COMPOSITIO MATHEMATICA

RICHARD LAVER

A saturation property on ideals

Compositio Mathematica, tome 36, nº 3 (1978), p. 233-242

<http://www.numdam.org/item?id=CM_1978__36_3_233_0>

COMPOSITIO MATHEMATICA, Vol. 36, Fasc. 3, 1978, pag. 233–242
Sijthoff & Noordhoff International Publishers – Alphen aan den Rijn
Printed in the Netherlands

A SATURATION PROPERTY ON IDEALS

Richard Laver

It is a theorem of Kunen [6] is that if κ is a real valued measurable cardinal, then for every $\alpha < \omega_1$,

$$\kappa \to (\kappa, \alpha)^2.$$

In this paper we consider a saturation property of ideals on κ which implies this partition relation, as well as generalizations to cases where $\kappa > 2^{\aleph_0}$. Accounts of the results on saturated ideals are given in Solovay [14] and Kunen [7].

For κ a cardinal, a κ-ideal is an ideal $\mathcal{I} \subseteq \mathcal{P}(\kappa)$ which is nontrivial ($\{\alpha\} \in \mathcal{I}$, all $\alpha < \kappa$, and $\kappa \notin \mathcal{I}$) and κ-complete. Define a κ-ideal $\mathcal{I}$ to be (λ, μ, ν)-saturated if whenever $\mathcal{H} \subseteq \mathcal{P}(\kappa) - \mathcal{I}$, Card $\mathcal{H} = \lambda$, there is a $\mathcal{G} \subseteq \mathcal{H}$, Card $\mathcal{G} = \mu$, such that $\cap \mathcal{F} \notin \mathcal{I}$ for every $\mathcal{F} \subseteq \mathcal{G}$ with Card $\mathcal{F} \leq \nu$. Thus $\mathcal{I}$ is λ-saturated if and only if $\mathcal{I}$ is $(\lambda, 2, 2)$-saturated.

The abovementioned generalizations of the above partition relation are of the form

$$\text{for all } \alpha < \gamma^+ \quad \kappa \to (\kappa, \alpha)^2,$$

where γ is an arbitrary regular cardinal. This relation holds in strong fashion for a wide class of κ with $2^\gamma < \kappa$ (for example, the Erdös–Rado theorem [2] implies that $(2^\gamma)^+ \to ((2^\gamma)^+, \gamma^+)^2$). This relation fails in strong fashion if $2^\theta \geq \kappa$ for some $\theta < \gamma$ (Sierpinski's theorem ([13]) states that $2^\theta \not\to (\theta^+, \theta^+)^2$). Our interest will be when γ is the least cardinal β such that $2^\beta \geq \kappa$.

It is consistent in this situation that $\kappa \not\to (\kappa, \gamma + 2)^2$, namely, Hajnal [4] showed that if γ is regular and $2^\gamma = \gamma^+$ then $2^\gamma \not\to (2^\gamma, (\gamma : 2))^2$. Martin's axiom $+ 2^{\aleph_0} = \aleph_2$ implies $2^\omega \not\to (2^\omega, (\omega : 2))^2$ [9]. It is consistent that $2^{<\gamma} = \gamma$ and $2^\gamma \not\to (2^\gamma, (\gamma : 2))^2$, where 2^γ is the successor of an arbitrary regular cardinal (Baumgartner [1]), and where 2^γ is weakly inaccessible ([9], relative to the consistency of a Mahlo cardinal). In the last result, 2^γ will be high in the weak Mahlo hierarchy if the large cardinal in the ground model was high in the Mahlo hierarchy. An account of these theorems and of related results is given in [9].[1]

In section 1 it is shown that if there is a (κ, κ, γ)-saturated κ-ideal (whence κ is weakly inaccessible ([15]), in fact high in the weak Mahlo hierarchy [14], and $\gamma > \kappa$) and if $\beta^{<\gamma} < \kappa$, all $\beta < \kappa$, then for all $\alpha < \gamma^+$, $\kappa \to (\kappa, \alpha)^2$.

In section 2, assuming the consistency of the existence of a measurable cardinal, the consistency of the above situation on κ and γ is shown, where γ is an arbitrary regular cardinal and $2^\gamma \geq \kappa$. Namely, if $\mathcal{M}$ is a countable transitive model of ZFC in which γ is regular, $\gamma < \kappa$, and κ is measurable, then there is a forcing extension $\mathcal{M}[G]$ in which no new $<\gamma$-sequences of ordinals have been added, $2^{<\gamma} = \gamma$, $\beta^{<\gamma} < \kappa$, all $\beta < \kappa$, $2^\gamma \geq \kappa$ (both $2^\gamma = \kappa$, $2^\gamma > \kappa$ can be arranged) and there is a (κ, κ, γ)-saturated normal κ-ideal. The ideal will be γ^{++}-saturated but not γ^+-saturated in $\mathcal{M}[G]$; if GCH holds in $\mathcal{M}$ the ideal will be $(\lambda, \lambda, \gamma)$-saturated, for all regular $\lambda > \gamma^+$ in $\mathcal{M}[G]$.

For example, the consistency of the existence of a measurable cardinal implies the consistency that there exists a $(2^{\aleph_1}, 2^{\aleph_1}, \aleph_1)$-saturated $2^{\aleph_1}$-ideal and that $\beta < 2^{\aleph_1} \Rightarrow \beta^{\aleph_0} < 2^{\aleph_1}$; these properties imply that $2^{\omega_1} \to (2^{\omega_1}, \alpha)^2$ for all $\alpha < \omega_2$.

Section 1

Assume κ and γ are cardinals, $\mathcal{I}$ is a (κ, κ, γ)-saturated κ-ideal and $\beta^{<\gamma} < \kappa$, all $\beta < \kappa$. Let $\alpha < \gamma^+$ and suppose by induction that $\kappa \to (\kappa, \alpha')^2$, all $\alpha' < \alpha$. Given $[\kappa]^2 = R \cup G$, we show that either there is an $X \in [\kappa]^\kappa$ with $[X]^2 \subset R$ or there is an $A \in [\kappa]^\alpha$ with $[A]^2 \subset G$.

Suppose first that there is a $Z \in \mathcal{P}(\kappa) - \mathcal{I}$ such that for every $z \in Z$, $\{w \in Z: \{z, w\} \in G\} \in \mathcal{I}$. Then an $X \in [Z]^\kappa$ with $[X]^2 \subseteq R$ may be constructed in κ steps using the assumption and the κ-completeness of $\mathcal{I}$.

For $Z \in \mathcal{P}(\kappa) - \mathcal{I}$, $y \in Z$, let

$$Z_y = \{z \in Z: \{y, z\} \in G\}.$$

Lest the above case obtain we may assume that for every $Z \in \mathcal{P}(\kappa) - \mathcal{I}$,

$$\mathrm{Card}\{y \in Z: Z_y \notin \mathcal{I}\} = \kappa.$$

[1] Correction to [9]: on page 1029, lines 8 and 9, change "cf $\kappa > \omega$" to "κ regular $> \omega$".

We claim

(*) If $Z \in \mathscr{P}(\kappa) - \mathscr{I}$, $\alpha' < \alpha$, then there is an $A' \in [Z]^{\alpha'}$ and a $Z' \subseteq Z$, $Z' \notin \mathscr{I}$, such that $[A']^2 \subseteq G$ and $A' \otimes Z' \subseteq G$.

Namely, let $\{y \in Z: Z_y \notin \mathscr{I}\} = Y \in [Z]^\kappa$. By (κ, κ, γ)-saturation there is a $Y' \in [Y]^\kappa$ such that if $B \subseteq Y'$ and Card $B = \gamma$ then $\bigcap_{y \in B} Z_y \notin \mathscr{I}$. By the induction hypothesis there is an $A' \in [Y']^{\alpha'}$ with $[A']^2 \subseteq G$. Then $Z' = \bigcap_{y \in A'} Z_y$ satisfies (*).

Let $\alpha = \Sigma_{0<\sigma<\delta}\, \alpha_\sigma$, where $\alpha_\sigma < \alpha$ and $\delta \leq \gamma$. We will construct a well founded tree $(T, <_T)$ of height δ. T_σ will be the σth level of T, and $T_{<\sigma}$, $T_{\leq\sigma}$ will be $\bigcup_{\theta<\sigma} T_\theta$, $\bigcup_{\theta\leq\sigma} T_\theta$. To each node t of T will be assigned a pair $(Z(t), A(t))$. For t the root node of T, $Z(t) = \kappa$, $A(t) = \phi$. In general,

(i) $Z(t) \notin \mathscr{I}$, $Z(t) \subseteq \bigcap_{u <_T t} Z(u)$, and $Z(t) \cap Z(u) = \phi$ ($t \neq u$, level t = level u),

(ii) (a) For each $u \in T$, if S_u is the set of immediate successors of u, then $(Z(u) - \bigcup_{t \in S_u} Z(t)) \in \mathscr{I}$.
(b) For each $\sigma < \delta$, $(\kappa - \bigcup_{t \in T_\sigma} Z(t)) \in \mathscr{I}$.

(iii) If $0 < \sigma < \delta$, $t \in T_\sigma$, then $A(t) \subseteq \bigcap_{u <_T t} Z(u)$, $tp(A(t)) = \alpha_\sigma$, $\min A(t) > \sup A(u)$ $(u <_T t)$, $[A(t)]^2 \subseteq G$, and $A(t) \otimes Z(t) \subseteq G$.

Suppose $T_{\leq\sigma}$ has been defined so that (i), (ii), and (iii) hold. Given u on level σ, pick a set S_u of immediate successors of u, together with pairs $(Z(t), A(t))$, $t \in S_u$, such that $\{Z(t), A(t)): t \in S_u\}$ forms a maximal family satisfying (i) and (iii). That $S_u \neq \phi$, and more generally that $\{Z(t): t \in S_u\}$ satisfies (ii)(a), follows from (*): if $Z = (Z(u) - \bigcup_{t \in S_u} Z(t)) \notin \mathscr{I}$, then by (*) there would be an $A \in [Z]^{\alpha_{\sigma+1}}$ and a $Z' \subseteq Z$, $Z' \notin \mathscr{I}$, with $\min A > \sup A(u)$, $[A]^2 \subseteq G$, $A \otimes Z' \subseteq G$; this would contradict the maximality of $\{(Z(t), A(t)): t \in S_u\}$. By (i), (ii) and the κ-saturation of $\mathscr{I}$, T_σ has power $<\kappa$. Thus (ii)(b) holds for level $\sigma + 1$ by κ-completeness of $\mathscr{I}$. So $T_{\leq\sigma+1}$ satisfies (i)–(iii).

Suppose $\lambda < \delta$ is a limit ordinal and $T_{<\lambda}$ has been defined. For each maximal branch b through $T_{<\lambda}$, let $Z(b) = \bigcap_{t \in b} Z(t)$. Let

$$B = \{b: Z(b) \notin \mathscr{I}\}.$$

Letting $\beta < \kappa$ be the cardinality of $T_{<\lambda}$, there are at most $\beta^\lambda \leq \beta^{<\gamma} < \kappa$ branches through $T_{<\lambda}$; this fact, with (i), (ii) and the κ-completeness of $\mathscr{I}$, implies that $(\kappa - \bigcup_{b \in B} Z(b)) \in \mathscr{I}$. We may then, for each $b \in B$, apply (*) to $Z(b)$ as in the $\sigma + 1$ case, choosing a set S_b of successors of b on level λ and pairs $(Z(t), A(t))$, $t \in S_b$, so that $T_{\leq\lambda}$ satisfies (i)–(iii).

This completes the construction of T. Since $\delta < \kappa$ there is, by κ-completeness of $\mathscr{I}$, a $y \in \bigcap_{\sigma<\delta} \bigcup_{t\in T_\sigma} Z(t)$. Let

$$b = \{t\colon y \in Z(t)\};$$

b is a branch through T of length δ. Let

$$A = \bigcup_{t\in b} A(t).$$

By construction, $tp(A) = \alpha$ and $[A]^2 \subseteq G$, completing the proof.

We note that the tree isn't needed in the case $\gamma = \aleph_0$. If $\mathscr{I}$ is normal then the X with $[X]^2 \subset R$ may be further required in the theorem to be in $\mathscr{P}(\kappa) - \mathscr{I}$.

Section 2

To force to get a model in which there is a (κ, κ, γ)-saturated ideal on κ, $\gamma < \kappa \leq 2^\gamma$, we blow up 2^γ with conditions having a "Δ-system" property such that if $\alpha < \gamma^{+\mathscr{M}[G]}$ the union of α conditions in the Δ-system is still a condition. There is more than one way to do this (see remarks at the end); we do it the following way here.

Let $\mathscr{M}$ be a countable transitive model of ZFC in which γ is a regular cardinal, $\kappa > \gamma$ is a measurable cardinal, and $\rho \geq \kappa$ satisfies $\rho^\gamma = \rho$. Let $\mathscr{I}$ be the ideal dual to a normal ultrafilter on κ. We define a partial ordering $\mathscr{P}$ in $\mathscr{M}$ such that if G is $(\mathscr{M}, \mathscr{P})$-generic then for some θ with $\gamma < \theta < \kappa$, $\mathrm{Cards}^{\mathscr{M}[G]} = \mathrm{Cards}^{\mathscr{M}} \cap ([0, \gamma] \cup [\theta, \infty))$, and in $\mathscr{M}[G]$, $2^\gamma = \rho$, $\beta^{<\gamma} < \kappa$, all $\beta < \kappa$, and

$$\bar{\mathscr{I}} =_{\mathrm{df}} \{X\colon \exists\, Y \in \mathscr{I} \quad X \subseteq Y\}$$

is a (κ, κ, γ)-saturated κ-ideal.

Remarks: $\mathscr{P}$ will have the σcc, some $\sigma < \kappa$. Prikry [11] proved that if in a ground model, σ, κ are cardinals, σ regular $< \kappa$, and $\mathscr{I}$ is a σ-saturated κ-ideal, then in any σcc forcing extension, $\bar{\mathscr{I}}$ will be a σ-saturated κ-ideal. Wagon ([16], Theorem 3.5), proved that conversely, if in a ground model, σ, κ are cardinals, with $\sigma < \kappa$, and in a σcc extension $\mathscr{J}$ is a κ-saturated κ-ideal, then for some $X \in \mathscr{J}$ and some κ-saturated κ-ideal $\mathscr{I}$ in the ground, $\mathscr{J} \cap \mathscr{P}(\kappa - X) =$

$\bar{\mathscr{I}} \cap \mathscr{P}(\kappa - X)$. The Kunen–Paris method [8] for constructing saturated ideals reduces in this case to Prikry's method. The proof below can be cast in terms of the Kunen–Paris definition of $\mathscr{I}$ ($X \in \mathscr{I} \Leftrightarrow \exists p \in G_p jp H_{jp} \kappa \notin jX$, where j is the elementary embedding associated with $\mathscr{I}$).

$\mathscr{P}$ is defined as follows. Let $\mathscr{Q}$ be the standard partial ordering $(2)^{<\gamma}$ for adding a subset of γ with conditions of power $<\gamma$. Fix a cardinal θ, $\gamma < \theta < \kappa$, such that

(*) If C is a closed unbounded subset of θ and $f\colon [C]^2 \to \theta$ satisfies $f(\{\alpha, \beta\}) < \min\{\alpha, \beta\}$, all $\alpha, \beta \in C$, then for some $B \in [C]^{(2^{<\gamma})^+}$, $\mathrm{Card}(f''[B]^2) = 1$.

(*) As a second order property true of κ, so there is a $\theta < \kappa$ satisfying (*)–(*) is equivalent to θ being subtle (Jensen–Kunen [5]) and $> \gamma$. Then let

$$\mathscr{P} = \{f\colon \mathrm{dom}\, f \in [\rho]^{<\theta}, \mathrm{rng}\, f \subseteq \mathscr{Q}\}$$

and order $\mathscr{P}$ by

$$f \leq g \text{ iff } \mathrm{dom}\, f \subseteq \mathrm{dom}\, g \text{ and } f(\alpha) \leq g(\alpha), \text{ all } \alpha \in \mathrm{dom}\, f.$$

$\mathscr{P}$ is γ-closed, so forcing with $\mathscr{P}$ adds no new $<\gamma$-sequences of ordinals and thus preserves all cardinals $\leq \gamma$. The standard application of the Marczewski–Erdös–Rado lemma on Δ-systems [10], [3], yields that $\mathscr{P}$ has the θ^+ chain condition, so forcing with $\mathscr{P}$ preserves cardinals $\geq \theta^+$. These two facts, plus the inaccessibility of κ, imply that if G is $(\mathscr{M}, \mathscr{P})$-generic then in $\mathscr{M}[G]$, $\beta^{<\gamma} < \kappa$ for all $\beta < \kappa$. In $\mathscr{M}[G]$, $2^\gamma = \rho$ by the usual genericity arguments. The last fact about cardinalities to show, then, is

LEMMA: $\gamma^{+\mathscr{M}[G]} = \theta$.

PROOF:
(1) $\gamma^{+\mathscr{M}[G]} \geq \theta$.

It suffices to show for each regular β in $\mathscr{M}$, with $\gamma < \beta < \theta$, that there is an onto map

$$j\colon \gamma \to 2^\beta$$

in $\mathscr{M}[\{g \restriction \beta\colon g \in G\}]$. Working in $\mathscr{M}$, let

$$\mathscr{F} = \{f \in \mathscr{P}\colon \mathrm{dom}\, f \subset \beta, \mathrm{Card}(\beta - \mathrm{dom}\, f) = \beta, \mathrm{rng}\, \mathrm{f} \subseteq \{0, 1\}\}$$

Using a well ordering of $\mathscr{F}$, assign to each $g \in (2)^\beta$ an ordinal $\sigma_g < 2^\beta$

so that for any $f \in \mathscr{F}$ and any $\sigma < 2^\beta$ there is a $g \in (2)^\beta$ with $\sigma_g = \sigma$ and $g \restriction \operatorname{dom} f = f$. In $\mathcal{M}[G]$ define $j\colon \gamma \to 2^\beta$ by $j(\delta) = \sigma_g$, where, for $\alpha < \beta$, $g(\alpha) = G_\alpha(\delta)$. To show j is onto, let $p \in \mathscr{P}$, $\sigma < 2^\beta$. Since β is regular $> \gamma$ there must be a $\delta < \gamma$ such that, letting $f = \{\langle \alpha, (p(\alpha))(\delta)\rangle\colon \alpha < \beta \text{ and } \delta \in \operatorname{dom} p(\alpha)\}$, we have $\operatorname{Card}(\beta - \operatorname{dom} f) = \beta$. Choose $g \in (2)^\beta$ with $\sigma_g = \sigma$ and $g \restriction \operatorname{dom} f = f$, and extend p to a condition q such that $(q(\alpha))(\delta) = g(\delta)$, all $\alpha < \beta$. Thus $q \Vdash j(\delta) = \sigma$.

(2) $\gamma^{+\mathcal{M}[G]} \leq \theta$.

Suppose $p \in \mathscr{P}$, $p \Vdash j\colon \gamma \to \theta$. We find an $r \geq p$ and an $\alpha < \theta$ such that $r \Vdash \operatorname{rng} j \subseteq \alpha$.

Claim: Let $q \in \mathscr{P}$, $q \geq p$, $\delta < \gamma$. Then there is a $q' \geq q$ and an $\alpha < \theta$ such that $q' \restriction \operatorname{dom} q = q$ and $q' \Vdash j(\delta) < \alpha$.

Proof: Suppose the claim is false. Construct sequences

$$q = q_0 \leq q_1 \leq \cdots \leq q_\nu \cdots (\nu < \theta,\ q_\nu \in \mathscr{P})$$
$$x_0, x_1, \ldots, x_\mu, \ldots (\mu < \theta,\ x_\mu < \rho)$$
$$y_0 \leq y_1 \leq \cdots \leq y_\nu \leq \cdots (\nu < \theta,\ y_\nu < \theta)$$
$$z_0, z_1, \ldots, z_\nu, \ldots (\nu < \theta,\ z_\nu < \theta,\ z_\nu \neq z_{\nu'} \text{ if } \nu \neq \nu')$$

such that

(a) $\{y_\nu\colon \nu < \theta\}$ is closed and unbounded in θ

(b) $\operatorname{dom} q_\nu = \{x_\mu\colon \mu < y_\nu\}$

(c) $\nu' < \nu$ implies $q_\nu \restriction \operatorname{dom} q_{\nu'} = q_{\nu'}$

(d) there exists a $\bar{q}_\nu \geq q_\nu$, $\operatorname{dom} \bar{q}_\nu = \operatorname{dom} q_\nu$, such that $\bar{q}_\nu \cup q_{\nu+1} \Vdash j(\delta) = z_\nu$.

Suppose we have defined q_ν with domain $\{x_\mu\colon \mu < y_\nu\}$; we define $\bar{q}_\nu$, $q_{\nu+1}$. By the assumption that the claim is false, q_ν does not force $j(\delta) \in \{z_{\nu'}\colon \nu' < \nu\}$, so pick a $z_\nu \neq z_{\nu'}$ $(\nu' < \nu)$ and an $r \geq q_\nu$ such that $r \Vdash j(\delta) = z_\nu$. Let

$$\bar{q}_\nu = r \restriction \operatorname{dom} q_\nu,$$

and let

$$q_{\nu+1} = r \restriction (\operatorname{dom} r - \operatorname{dom} q_\nu) \cup q_\nu.$$

Then enumerate $\operatorname{dom} q_{\nu+1} - \operatorname{dom} q_\nu$ as $\{x_\mu\colon y_\nu \leq \mu < y_{\nu+1}\}$.

Suppose $\lambda < \theta$ is a limit ordinal and the q, x, y and z sequences have been defined for all $\eta < \lambda$. By (c),

$$\bigcup_{\eta<\lambda} q_\eta \in \mathscr{P}.$$

Let $\bigcup_{\eta<\lambda} q_\eta = q_\lambda$, and let $y_\lambda = \sup_{\eta<\lambda} y_\eta$. This completes the construction.

Let $\nu' < \nu < \theta$. $\bar{q}_{\nu'} \cup q_{\nu'+1}$ is incompatible with $\bar{q}_\nu \cup q_{\nu+1}$. Since $\bar{q}_\nu \geq q_{\nu'+1}$ and $(\operatorname{dom} q_{\nu+1} - \operatorname{dom} \bar{q}_\nu) \cap \operatorname{dom} q_{\nu'+1} = \varnothing$, there must be an $f(y_{\nu'}, y_\nu) < y_{\nu'}$ such that $\bar{q}_\nu(x_{f(y_{\nu'},y_\nu)})$ is incompatible with $q_\nu(x_{f(y_{\nu'},y_\nu)})$.

By (*), find

$$B \in [\{y_\nu \colon \nu < \theta\}]^{(2^{<\gamma})^+}$$

such that

$$f''(B) = \{\mu\},$$

some μ. Thus for y_ν, $y_{\nu'} \in B$, $y_\nu \neq y_{\nu'}$, $\bar{q}_\nu(\mu) \neq \bar{q}_{\nu'}(\mu)$. This contradicts that $\operatorname{Card}\{\bar{q}_\nu(\mu)\colon y_\nu \in B\} \leq \operatorname{Card} \mathscr{Q} = 2^{<\gamma}$, and proves the claim.

To prove (2), use the claim γ times to construct a sequence

$$p \leq p_0 \leq p_1 \leq \cdots \leq p_\delta \leq \cdots (\delta < \gamma)$$

such that $\delta' < \delta$ implies $p_{\delta'} \restriction \operatorname{dom} p_{\delta'} = p_{\delta'}$ and such that for each δ there is an $\alpha_\delta < \theta$ with $p_\delta \Vdash j(\delta) < \alpha_\delta$. Then $r = \bigcup_\delta p_\delta$ forces $\operatorname{rng} j \subseteq \sup_\delta \alpha_\delta < \theta$.

This proves the lemma.

Since $\mathscr{P}$ has the $\theta^+ cc$ and $\theta = \gamma^{+\mathscr{M}[G]}$, Prikry's theorem yields that $\bar{\mathscr{I}}$ is a γ^{++}-saturated normal κ-ideal in $\mathscr{M}[\check{G}]$. ($\bar{\mathscr{I}}$ is not γ^+-saturated, for instance the sets $X_\alpha\colon \alpha < \theta$ are disjoint members of $\mathscr{P}(\kappa) - \bar{\mathscr{I}}$, where $\beta \in X_\alpha$ iff $\beta + \alpha$ is the least $\lambda > \beta$ with $G_\lambda(0) = 0$.) We show that $\bar{\mathscr{I}}$ is (κ, κ, γ)-saturated.

Let $\dot{Z}_\alpha$, $\alpha < \kappa$, be terms which are forced (assume without loss of generality by the empty condition) to stand for members of $\mathscr{P}(\kappa) - \bar{\mathscr{I}}$. We find a κ-subsequence of the Z_α's in $\mathscr{M}[G]$ and a condition which forces that any $\leq\gamma$-intersection from the subsequence belongs to $\mathscr{P}(\kappa) - \bar{\mathscr{I}}$.

Note that $p \Vdash \dot{X} \in \mathscr{P}(\kappa) - \bar{\mathscr{I}}$ just in case for all $p' \geq p$

$$\{\beta\colon \exists p'' \geq p' \quad p'' \Vdash \beta \in \dot{X}\} \notin \mathscr{I}.$$

Call $\mathscr{R} \subseteq \mathscr{P}$ a Δ-system with kernel p if for each $r \in \mathscr{R}, r \restriction \operatorname{dom} p =$

p, and if $r, s \in \mathcal{R}$, $r \neq s$, then $\operatorname{dom} r \cap \operatorname{dom} s = \operatorname{dom} p$. An application of the normality of $\mathcal{I}$ in the usual way gives that in $\mathcal{M}$, if $X \in \mathcal{P}(\kappa) - \mathcal{I}$ and for each $\beta \in X$, $p_\beta \in \mathcal{P}$, then there is a $Y \subseteq X$, $Y \notin \mathcal{I}$, with $\{p_\beta : \beta \in Y\}$ a Δ-system.

Work in $\mathcal{M}$. For each $\alpha < \kappa$ pick $Y_\alpha \in \mathcal{P}(\kappa) - \mathcal{I}$ and $p_{\alpha\beta}$, $\beta \in Y_\alpha$, such that

$$p_{\alpha\beta} \Vdash \beta \in \dot{Z}_\alpha.$$

Pick $X_\alpha \subseteq Y_\alpha$, $X_\alpha \notin \mathcal{I}$, and p_α such that $\{p_{\alpha\beta} : \beta \in X_\alpha\}$ forms a Δ-system with kernel p_α. Pick $A \in [\kappa]^\kappa$ and p such that $\{p_\alpha : \alpha \in A\}$ forms a Δ-system with kernel p.

If $\alpha, \alpha' \in A$, $\alpha \neq \alpha'$, call $\{\alpha, \alpha'\}$ good if

$$\{\beta \in X_\alpha \cap X_{\alpha'} : p_{\alpha\beta} \text{ is compatible with } p_{\alpha'\beta}\} \notin \mathcal{I}.$$

We claim that there is an $A' \subseteq A$, $\operatorname{Card} A' = \kappa$, such that every $\{\alpha, \alpha'\} \in [A']^2$ is good. For if no such A' existed, the partition relation $\kappa \to (\kappa, \theta^+)^2$ would give a $C \subseteq A$, $\operatorname{Card} C = \theta^+$, such that for each $\{\alpha, \alpha'\} \in [C]^2$ there is a $W_{\alpha\alpha'} \in \mathcal{P}(\kappa) - \mathcal{I}$ with $p_{\alpha\beta}$ incompatible with $p_{\alpha'\beta}$, all $\beta \in W_{\alpha\alpha'}$. Pick $\beta \in \bigcap_{\{\alpha,\alpha'\} \in [C]^2} W_{\alpha\alpha'}$; then $\{p_{\alpha\beta} : \alpha \in C\}$ would be a set of θ^+ incompatible elements of $\mathcal{P}$, a contradiction.

Define $A'' \in \mathcal{M}[G]$ by

$$A'' = \{\alpha \in A' : p_\alpha \in G\}.$$

Clearly $p \Vdash \operatorname{Card} A'' = \kappa$. We claim that p forces that any intersection of $\leq \gamma$ members of $\{\dot{Z}_\alpha : \alpha \in \dot{A}''\}$ is in $\mathcal{P}(\kappa) - \bar{\mathcal{I}}$, which will prove the theorem. Suppose $q \geq p$ and $\dot{C}$ is a term such that

$$q \Vdash (\dot{C} \subseteq \dot{A}'' \text{ and } \operatorname{Card} \dot{C} \leq \gamma).$$

Then there is a $D \subseteq A'$, $D \in \mathcal{M}$, $\operatorname{Card} D < \theta$, and an $r \geq q$ such that $r \Vdash \dot{C} \subseteq D$ (the existence of such a D and r is equivalent to the statement that $q \Vdash \gamma^+ \leq \theta$). Since $D \subseteq A'$ and $\operatorname{Card} D < \theta$.

$$B = \{\beta \in \bigcap_{\alpha \in D} X_\alpha : \bigcup_{\alpha \in D} p_{\alpha\beta} \in \mathcal{P}\} \in \mathcal{P}(\kappa) - \mathcal{I}.$$

For $\beta \in B$, let

$$S_\beta = (\bigcup_{\alpha \in D} p_{\alpha\beta} - \bigcup_{\alpha \in D} p_\alpha).$$

Define in $\mathcal{M}[G]$,

$$Z = \{\beta \in B\colon s_\beta \in G\}.$$

By construction,

$$r \Vdash \dot{Z} \subseteq \bigcap_{\alpha \in \dot{C}} \dot{Z}_\alpha.$$

Since any refinement of the s_β's to a Δ-system will be with kernel $\emptyset$, and since $\{\beta \in B\colon \operatorname{dom} s_\beta \cap \operatorname{dom} r \neq \emptyset\}$ 3 has power $<\kappa$,

$$r \Vdash \dot{Z} \in \mathcal{P}(\kappa) - \bar{\mathcal{I}}.$$

So $r \Vdash \bigcap_{\alpha \in \dot{C}} \dot{Z}_\alpha \in \mathcal{P}(\kappa) - \bar{\mathcal{I}}$, which completes the proof.

The proof shows that $\bar{\mathcal{I}}$ is $(\lambda, \lambda, \gamma)$-saturated for any λ such that in $\mathcal{M}$, λ is a regular cardinal satisfying $\lambda \to (\lambda, \theta^+)^2$. If, e.g., the GCH holds in $\mathcal{M}$ this relation holds for all regular $\lambda > \theta^+$.

Another way to force to get a (κ, κ, γ)-saturated κ-ideal is to blow up 2^γ to ρ by adding ρ Sacks subsets of γ (analogs for γ of Sacks subsets of ω ([12])). Regarding the method used in this paper, it is necessary to collapse an inaccessible cardinal to take care of the case $\gamma = \omega$, but for γ regular $>\omega$ an alternative proved by Baumgartner is to arrange that $2^\gamma = \gamma^+$ and $\Diamond_\gamma$ hold, and then force to add ρ subsets of γ taking $<\gamma^+$ supports.

The existence of a (κ, γ, γ)-saturated κ-ideal (γ least such that $2^\gamma \geq \kappa$) does not follow from the existence of a γ^+-saturated κ-ideal. For instance, if 2^γ is blown up to κ in the standard way ($2^{<\gamma} = \gamma$ and κ measurable in the ground model), then in the extension there is a γ^+-saturated 2^γ-ideal, but there is a family $\{X_\alpha \subseteq 2^\gamma\colon \alpha < 2^\gamma\}$ such that the intersection of any γ of the X_α's or their complements has power $\leq \gamma$. This property implies that there are no $(2^\gamma, \gamma, \gamma)$-saturated 2^γ-ideals. Similarly, in the model of Kunen and Paris [8] which has a 2^γ-saturated 2^γ-ideal but no λ-saturated 2^γ-ideals for any $\lambda < 2^\gamma$, there are no $(2^\gamma, \gamma, \gamma)$-saturated 2^γ-ideals; if those forcing conditions are replaced by the ones of ([9], page 1036), one obtains a model with those properties in which $2^\gamma \not\to (2^\gamma, (\gamma : 2))^2$.

REFERENCES

[1] J. Baumgartner: Almost disjoint sets, the dense set problem, and the partition calculus. *Ann. Math. Logic. 10* (1976) 401–439.

[2] P. Erdös and R. Rado: A partition calculus in set theory. *Bull. Amer. Math. Soc. 62* (1956) 427–489.

[3] P. ERDÖS and R. RADO: Intersection theorems for systems of sets. *J. London Math. Soc. 35* (1960) 85–90.

[4] A. HAJNAL: Some results and problems in set theory. *Acta Math. Acad. Sci. Hungar. 11* (1960) 277–298.

[5] R. JENSEN and K. KUNEN: *Some combinatorial properties of L and V*, mimeographed.

[6] K. KUNEN: A partition theorem. *Notices of the AMS 18* (1971) 425.

[7] K. KUNEN: Saturated Ideals. *J. Symb. Logic*,

[8] K. KUNEN and J. PARIS: Boolean extensions and measurable cardinals. *Ann. Math. Logic 2* (1971) 359–377.

[9] R. LAVER: Partition relations for uncountable cardinals $\leq 2^{\aleph_0}$. *Colloquia Mathematica Societatis Janos Bolyai 10* (Infinite and finite sets, Keszthely [Hungary], 1973) North Holland, 1975, 1029–1042.

[10] M. MARCZEWSKI: Séparabilité et multiplication cartésienne des espaces topologiques. *Fund. Math. 34* (1947) 127–143.

[11] K. PRIKRY: Changing measurable into accessible cardinals. *Dissertationes Math.* (Rozprawy Mat.) *68*, 1970.

[12] G. SACKS: Forcing with perfect closed sets. *Proceedings of AMS Symposia in Pure Mathematics 13*, Part 1, Providence, 1971, 331–355.

[13] W. SIERPINSKI: Sur un problème de la théorie des relations. *Ann. Scuola Norm. Sup. Pisa 2* (1933) 285–287.

[14] R.M. SOLOVAY: Real valued measurable cardinals. *Proceedings of Symposia in pure mathematics, 13*, Part 1, Providence, 1971, 397–428.

[15] S. ULAM: Zur Masstheorie in der allgemeine Mengenlehre. *Fund. Math. 16* (1930) 140–150.

[16] S. WAGON: Decompositions of saturated ideals. Thesis, Dartmouth College, 1975.

(Oblatum 12–IV–1976)

University of Colorado
Boulder, Colorado 80302, USA