n$'s uniformly, then $\exists_q < \infty \exists C$ such that $\pi_q(A) \leq C\gamma_\infty(A)$, for any E-valued operator A. By Lemma 1.1 we have $\gamma_1({}^tA) \leq \chi_u(E)\pi_1({}^tA)$ therefore $\gamma_\infty(A) \leq \gamma_\infty({}^{tt}A) \leq \gamma_1({}^tA) \leq \chi_u(E)\pi_1({}^tA)$ and we conclude that $\pi_q(A) \leq C\chi_u(E)\pi_1({}^tA)$.

PROOF OF (ii): By [12] (or also [20]) the hypothesis on E and E^* imply that $\exists p \in]1,2]$ $\exists q \in [2,\infty[$ and there are constants C and C' such that for any E-valued operator A one has:

$$\pi_q(A) \leq C\gamma_\infty(A) \qquad \pi_1({}^tA) \leq C'\pi_p({}^tA). \tag{1.2}$$

By part (i) we have

$$\pi_q(A) \leq C\chi_u(E)\pi_1({}^tA).$$

therefore by (1.2)

$$\pi_q(A) \leq CC'\chi_u(E)\pi_p({}^tA). \qquad \text{Q.E.D.}$$

REMARK 1.2: Let $1 \leq p,\ q < \infty$. It is not difficult to check that a Banach space E with an unconditional basis which is p-convex and q-concave in the sense of [2] satisfies property II in theorem 1.1 for some constant C. The same applies to a Banach lattice E which is of type $\geq p$ and of type $\leq q$ in the sense of [7].

For the case $q = p$ in property II above, the reader should consult [9].

§2. Applications to spaces of operators

In this section, we study some Banach space properties of the space $c_p(H)$ of compact operators T on the Hilbert space H such that $\mathrm{tr}(T^*T)^{p/2} < \infty$; equipped with the norm $T \to \{\mathrm{tr}(T^*T)^{p/2}\}^{1/p}$ this space becomes, for $1 \leq p < \infty$, a Banach space. The space $c_2(H)$ is the space of Hilbert Schmidt operators on H, it is a Hilbert space (therefore it has an unconditional basis!).

By $c_\infty(H)$, we will denote the Banach space of all compact operators on H with the usual operator norm. If H is the separable Hilbert space l_2 we write simply c_p instead of $c_p(l_2)$. For details on these spaces we refer the reader to [4]. More information on the geometry of the spaces c_p will be found in [1].

Throughout this section, we denote (r_n) the Rademacher functions on the Lebesgue interval. Essentially to shorten the statements, we introduce the

DEFINITION 2.1: *Let E be a Banach space, we define the number $\alpha(E)$ as the smallest of the constants α with the following property: the inequality*

$$\int \Big\| \sum_{1 \leq i,j \leq n} \epsilon_{ij} r_i(s) r_j(t) x_{ij} \Big\|^2 \mathrm{d}s\, \mathrm{d}t \leq \alpha^2 \int \Big\| \sum r_i(s) r_j(t) x_{ij} \Big\|^2 \mathrm{d}s\, \mathrm{d}t \tag{2.1}$$

holds for all integers n, all $n \times n$ matrices (x_{ij}) of elements of E and for all choices of signs (ϵ_{ij}) in $\{-1, +1\}^{n^2}$.

We will consider the class of Banach spaces E such that $\alpha(E) < \infty$. Clearly this property depends only on the isomorphic structure of the space E.

REMARK 2.1: For any Banach space E, one can define the space $\mathrm{Rad}(E)$ as the set of sequences (x_n) of elements of E such that $\Sigma\, x_n r_n(t)$ is a convergent series in E for almost every t in $[0, 1]$. One can check that equipped with the norm

$$(x_n)_n \to \sup_n \left(\int \left\| \sum_1^n r_i(t) x_i \right\|^2 \mathrm{d}t\right)^{1/2},$$

the space $\mathrm{Rad}(E)$ becomes a Banach space. As is easily seen by averaging over the ϵ_{ij}'s in (2.1), the condition $\alpha(E) < \infty$ means precisely that $\mathrm{Rad}(E)$ and Rad $(\mathrm{Rad}(E))$ can be identified in a natural way. We are grateful to L. Tzafriri for raising a question about $\mathrm{Rad}(\mathrm{Rad}(E))$ which actually lead us to the present note. (For more details on $\mathrm{Rad}(E)$ see [20]).

We will need the following lemma which is a well known consequence of Khintchine's inequality.

LEMMA 2.1: (i) *If* $0 < p < q < \infty$, *there is a constant* $K(p, q)$ *such that for all infinite matrices of complex numbers with only finitely many non zero entries, one has*:

$$(2.2) \qquad \left(\int \left| \sum r_i(s) r_j(t) \alpha_{ij} \right|^q \mathrm{d}s\, \mathrm{d}t\right)^{1/q} \leq K(p, q) \int \left| \sum r_i(s) r_j(t) \alpha_{ij} \right|^p \mathrm{d}s\, \mathrm{d}t\bigg)^{1/p}.$$

(ii) *If* $1 < p < \infty$, *the orthogonal projection* P *is bounded from* $L_p(\mathrm{d}s, \mathrm{d}t)$ *onto the closed linear span of* $\{r_i(s) r_j(t) \mid i, j = 1, 2, \ldots\}$ *in* $L_p(\mathrm{d}s, \mathrm{d}t)$; *we denote its norm by* $\|P\|_p$.

Note that part (ii) is an easy consequence of (i).

PROPOSITION 2.1: *Let E be a Banach space with l.u.st., and assume that E does not contain l_∞^n's uniformly (resp. that both E and E^* do not contain l_∞^n's uniformly) then for any space G which is isometric to a subspace of E (resp. to a subspace of a quotient of E) one has*

$$\alpha(G) \leq \psi_E < \infty$$

where ψ_E is a constant depending only on E.

PROOF: We use the following notations: whenever $1 \leq \alpha < \infty$, we write simply L_α for $L_\alpha([0,1]\times[0,1],\ \mathrm{d}s\,\mathrm{d}t)$. If $(s,t)\to Z(s,t)$ is a Banach space valued random variable on $[0,1]\times[0,1]$, we write $\|Z\|_\alpha$ for $(\int\|Z(s,t)\|^\alpha\,\mathrm{d}s\,\mathrm{d}t)^{1/\alpha}$. Lastly, the function $(s,t)\to r_i(s)r_j(t)$ will be denoted $r_i\otimes r_j$.

Assume first that $E\not\supset l_\infty^n$'s uniformly so that we may assume that E satisfies property I in theorem 1.1.

Let $\{x_{ij}\}\subset E$ and $\{\epsilon_{ij}\}$ be as in definition 2.1. Throughout the proof, Σ will mean: $\Sigma_{i=1}^n\Sigma_{j=1}^n$.

We define $X(s,t)=\Sigma\,\epsilon_{ij}x_{ij}r_i(s)r_j(t)$, $Y(s,t)=\Sigma\,x_{ij}r_i(s)r_j(t)$. Consider the operators $u:E^*\to L_q$, $v:E^*\to L_1$ defined respectively by

$$\forall\xi\in E^*,\quad u(\xi)=\langle\xi,X\rangle,\quad v(\xi)=\langle\xi,Y\rangle.$$

By lemma 2.1(i) we have:

$$\forall\xi\in E^*\quad \|u(\xi)\|_q\le K(2,q)K(1,2)\|v(\xi)\|_1$$

therefore: $$\pi_1(u)\le K(2,q)K(1,2)\pi_1(v)$$

hence by I): $$\pi_q({}^t u)\le CK(2,q)K(1,2)\pi_1(v)$$

which gives by (1.1):

$$\|X\|_2\le\|X\|_q\le CK(2,q)K(1,2)\|Y\|_1\le CK(2,q)K(1,2)\|Y\|_2.$$

Since this is true for any (ϵ_{ij}), we conclude that $\alpha(E)\le CK(2,q)K(1,2)$. Moreover, if G is a subspace of E, it is clear that $\alpha(G)\le\alpha(E)$.

Now assume that both E and E^* do not contain l_∞^n's uniformly and therefore that E satisfies property II in theorem 1.1. Let F be a quotient of E and let σ denote the quotient map $E\to F$. Let $\{x_{ij}\}$ be elements of F and let ϵ_{ij} be as in definition 2.1. We define X, Y as above and $u:F^*\to L_q$ $v:F^*\to L_p$ by: $\forall\eta\in F^*$ $u(\eta)=\langle\eta,X\rangle$, $v(\eta)=\langle\eta,Y\rangle$.

By an easy lifting argument, we see that, given $\epsilon>0$, there exists an E-valued random variable $\tilde{Y}$ on $[0,1]\times[0,1]$ such that $\|\tilde{Y}\|_p\le\|Y\|_p+\epsilon$ and $\sigma(\tilde{Y})=Y$. Clearly, v extends to an operator $\tilde{v}:E^*\to L_p$ defined by: $\forall\xi\in E^*$ $\tilde{v}(\xi)=\langle\xi,\tilde{Y}\rangle$.

Consider the operator $w:L_p\to L_q$ defined by

$$\forall\varphi\in L_p\qquad w(\varphi)=\sum\epsilon_{ij}\langle r_i\otimes r_j,\ \varphi\rangle r_i\otimes r_j.$$

It follows from lemma 2.1 that $\|w\| \leq \|P\|_p K(p, 2) K(2, q)$; moreover, we clearly have $u = wv$. Using property II, we get:

$$\pi_q({}^t u) \leq \pi_q({}^t(w\tilde{v})) \leq C\pi_p(w\tilde{v}) \leq C\|w\|\pi_p(\tilde{v})$$

hence, by (1.1):

$$\|X\|_2 \leq \|X\|_q \leq \pi_q({}^t u) \leq C\|w\|\pi_p(\tilde{v}) \leq C\|w\|(\|Y\|_p + \epsilon) \leq C\|w\|(\|Y\|_2 + \epsilon).$$

Finally, we obtain:

$$\alpha(F) \leq C\|w\| \leq C\|P\|_p K(p, 2) K(2, q).$$

In conclusion, if G is a subspace of F, then $\alpha(G) \leq \alpha(F)$. q.e.d.

For the sake of generality, we introduce the

DEFINITION 2.2: *A finite or infinite matrix of elements (x_{ij}) of a Banach space will be called $\otimes$-unconditional if there exists a constant K such that*

$$\frac{1}{K}\left\|\sum \alpha_{ij} x_{ij}\right\| \leq \left\|\sum a_{ij}\epsilon'_i\epsilon''_j x_{ij}\right\| \leq K\left\|\sum \alpha_{ij} x_{ij}\right\| \tag{2.3}$$

for arbitrary matrices of scalars (α_{ij}) with finitely many non zero entries and arbitrary sequences of signs (ϵ'_n), (ϵ''_n) in $\{-1, +1\}^{\mathbb{N}}$. We will denote $K\{x_{ij}\}$ the smallest constant K for which 2.3 holds. For example, if E and F are two spaces with unconditional bases (e_n) and (f_n) respectively then $\{e_i \otimes f_j \mid i, j \in \mathbb{N}\}$ is a $\otimes$-unconditional basis of $E \hat{\otimes}_\lambda F$ for any tensor norm λ.

REMARK 2.2: It is worthwhile to point out that if $\alpha(E) < \infty$ then necessarily E does not contain l^n_∞'s uniformly; indeed, it is easily checked that

$$\alpha(l^n_1 \otimes_\epsilon l^n_1) \to \infty \qquad \text{when } n \to \infty$$

and $l^n_1 \otimes_\epsilon l^n_1$ is isometric (in the real case) to a subspace of $l^{2^{2n}}_\infty$.

The reason for the introduction of $\otimes$-unconditionality is

PROPOSITION 2.2: *Let (x_{ij}) be a $\otimes$-unconditional matrix of elements of E; then, if $\alpha(E) < \infty$, $(x_{ij})_{i,j}$ is an unconditional basic* *sequence and*:

$$\chi\{x_{ij}\} \leq (K\{x_{ij}\})^2 \alpha(E).$$

PROOF: By (2.3), we have with the notation of definition 2.1: on one hand

$$\frac{1}{K\{x_{ij}\}} \left\| \sum \alpha_{ij}\epsilon_{ij}x_{ij} \right\| \leq \left(\int \left\| \sum \alpha_{ij}\epsilon_{ij}r_i(s)r_j(t)x_{ij} \right\|^2 \mathrm{d}s\, \mathrm{d}t \right)^{1/2},$$

on the other hand

$$\left(\int \left\| \sum r_i(s)r_j(t)\alpha_{ij}x_{ij} \right\|^2 \mathrm{d}s\, \mathrm{d}t \right)^{1/2} \leq K\{x_{ij}\} \left\| \sum \alpha_{ij}x_{ij} \right\|.$$

We immediately conclude:

$$\left\| \sum \alpha_{ij}\epsilon_{ij}x_{ij} \right\| \leq K\{x_{ij}\}\alpha(E)K\{x_{ij}\} \left\| \sum \alpha_{ij}x_{ij} \right\|$$

which proves the assertion of the proposition.

A norm λ on $l_2 \otimes l_2$ is called unitarily invariant if $\lambda(\Sigma\, ux_i \otimes vy_i) = \lambda(\Sigma\, x_i \otimes y_i)$ whenever u and v are unitary operators on l_2 and (x_i) and (y_i) are arbitrary finite sequences of elements of l_2.

In [11], Lewis proved that, for such norms λ, $l_2 \hat{\otimes}_\lambda l_2$ has l.u.st. if and only if λ is equivalent to the Hilbert Schmidt norm. The first part of the next theorem extends his result in certain cases.

THEOREM 2.1: *Let E be a space with l.u.st.; assume that E does not contain l_∞^n's uniformly (resp. that both E and E^* do not contain l_∞^n's uniformly).*

A. *Let λ be a unitarily invariant crossnorm on $l_2 \otimes l_2$, if $l_2 \hat{\otimes}_\lambda l_2$ is isomorphic to a subspace of E (resp. to a quotient of a subspace of E) then necessarily λ is equivalent to the Hilbert Schmidt norm.*

B. *In particular, this happens for the space c_p only if $p = 2$.*

C. *Moreover, there exists a constant $\delta_E > 0$ depending only on E such that the shortest distance Δ_n^p of the space $c_p(l_2^n)$ – the Schatten p-class on n-dimensional Hilbert space – to any subspace of E (resp. to a quotient of a subspace of E) satisfies*:

$$\forall n \in \mathbb{N} \qquad \Delta_n^p \geq \delta_E n^{|(1/p)-(1/2)|}.$$

PROOF: A. By proposition 2.1 we can assume that $\alpha(l_2 \hat{\otimes}_\lambda l_2) < \infty$. Let (e_n) be the canonical basis of l_2. Clearly, $\{e_i \otimes e_j\}$ is $\otimes$-unconditional in $l_2 \hat{\otimes}_\lambda l_2$, therefore by proposition 2.2, $\{e_i \otimes e_j\}$ is an unconditional basis of $l_2 \hat{\otimes}_\lambda l_2$, and this happens only if λ is equivalent to the Hilbert Schmidt norm (cf. theorem 2.2 in [10]).

B. Clearly follows from A.

C. The proof of theorem 2.2 in [10] actually shows that if χ_n^p denotes the unconditionality constant of $\{e_i \otimes e_j \mid i, j = 1, 2, \ldots, n\}$ in $c_p(l_2^n)$ then:

$$2\chi_n^p \geq n^{|(1/p)-(1/2)|}.$$

By proposition 2.1 we have: $\alpha(c_p(l_2^n)) \leq \Delta_n^p \psi_E$ hence by proposition 2.2:

$$\chi_n^p \leq \Delta_n^p \psi_E$$

and we conclude that

$$\forall n \in \mathbb{N} \qquad \Delta_n^p \geq (1/2\psi_E) n^{|(1/p)-(1/2)|}. \qquad \text{q.e.d.}$$

REMARK 2.3: Part C above when $E = L_p$ completes the proof of a conjecture (rather an 'expectation') of Mac-Carthy in [14]; the case when $1 \leq p \leq 2$ was already settled in [5]. Obviously $d(c_p(l_2^n), c_2(l_2^n)) \leq n^{|(1/p)-(1/2)|}$; therefore, by Dvoretzky's theorem [16], we have: $\Delta_n^p \leq n^{|(1/p)-(1/2)|}$ when the dimension of E is infinite.

REMARK 2.4: It should be clear to the reader that the method of §2 applies as well to spaces of operators between banach spaces with unconditional bases. For example: if E is as in theorem 2.1 above, if F and G are spaces with unconditional bases $(f_n)(g_n)$ respectively and if λ is a tensor norm on $F \otimes G$, then $F \hat{\otimes}_\lambda G$ is isomorphic to a subspace of E (resp. a quotient of a subspace of E) only if $\{f_n \otimes g_k \mid n, k = 1, 2, \ldots\}$ is already an unconditional basis of $F \hat{\otimes}_\lambda G$. Actually, all that is required on λ is that $\{f_n \otimes g_k\}$ be $\otimes$-unconditional in $F \hat{\otimes}_\lambda G$.

REMARK 2.5: Maurey and Rosenthal have constructed rather recently [13], the first example of a uniformly convex space which does not embed in a uniformly convex space with unconditional basis (See also [18]). Our theorem shows that if $1 < p \neq 2 < \infty$, the space c_p is an example of the same nature, with even a stronger property: subspace can be replaced by quotient of a subspace.

REMARK 2.6: Finally, we indicate how to deduce some of the results of [5] from the preceding ones. If $1 < p < \infty$, it is well known that c_p is uniformly convex, therefore does not contain l_∞^n's uniformly; hence it follows immediately from theorem 2.1 that if $1 < p \neq 2 < \infty$ the space c_p itself fails to have l.u.st. This can be extended to the case $p = 1$ by using proposition 3.2 in [15] which shows that c_1 does not contain l_∞^n's uniformly. Lastly, since $c_\infty^* = c_1$, we also obtain by remark 1.1 that c_∞ fails to have l.u.st. although it does contain l_∞^n's uniformly.

§3. Applications to invariant subspaces of L_p or C

In this section, we again work with the following property of a Banach space E which we call property G-L: Whenever A is a 1-absolutely summing operator on E^*, its adjoint factors through L_∞.

In addition to spaces with l.u.st. (see lemma 1.1), Gordon and Lewis pointed in [5] that any subspace of L_1 and any quotient of a $C(K)$ space have the above property. One can slightly extend this remark: *we will say that a Banach space E has property $\mathcal{M}$ if: 'any 2-absolutely summing operator with domain space E is actually 1-absolutely summing. Then, we can prove easily*

LEMMA 3.1: *Assume that L is a Banach space with l.u.st. verifying property $\mathcal{M}$, then any subspace E of L as well as any quotient of L^* has the above property G-L.*

PROOF: Let A be a 1-absolutely summing operator on E; a fortiori A is 2-absolutely summing. By a well known property of such operators (cf. [22]) A extends to a 2- absolutely summing operator $\tilde{A}$ defined on the whole of L. By our assumption on L, we know that $\tilde{A}$ must be also 1- absolutely summing; since L has l.u.st., lemma 1.1 ensures that $\gamma_1(\tilde{A}) < \infty$, therefore $\gamma_1(A) < \infty$, and we conclude that ${}^tA$ factors through L_∞. The case of a quotient of L^* is treated similarly.

REMARK 3.1: By a theorem of Grothendieck, the space L_1 verifies property $\mathcal{M}$. In [12], it is proved that any space "of cotype 2" verifies property $\mathcal{M}$.

We will need some standard

NOTATIONS: Throughout this section we consider a compact abelian group G with dual group Γ. m will denote the normalised Haar measure on G. For any χ in Γ and any f in $L_1(m)$, we write $\hat{f}(\chi)$ for $\int f(g)\bar{\chi}(g)m(\mathrm{d}g)$.

If $F \subset \Gamma$ is given, if $1 \leq p \leq \infty$, we will denote L_F^p (resp. C_F) the subspace of $L_p(m)$ (resp. $C(G)$) consisting of those f in $L_p(m)$ (resp. $C(G)$) such that: $\forall \chi \notin F$, $\hat{f}(\chi) = 0$.

For any set S, $l_p(S)$ denotes the space of those x in $\mathbb{C}^S$ for which $\Sigma_{s\in S}|x(s)|^p < \infty$, equipped with the norm $x \to (\Sigma_{s\in S}|x(s)|^p)^{1/p}$. Recall that, if $p > 1$, a subset F of Γ is called a $\Lambda(p)$ set if there exists a constant λ such that:

$$\forall f \in L_F^p \qquad \|f\|_p \leq \lambda \|f\|_1;$$

the smallest among such constants λ is denoted $\Lambda_p(F)$.

Also, recall that $F \subset \Gamma$ is called a Sidon set if there exists a constant λ such that:

$$\forall f \in C_F \qquad \sum_{\chi \in F} |\hat{f}(\chi)| \leq \lambda \|f\|_\infty.$$

These notions are related to the notion of unconditionality via the following simple and well known fact: If $p > 2$, a subset F of Γ is a $\Lambda(p)$-set (resp. a Sidon set) if and only if $\{\chi \mid \chi \in F\}$ forms an unconditional basis of L_F^p (resp. C_F); in that case L_F^p (resp. C_F) is actually isomorphic to $l_2(F)$ (resp. $l_1(F)$).

Our main result is the following.

THEOREM 3.1: *Let $F \subset \Gamma$ be a $\Lambda(2)$ set. Assume $2 < p \leq \infty$. If L_F^p (resp. C_F) has the above property G-L, then $\{\chi \mid \chi \in F\}$ is an unconditional basis of L_F^p (resp. C_F).*

PROOF: Assume that $2 < p < \infty$ and set $X = L_F^p$ or C_F. By our assumption on X (and the closed graph theorem), there is a constant K such that $\gamma_\infty({}^tA) \leq K\pi_1(A)$ whenever A is a 1-absolutely summing operator on X^*.

Let $\{\epsilon_\chi \mid \chi \in F\}$ be an arbitrary family indexed by F such that $|\epsilon_\chi| = 0$ or 1 with only finitely many non zero terms. For g in G, we denote T_g the translation operator defined by:

$$\forall f \in L_p(m), \forall g' \in G, (T_g f)(g') = f(gg').$$

Fix x in X and ξ in X^* and let x_0 be the element of X defined by:

$$\forall \chi \in \Gamma, \quad \hat{x}_0(\chi) = \epsilon_\chi \hat{x}(\chi).$$

We claim that:

$$|\langle \xi, x_0 \rangle| \leq K\Lambda_2(F)^2 \|x\|_X \|\xi\|_X.$$

Clearly this claim implies that the unconditionality constant of $\{\chi \mid \chi \in F\}$ in X is majorized by $K\Lambda_2(F)^2$ and this settles theorem 3.1. We now proceed to prove the above claim: we introduce an operator $A : X^* \to L^2_F$ defined by:

$$\forall \eta \in X^* \qquad \forall \chi \in \Gamma \qquad \widehat{A\eta}(\chi) = \hat{x}_0(\chi)\langle \eta, \chi \rangle.$$

Similarly we introduce $B : X \to L^2_F$ defined by:

$$\forall y \in X \qquad \forall \chi \in \Gamma \qquad \widehat{By}(\chi) = \hat{y}(\chi)\langle \xi, \chi \rangle.$$

Now fix η in X^*; if ψ is defined by: $\forall g \in G \; \psi(g) = \langle \eta, T_g x \rangle$, it is easy to see that $\psi \in L^\infty_F$ and:

$$\forall \chi \in \Gamma \qquad \hat{\psi}(\chi) = \hat{x}(\chi)\langle \eta, \chi \rangle.$$

Therefore we can write:

$$\begin{aligned}\|A\eta\|_2 &\leq \Big(\sum_{\chi \in F} |\hat{x}(\chi)\langle \eta, \chi \rangle|^2\Big)^{1/2} \\ &\leq \Lambda_2(F) \int |\langle \eta, T_g x \rangle| m(\mathrm{d}g)\end{aligned}$$

and consequently:

$$\pi_1(A) \leq \Lambda_2(F) \int \|T_g x\|_X m(\mathrm{d}g) = \Lambda_2(F)\|x\|_X.$$

Similarly, we have: $\forall y \in X$

$$\begin{aligned}\|By\|_2 &\leq \Lambda_2(F) \int |\langle \xi, T_g y \rangle| m(\mathrm{d}g) \\ &\leq \Lambda_2(F) \int |\langle {}^t T_g \xi, y \rangle| m(\mathrm{d}g), \text{ hence:}\end{aligned}$$

$$\pi_1(B) \leq \Lambda_2(F) \int \|{}^t T_g \xi\|_{X^*} m(\mathrm{d}g) = \Lambda_2(F)\|\xi\|_{X^*}.$$

Clearly, A is the adjoint of an operator ${}^tA: L_F^2 \to X$ (actually the convolution by x_0), and the operator $B \circ {}^tA: L_F^2 \to L_F^2$ is simply a diagonal multiplication:

$$\forall \chi \in F \qquad B \circ {}^tA(\chi) = \langle \xi, \chi \rangle \hat{x}_0(\chi)\chi.$$

By a well known result ([22]), the nuclear norm, $\nu_1(B \circ {}^tA)$, of $B \circ {}^tA$ satisfies:

$$\nu_1(B \circ {}^tA) \leq \gamma_\infty({}^tA)\pi_1(B).$$

Therefore, we have:

$$|\mathrm{tr}(B \circ {}^tA)| \leq \nu_1(B \circ {}^tA) \leq K\pi_1(A)\pi_1(B)$$

and this gives using the preceding estimates:

$$\Big|\sum_{\chi \in F} \hat{x}_0(\chi)\langle \xi, \chi \rangle\Big| \leq K\Lambda_2(F)^2 \|x\|_X \|\xi\|_{X^*}$$

which proves the above claim. The case $X = L_F^\infty$ is treated with almost the same proof.

To summarize the non trivial consequences of theorem 3.1 let us state:

THEOREM 3.2: *Let $F \subset \Gamma$ be a $\Lambda(2)$ set and assume $p > 2$. The following are equivalent*:

(i) *F is a $\Lambda(p)$ set.*

(ii) *L_F^p has l.u.st.*

(iii) *$(L_F^p)^*$ embeds in a space with l.u.st. and with property $\mathcal{M}$.*

PROOF: i $\Rightarrow$ ii, i $\Rightarrow$ iii are trivial (it is well known that a Hilbert space satisfies property $\mathcal{M}$). The converse implications all follow from theorem 3.1 combined with lemmas 1.1 and 3.1.

As a corollary, we obtain the announced result: for each p with $4 < p < \infty$, there are (invariant) subspaces of $L_p(T)$ which fail to have l.u.st.; indeed, Rudin constructed in [23] a $\Lambda(2)$ set $R \subset \mathbb{Z}$ which is not a $\Lambda(p)$ set if $p > 4$; by the above theorem, L_R^p fails to have l.u.st. for $p > 4$.

It is generally conjectured (following [23]) that there are $\Lambda(2)$ sets which are not $\Lambda(p)$ sets if $p > 2$; such a conjecture implies the existence – for $2 < p \leq 4$ – of a subspace of L_p failing to have l.u.st. (which I do not see how to prove).

Theorem 3.1 allows us to state:

THEOREM 3.3: *Let $F \subset \Gamma$ be a $\Lambda(2)$ set. The following are equivalent*:

(i) *F is a Sidon set*

(ii) (*resp.* ii′) *C_F (resp. L^∞_F) has l.u.st.*

(iii) (*resp.* iii′) *C_F (resp. L^∞_F) embeds in a space L with l.u.st. and with property $\mathcal{M}$.*

REMARK 3.2: Assume again that F is a $\Lambda(2)$ set. If C_F or L^∞_F is isomorphic to a quotient of L^*, with L as in lemma 3.1, then F must be a *finite* set. Indeed, we deduce from theorem 3.1 and lemma 3.1 that F must be a Sidon set, hence we must have $C_F \cong L^\infty_F \cong l_1(F)$, but $l_1(F)$ is isomorphic to a quotient of such a space L^* only if F is finite.

Varopoulos proved in [24] that if C_F or L^∞_F is a $\mathcal{L}_1$ space then F is a Sidon set. To extend this result, we will need the following well known fact:

LEMMA 3.2: *If C_F or L^∞_F has property $\mathcal{M}$, then F is a $\Lambda(2)$ set.*

PROOF: The natural injections $C_F \to L^2_F$ and $L^\infty_F \to L^2_F$ are obviously 2- absolutely summing. Since $C_F \to L^2_F$ is a restriction of $L^\infty_F \to L^2_F$, we conclude in both cases that $j: C_F \to L^2_F$ is 1- absolutely summing. By Pietsch's factorisation theorem ([19], prop. 3.1), there exists a probability measure μ on G such that:

$$\forall f \in C_F \qquad \|f\|_2 \leq \pi_1(j) \int |f(g')| \mu(\mathrm{d}g').$$

Therefore

$$\forall g \in G \qquad \|f\|_2 \leq \pi_1(j) \int |f(gg')| \mu(\mathrm{d}g'),$$

and integrating with respect to $m(\mathrm{d}g)$ we obtain by the invariance of m: $\|f\|_2 \leq \pi_1(j) \int |f(g)| m(\mathrm{d}g)$, which proves that F is a $\Lambda(2)$-set.

A typical space with property $\mathcal{M}$ is L_1 (this is a form of Grothendieck's theorem cf. [19]); since $\mathcal{L}_1$ spaces embed in L_1 (see [19]), the following generalises Varopoulos's result:

COROLLARY 3.1: *If C_F or L_F^∞ embeds in a space with l.u.st. and with property $\mathcal{M}$ then F is a Sidon set.*

PROOF: Since property $\mathcal{M}$ is clearly inherited by subspaces, we deduce from lemma 3.2 that F is necessarily a $\Lambda(2)$ set, and we conclude by theorem 3.3.

In conclusion, we wish to present the following questions a priori in increasing order of difficulty:

QUESTION 1: Is there a space verifying property G-L and yet failing l.u.st.?

Is there a subspace of a space with l.u.st. and with property $\mathcal{M}$ which fails l.u.st.?

Is there a subspace of L_1 failing l.u.st.?

QUESTION 2: If a Banach space X is not isomorphic to a Hilbert space, does there exist a space which is finitely representable (cf. [17]) in X, and which fails l.u.st.? Does there even exist a subspace of X failing l.u.st.?

REFERENCES

[1] J. ARAZY and J. LINDENSTRAUSS: Some linear topological properties of the spaces c_p of operators on Hilbert space. *Compositio Math. 30* (1975) 81–111.

[2] T. FIGIEL, W. JOHNSON: A uniformly convex space which contains no l_p. *Compositio Math 29* (1974) 179–190.

[3] T. FIGIEL, W. JOHNSON, L. TZAFRIRI: On Banach lattices and spaces having local unconditional structure, with applications to Lorentz function spaces. *Journal of Approximation theory 13* (1975) 395–412.

[4] I.C. GOHBERG, M.G. KREIN: *Introduction to the theory of linear non self adjoint operators.* Translations of Mathematical Monographs, 18, A.M.S. (1969).

[5] Y. GORDON, D. LEWIS: Absolutely summing operators and local unconditional structures. *Acta Math. 133* (1974) 27–48.

[6] Y. GORDON, D. LEWIS: Banach ideals on Hilbert spaces. *Studia Math. 54* (1975) 161–172.

[7] J.L. KRIVINE: Théorèmes de factorisation dans les espaces réticulés. Séminaire Maurey-Schwartz 73-74-exp. XXII. Ecole Polytechnique, Paris.

[8] S. KWAPIEŃ: On a theorem of L. Schwartz and its applications to analysis. *Studia Math. 38* (1969) 193–201.

[9] S. KWAPIEŃ: On operators factorizable through L_p-space. *Bull. Soc. Math. France Mémoire, 31–32* (1972) 215–225.

[10] S. KWAPIEŃ, A. PEŁCZYŃSKI: The main triangle projection in matrix spaces and its applications. *Studia Math. 34* (1970) 43–68.

[11] D. LEWIS: An isomorphic characterization of the Schmidt class. *Composito Math 30* (1975) 293–297.

[12] B. MAUREY: Théorèmes de factorisation pour les opérateurs à valeurs dans un espace L_p. *Soc. Math. France Astérisque 11* (1974).
[13] B. MAUREY, H.P. ROSENTHAL: Normalized weakly null sequences with no unconditional subsequences. *Studia Math. 61* (1977) 77–98.
[14] C. MC-CARTHY: -c_p. *Israel J. Math. 5* (1967) 249–271.
[15] N. TOMCZAK-JAEGERMAN: The moduli of smoothness and convexity and the Rademacher averages of the trace classes S_p ($1 \leq p < \infty$). *Studia Math., 50* (1974) 163–182.
[16] A. DVORETZKY: *Some results on convex bodies and Banach spaces.* Proc. Symp. on Linear Spaces, Jerusalem (1961), 123–174.
[17] R.C. JAMES: Some self dual properties of normed linear spaces, *Ann. of Math. Studies 69* (1972).
[18] W.B. JOHNSON and L. TZAFRIRI: Some more Banach spaces which do not have l.u.st. *Houston Journal of Maths. 3* (1977) 55–60.
[19] J. LINDENSTRAUSS and A. PELCZYŃSKI: Absolutely summing operators in $\mathscr{L}_p$ spaces and their applications. *Studia Math. 29* (1968) 275–326.
[20] B. MAUREY et G. PISIER: Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach. *Studia Math. 58* (1976) 45–90.
[21] A. PELCZYŃSKI: Sur certaines propriétés isomorphiques nouvelles des espaces de Banach de fonctions holomorphes A et H^∞. *C.R. Acad. Sc. Paris A-279* (1974) 9–12.
[22] A. PIETSCH: Absolut p. summierende Abbildungen in normierten. Räumen. *Studia Math 28* (1967) 333–353.
[23] W. RUDIN: Trigonometric series with gaps. *Journal of Mathematics and Mechanics 9* (1960) 203–227.
[24] N.TH. VAROPOULOS: Sous espaces de $C(G)$ invariants par translation et de type $\mathscr{L}_1$. Séminaire Maurey-Schwartz 75–76. Exposé XII. Ecole Polytechnique. Paris.

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Added in proof

1° The result of Lindenstrauss mentioned in the introduction appears in the book "Classical Banach spaces" (Vol. I) by J. Lindenstrauss and L. Tzafriri (Ergebnisse der Matematik, band 92, Springer Verlag 1977) on page 51, theorem 1.g.5.

2° The results of §3 concerning Sidon sets have been extended by the author (cf. Ensembles de Sidon et espaces de cotype 2, Exposé N° 14, Séminaire sur la Géométrie des Espaces de Banach 1977–1978, Ecole Polytechnique, Palaiseau (France)).

3° There exist subspaces of L_p or l_p without l.u.st. for any $p > 2$ (and not only $p > 4$). This improvement was obtained independently and differently by Figiel, Kwapień and Pełczynski (cf. Sharp estimates for the constants of local unconditional structure of Minkovski spaces, Bull. Acad. Pol. Sci. to appear) and by the author (yet unpublished).