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ON THE ESSENTIAL HEIGHT OF HOMOTOPY TREES WITH FINITE FUNDAMENTAL GROUP

Micheal N. Dyer

1. Introduction

Let G be a group. A (G, i)-complex X is a finite, connected CW complex with dimension $\leq i$ having $\pi_1 X$ isomorphic to G and $\pi_j X = 0$ for $1 < j < i$. The homotopy tree $HT(G, i)$ is a directed tree whose vertices $[X]$ consist of the homotopy classes of (G, i)-complexes X; a vertex $[X]$ is connected by an edge to vertex $[Y]$ iff Y has the homotopy type of the sum $X \vee S^i$ of X and an i-sphere S^i. Let $\boldsymbol{\chi}(X) = (-1)^i \chi(X)$ be the *directed* Euler characteristic of a (G, i)-complex X; $\chi_{\min} = \chi_{\min}(G, i) = \min\{\boldsymbol{\chi}(X) \mid X$ is a (G, i)-complex$\}$. The *level* of a vertex $[X]$ is the number $\boldsymbol{\chi}(X) - \chi_{\min}$. A (G, i)-complex X is a *root* provided $[X]$ has no predecessors in the tree; X is a *minimal root* iff $[X]$ is at level 0.

DEFINITION: We say that $HT(G, i)$ has *essential height* $\leq k$ iff for any two (G, i)-complexes X, Y such that $\boldsymbol{\chi}(X) = \boldsymbol{\chi}(Y) \geq \chi_{\min} + k$, X has the same homotopy type as Y [4].

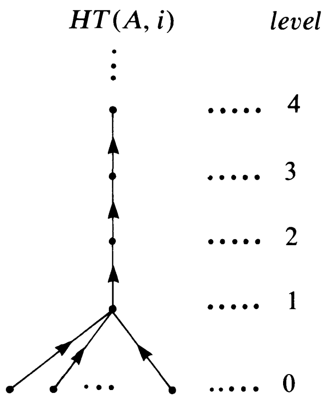
THEOREM 1: *For any finite group π and integer $i \geq 2$, the homotopy tree $HT(\pi, i)$ has essential height ≤ 2.*

This is an easy consequence of R. Williams' generalization [15, theorem 4.6] of the cancellation theorem of H. Bass to the category of pointed modules. The proof is given in section 4.

THEOREM 2: *For any finite abelian group A, the homotopy tree $HT(A, i)$ has essential height ≤ 1.*

Throughout this paper π will denote a finite group and A a finite abelian group. In general, these are the best possible results. It is shown in [5], that for π equal to the generalized quaternion group of order 32, the tree $HT(\pi, 3)$ has essential height equal to two. For A a finite abelian group, W. Metzler [8] and A. Sieradski [9] show that there are often distinct minimal roots in these trees. The only remaining question for $HT(A, i)$ is the number of minimal roots.

A picture of the trees $HT(A, i)$ would be:



It would be very interesting to know about the height of the *simple* homotopy tree $SHT(\pi, i)$ as well.

The outline of the paper is as follows. In section 2 key isomorphisms are isolated, which are used in section 3 to show that we may "shuffle k-invariants" via certain automorphisms of the homotopy modules of minimal roots. The proofs of theorems 1 and 2 are found in section 4. In section 5 we apply our results to the problem of C.T.C. Wall concerning spaces dominated by finite two-dimensional complexes.

For example, we show the following theorem.

THEOREM 3: *Let X be any connected CW-complex which is dominated by a finite, connected 2-complex and suppose that the Wall invariant of X vanishes. If $\pi_1 X$ is a finite abelian group, then $X \vee S^2$ has the homotopy type of a finite 2-complex.*

A sharper (but more technical) version of theorem 3 is proved in section 5.

2. Certain isomorphisms

In this section we develop certain technical results necessary for the proof of theorem 2.

Let π be a finite group of order n and let $N = \Sigma_{x\in\pi} x$ be the (norm) element in $Z\pi$ consisting of the sum of all the group elements. A *unit mod* N is an element $u \in Z\pi$ for which there is an element $u' \in Z\pi$ such that $u'u$ and uu' are congruent to 1 modulo the ideal (N) generated by N. Equivalently, $u+(N)$ is a unit in the augmentation ring $Z\pi/(N)$.

The augmentation of units mod N is of some interest. The augmentation $\epsilon: Z\pi \to Z$ induces $\epsilon': Z\pi/(N) \to Z_n$. There is a homomorphism

$$\partial: Z_n^* \longrightarrow \tilde{K}_0 Z\pi$$

from the group of units in the ring Z_n to the reduced projective class group $\tilde{K}_0 Z\pi$ of $Z\pi$, defined by carrying the residue class $p + nZ = [p]$ modulo n (p is prime to n) to the class $\{(p, N)\} \in \tilde{K}_0 Z\pi$ of the projective ideal (p, N) generated by p and N (see [10, §6] and [3, sections 2–4]). For A a finite abelian group, $[p] \in \ker \partial$ iff the ideal (p, N) is isomorphic (as an A-module) to ZA [12, theorem 19.8 and the discussion following]. The following is proved in [10, lemma 6.3, page 279].

PROPOSITION 2.1. *Let A be a finite abelian group. If $u \in ZA$ is a unit* mod N, *then $\epsilon'(u) \in \ker \partial$. Furthermore, given any $[p] \in \ker \partial$, then there is a unit u* mod N *such that $\epsilon(u) = p$.* □

Consider a free π-module $(Z\pi)^t$ of rank t and the (ring) homomorphism

$$\epsilon: (Z\pi)^t \longrightarrow Z^t$$

given by $\epsilon(\alpha_1, \ldots, \alpha_t) = (\epsilon(\alpha_1), \ldots, \epsilon(\alpha_t))$. We have now the following crucial lemma.

LEMMA 2.2: *Let A be any finite abelian group and K be any submodule of $(ZA)^t$ such that $\epsilon(K) = 0$. For any unit u* mod N *in ZA, the homomorphism*

$$\bar{u}: K \longrightarrow K$$

given by multiplication by u is an isomorphism.

PROOF: u is a unit mod N implies the existence of a $u' \in ZA$ such that $u'u = uu' = 1 + \alpha N$ ($\alpha \in Z$). $x = (x_1, \ldots, x_t)$ is a member of K iff $\epsilon(x) = (\epsilon(x_1), \ldots, \epsilon(x_t)) = 0$ iff $N \cdot x = 0$. Thus $u'ux = uu'x = x + \alpha Nx = x$ for all $x \in K$. $\square$

Now consider the $(k+1) \times (k+1)$ integer matrix ($k \geq 1$)

$$M_{k+1} = \begin{bmatrix} q & 0 & \cdots & & 0 & t \\ p_1 & q & & & & 0 \\ 0 & p_2 & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & q & \vdots \\ \vdots & & & & p_{k-1} \quad q & 0 \\ 0 & \cdots & & & 0 \quad b & c \end{bmatrix}.$$

A straightforward induction argument shows that

$$\det M_{k+1} = cq^k + (-1)^k p_1 \cdots p_{k-1} tb.$$

PROPOSITION 2.3: *Let π be any finite group of order n and Z the trivial π-module. Let v be a unit* mod N *in $Z\pi$ having $\epsilon(v) = c$. Suppose that $c, q, b, t, p_1, \ldots, p_{k-1}$ are integers such that*

$$cq^k + bp_1 \cdots p_{k-1} tn = 1.$$

Then the left π-homomorphism

$$\alpha : Z^k \oplus Z\pi \longrightarrow Z^k \oplus Z\pi$$

with matrix

$$\alpha = \begin{bmatrix} q & 0 & \cdots & & 0 & (-1)^k \cdot t \\ p_1 & q & & & & 0 \\ 0 & p_2 & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & p_{k-1} & q & 0 \\ 0 & \cdots & & 0 & bN & v \end{bmatrix}$$

is an isomorphism.

PROOF: Let $\iota : Z\pi N \to Z\pi$ denote the natural inclusion and consider the following exact ladder of modules:

$$\begin{tikzcd}
0 \arrow[r] & (Z)^k \oplus Z\pi N \arrow[r, hook, "id \oplus \iota"] \arrow[d, "\alpha'"] & (Z)^k \oplus Z\pi \arrow[r] \arrow[d, "\alpha"] & Z\pi/(N) \arrow[r] \arrow[d, "\alpha''"] & 0 \\
0 \arrow[r] & (Z)^k \oplus (Z\pi N) \arrow[r, hook, "id \oplus \iota"] & (Z)^k \oplus Z\pi \arrow[r] & Z\pi/(N) \arrow[r] & 0
\end{tikzcd}$$

with α', α'' the appropriate maps induced by α. The image of α restricted to $Z^k \oplus Z\pi N$ is contained in $Z^k \oplus Z\pi N$ because this is the submodule $(Z^k \oplus Z\pi)^\pi$ of elements fixed under the action of π. The matrix of α' is given by

$$\begin{bmatrix}
q & 0 & \cdots & \cdots & 0 & (-1)^k nt \\
p_1 & q & & & & 0 \\
0 & p_2 & q & & & \vdots \\
\vdots & & \ddots & \ddots & & \vdots \\
\vdots & & & p_{k-1} & q & 0 \\
0 & \cdots & \cdots & 0 & b & c
\end{bmatrix} \qquad (\epsilon(v) = c)$$

which has determinant $ca^k + bp_1 \cdots p_{k-1}tn = 1$ and hence α' is an isomorphism. α'' is simply multiplication by the element $v \in Z\pi$ and is an isomorphism because v is a unit mod N. By the five lemma, α is an isomorphism. □

3. Shuffling k-invariants

Let n denote the order of the group π. For any (π, i)-complex X, it is known from [3, §2] that the $(i+1)$-cohomology group $H^{i+1}(\pi, \pi_i)$ (with coefficients in the π-module $\pi_i X = \pi_i$) is isomorphic to Z_n. Let $\iota: \pi_i \hookrightarrow C_i = C_i(\tilde{X})$, where C_i is the free π-module which is the cellular chain module of the universal cover $\tilde{X}$ of X. We use the fact that

$$H^{i+1}(\pi; \pi_i) \cong \mathrm{End}_\pi(\pi_i)/B^i,$$

with $B^i = \mathrm{im}\{\mathrm{Hom}_\pi(C_i, \pi_i) \xrightarrow{\iota^*} \mathrm{End}_\pi(\pi_i)\}$, to identify $H^{i+1}(\pi; \pi_i)$ with Z_n via $(\bar{q}: \pi_i \to \pi_i) \to q + (n)$, $q \in Z$. Thus $[1] \in Z_n$ corresponds to the class of $\mathrm{id} \in \mathrm{End}(\pi_i)$. Notice also that if $\ell: \pi_i \to \pi_i \oplus Z\pi^j$ is the natural inclusion ($Z\pi^j$ the direct sum of j copies) then $\ell_*: H^{i+1}(\pi; \pi_i) \to H^{i+1}(\pi; \pi_i \oplus Z\pi^j)$ is an isomorphism because $H^{i+1}(\pi; Z\pi) = 0$ for any finite group π $(i \geq 0)$. We identify all these groups using ℓ_*.

We say that an isomorphism $\alpha: \pi_i \oplus Z\pi^j \xrightarrow{\cong} \pi_i \oplus Z\pi^j$ has *degree* $q \in Z_n^*$ if $\alpha_*(1) = q \in H^{i+1}(\pi; \pi_i)$.

PROPOSITION 3.1: (Bass-Williams [15]). *For each finite group π, $i \geq 2$, each minimal root $X \in HT(\pi, i)$ and each $[q] \in \ker\{\partial: Z_n^* \to \tilde{K}_0 Z\pi\}$ there exists an automorphism $\pi_i \oplus Z\pi^2 \to \pi_i \oplus Z\pi^2$ of degree $[q]$.*

PROOF. For each $X \in HT(\pi, i)$ and each $[q] \in \ker \partial$ it is proved in [4, page 309] that there is an integer $j \geq 2$ and an automorphism

$$\alpha: \pi_i X \oplus (Z\pi)^j \longrightarrow \pi_i X \oplus (Z\pi)^j$$

having degree $[q]$. However, by J. Williams' generalization [15, theorem 4.6 and the remark following 4.9] of the Bass cancellation theorem to the category of pointed modules, one may "cancel" all but two factors of $Z\pi$ while preserving the degree; i.e., there is an automorphism

$$\alpha': \pi_i X \oplus (Z\pi)^2 \longrightarrow \pi_i X \oplus (Z\pi)^2$$

also having degree $[q]$. $\square$

PROPOSITION 3.2: *Let A be a finite abelian group of order n and Y be any minimal root in $HT(A, i)$. Let A_i denote the A-module $\pi_i(Y)$. For each $[q] \in \ker\{\partial: Z_n^* \to \tilde{K}_0 ZA\}$ there exists an automorphism $A_i \oplus ZA \to A_i \oplus ZA$ of degree $[q]$.*

PROOF: Consider $A = Z_{\tau_1} \times \cdots \times Z_{\tau_s}$ $(\tau_1 | \tau_2 | \cdots | \tau_s)$ and let $n = \tau_1 \cdots \tau_s$ denote the order of the group. Let Y denote any minimal root of $HT(A, i)$. We consider the standard A-module

$$A_i = \pi_i(Y) \to C_i(\tilde{Y}) = C_i,$$

where C_i is the (finitely generated) free A-module which is the cellular chain module of the universal cover $\tilde{Y}$ of Y. Let $\nu = \operatorname{rank}_A C_i$, $\{e_j\}$ be a ZA-basis for C_i, and ψ designate the $\operatorname{rank}_Z \Sigma_i = \operatorname{im}\{\epsilon | A_i: A_i \to Z^\nu\}$ where $\epsilon: C_i \to Z^\nu$ is the augmentation on each coordinate and Σ_i is the subgroup of spherical homology classes of $H_i(Y)$.

As $\Sigma_i \hookrightarrow Z^\nu$, use the fundamental theorem of finitely generated free abelian groups to choose a new basis for Z^ν

$$\{a_1, \ldots, a_\psi, a_{\psi+1}, \ldots, a_\nu\}$$

so that the set $\{\alpha_1 a_1, \ldots, \alpha_\psi a_\psi\}$ $(\alpha_j \geq 1)$ is a basis for Σ_i.

Note that each α_j can be chosen so that $\alpha_j = \tau_{k(j)}$. We do this as follows. There is an isomorphism

$$H_i(Y)/\Sigma_i \cong H_i(A),$$

this last being a finite abelian group. Let Y^{i-1} denote the $(i-1)$-skeleton of Y and Σ_{i-1} denote the image of $\pi_{i-1}Y^{i-1}$ in $H_{i-1}(Y^{i-1})$ under the Hurewicz homomorphism. Then the following lower sequence is an exact sequence of free abelian groups

$$\begin{tikzcd}
 & & C_i \arrow[r, two heads] \arrow[d, "\epsilon"] & \pi_{i-1}Y^{i-1} \arrow[r] \arrow[d, "\epsilon"] & 0 \\
0 \arrow[r] & H_i(Y) \arrow[r] & Z^\nu \arrow[r] & \Sigma_{i-1} \arrow[r] & 0
\end{tikzcd}$$

obtained by applying the augmentation homomorphism to the upper sequence. As Σ_{i-1} is free we have $Z^\nu \cong H_i(Y) \oplus \Sigma_{i-1}$. Since the $\operatorname{rank}_Z H_i(Y) = \operatorname{rank}_Z \Sigma_i$, $a_1, \ldots, a_\psi$ may be chosen as a basis for $H_i(Y)$ and $\alpha_1, \ldots, \alpha_\psi$ will be the torsion coefficients of $H_i(A)$, each of which (by the Künneth formula) is one of the torsion coefficients of A itself.

Express the new basis $\{a_j\}$ in terms of the old basis $\{\epsilon(e_j)\}$ as follows:

$$a_j = \sum_{k=1}^{\nu} b_{jk} \cdot \epsilon(e_k) \quad (b_{jk} \in Z).$$

Use the invertible $\nu \times \nu$ integral matrix $B = (b_{jk})$ to determine a new basis of C_i

$$f_j = \sum_{k=1}^{\nu} b_{jk} e_k \quad (j = 1, \ldots, \nu).$$

With respect to this basis $\{f_j\}_{j=1}^{\nu}$ for C_i, $\{\alpha_j \cdot \epsilon(f_j)\}_{j=1}^{\psi}$ is a basis for $\Sigma_i \hookrightarrow Z^\nu$.

Because $\epsilon(A_i) = \Sigma_i$, we may choose elements $\mu_1, \mu_2, \ldots, \mu_\psi$ of A_i such that $\epsilon(\mu_j) = \alpha_j \cdot \epsilon(f_j)$ $(j = 1, \ldots, \psi)$.

For each $k = 1, \ldots, \psi - 1$, define a homomorphism

$$r_k\colon C_i \longrightarrow C_i$$

by $\quad r_k(f_j) = \begin{cases} 0 & \text{if } k \neq j \\ N \cdot \mu_{k+1} & \text{if } k = j \end{cases} \quad (j = 1, \ldots, \nu).$

Let $E^j_{\ell m}$ denote the elementary $j \times j$ matrix with a 1 in the ℓth row and the mth column and zeros elsewhere. Notice that the matrix of r_k with respect to $\{f_j\}$ is given by $N \cdot \alpha_{k+1} E^{\nu}_{k+1,k}$ and the matrix of the map $\epsilon(r_k)$ defined by r_k on Σ_i with respect to the basis $\{\alpha_j \cdot \epsilon(f_j)\}$ is given by $\alpha_k \cdot n \cdot E^{\psi}_{k+1,k}$. This last follows because $r_k(f_k) = \alpha_{k+1} N f_{k+1}$ which implies that $\epsilon(r_k)(\epsilon(f_k)) = \alpha_{k+1} \cdot n \cdot \epsilon(f_{k+1})$. Hence, $\epsilon(r_k)(\alpha_k \cdot \epsilon(f_k)) =$

$\alpha_k \cdot \alpha_{k+1} \cdot n \cdot \epsilon(f_{k+1}) = (\alpha_k \cdot n)(\alpha_{k+1} \cdot \epsilon(f_{k+1}))$. Notice also that r_k has image in A_i, hence $r_k \mid A_i \colon A_i \to A_i$ is a map of degree 0.

Now choose a unit $u \bmod N$ in ZA with $\epsilon(u) = q$. q is prime to n implies that q is prime to each τ_j $(j = 1, \ldots, s)$ and hence to each α_j $(j = 1, \ldots, \psi)$. Thus q^{ψ} is prime to $(\alpha_1 \cdots \alpha_{\psi-1})n^{\psi+1}$, so choose integers b, c such that

$$cq^{\psi} + b(\alpha_1 \cdots \alpha_{\psi-1})n^{\psi+1} = 1.$$

The above equation yields

$$c \equiv q^{-\psi} \pmod{n}$$

and hence $[c]$ is a member of $\ker \partial$ also. Choose a unit $v \bmod N$ in ZA such that $\epsilon(V) = c$ (see 2.1).

With all these data, we may define the isomorphism

$$\alpha \colon A_i \oplus ZA \longrightarrow A_i \oplus ZA$$

of degree $[q]$: α is given by a (2×2)-matrix of homomorphisms

$$\alpha = \left(\begin{array}{c|c} \alpha_{11} \colon A_i \longrightarrow A_i & \alpha_{12} \colon ZA \longrightarrow A_i \\ \hline \alpha_{21} \colon A_i \longrightarrow ZA & \alpha_{22} \colon ZA \longrightarrow ZA \end{array}\right).$$

Let $\alpha_{11} = \bar{u} + \Sigma_{k=1}^{\psi-1} r_k \mid A_i$, $\alpha_{12}(1) = (-1)^{\psi} N \cdot f_1$, $\alpha_{21} = bNp_{\psi} \mid A_i$, and $\alpha_{22} = \bar{v}$, where $p_j \colon C_i \to ZA$ is the projection on the jth coordinate. Recall that $\bar{u} \colon A_i \to A_i$ means right multiplication by u.

To show that α is an isomorphism we decompose $A_i \hookrightarrow C_i$ by applying $\epsilon \colon (ZA)^{\nu} \to Z^{\nu}$ to A_i. Thus we have

$$\begin{tikzcd} & & C_i \arrow[r, "\epsilon"] & (Z)^{\nu} \arrow[r] & 0 \\ 0 \arrow[r] & K \arrow[r] & A_i \arrow[r, "\epsilon"] \arrow[u, hook] & \Sigma_i \arrow[r] \arrow[u, hook] & 0. \end{tikzcd}$$

$\alpha \mid K$ is simply multiplication by u because $r_k \mid K = 0 = bNp_{\psi} \mid K$. Thus $\alpha \mid K$ is an isomorphism by lemma 2.2. α also induces a map $\alpha' \colon \Sigma_i \oplus ZA \to \Sigma_i \oplus ZA$.

It is clear (using the basis $\{\alpha_j \cdot \epsilon(f_j)\}$) that the matrix of the map $\alpha' \colon \Sigma_i \oplus ZA \to \Sigma_i \oplus ZA$ induced by α is given by

$$\begin{array}{rc}
\begin{array}{r} \\ \\ \\ \\ \\ \psi \\ \\ \end{array} &
\begin{array}{c}
\left[\begin{array}{cccccc|c}
q & 0 & \cdots & \cdots & \cdots & 0 & (-1)^{\psi} n \\
\alpha_1 n & q & \ddots & & & \vdots & 0 \\
\vdots & \alpha_2 n & \ddots & \ddots & & \vdots & \vdots \\
\vdots & & \ddots & q & & \vdots & \vdots \\
\vdots & & & \alpha_{\psi-2}\cdot n & q & 0 & \vdots \\
0 & \cdots & \cdots & 0 & \alpha_{\psi-1}\cdot n & q & 0 \\
\hline
0 & & & 0 & 0 & bN & v
\end{array}\right] \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \psi
\end{array}
\end{array}$$

Because v is a unit mod N and

$$c \cdot q^{\psi} + b(\alpha_1 \cdots \alpha_{\psi-1})n^{\psi+1} = 1 \quad (\epsilon(v) = c)$$

we have α' is an isomorphism (Proposition 2.3). Hence, by the five lemma, α is an isomorphism.

To see that α has degree $[q]$, observe that degree α = degree α_{11} = degree $\bar{u}$ because α_{11} is $\bar{u}$ plus maps of degree 0. But degree $\bar{u}$ = degree $\bar{q} = [q]$ because $\epsilon(u - q) = 0$. □

4. Proof of Theorems 1 and 2

The proof of the main results require the use of the theory of algebraic i-types, which we now outline.

An algebraic i-type is a triple (G, π_i, k), where G is a group, π_i a G-module and k is an element of the group $H^{i+1}(G; \pi_i)$. Such triples form the objects of a category $\mathcal{T}(i)$, the category of i-types. A morphism in $\mathcal{T}$ is a pair of maps (α, β): $(G, \pi_i, k) \to (G', \pi_i', k')$ where $\alpha: G \to G'$ is a group homomorphism, $\beta: \pi_i \to \pi_i'$ is an α-homomorphism ($\beta(x \cdot y) = \alpha(x) \cdot \beta(y)$ for any $x \in G$, $y \in \pi_i$) and $\alpha^*(k') = \beta_*(k)$ in the following diagram:

$$H^{i+1}(G; \pi_i) \xrightarrow{\beta_*} H^{i+1}(G; {}_\alpha\pi_i') \xleftarrow{\alpha^*} H^{i+1}(G', \pi_i')$$

where ${}_\alpha\pi_i'$ is the G-module with structure induced by α. (α, β) is an *isomorphism* iff both α and β are bijective. We denote by $\mathcal{T}(G, i)$ the full subcategory of $\mathcal{T}(i)$ whose objects (G', π_i, k) have G' isomorphic to G.

Let $\mathcal{C}(G, i)$ denote the full subcategory of TOP whose objects are (G, i)-complexes. By a theorem of S. MacLane and J.H.C. Whitehead, there is (homotopy) functor $\mathbb{T}: \mathscr{C}(G, i) \to \mathscr{T}(G, i)$ defined by $\mathbb{T}(X) = (\pi_1 X, \pi_i X, kX)$, where $kX \in H^{i+1}(\pi_1 X, \pi_i X)$ is the first k-invariant of X [7]. $\mathbb{T}(f: X \to Y) = (f_{1\#}, f_{i\#})$ and for each pair of objects $X, Y \in \mathscr{C}(G, i)$, $\mathbb{T}$: Map$(X, Y) \to$ Hom$(\mathbb{T}(X), \mathbb{T}(Y))$ is surjective. This functor is not an equivalence of categories, but it is strong enough that any two (G, i)-complexes X and Y have the same homotopy type iff $\mathbb{T}(X)$ is isomorphic to $\mathbb{T}(Y)$ [7, theorem 1, page 42].

DEFINITION: Let π be a finite group and M be a π-module. M has the *cancellation property* iff for any module M' with

$$M' \oplus (Z\pi)^\alpha \cong M \oplus (Z\pi)^\beta \quad (\beta \geq \alpha)$$

we have $M' \cong M \oplus (Z\pi)^{\beta-\alpha}$.

For any module M over π, $M \oplus (Z\pi)^2$ has the cancellation property, by the theorem of H. Bass [12, §9]. If A is a finite abelian group and $A_i = \pi_i Y$, where Y is any (A, i)-complex, then $A_i \oplus Z\pi$ has the cancellation property [12, theorem 19.8], [3, page 267]. If π is any finite group and X is a $(\pi, 2i)$-complex, then $\pi_{2i}X \oplus Z\pi$ has the cancellation property [3, corollary 4.2, page 267]. These last two statements are corollaries to the powerful theorem of H. Jacobinski [12, theorem 19.8].

Using propositions 3.1 and 3.2, we now give a

PROOF OF THE MAIN THEOREMS: Let X be any (π, i)-complex and Y be a minimal root of $HT(\pi, i)$. Consider the algebraic i-type $\mathbb{T}(X) = (\pi_1 X, \pi_i X, kX)$ of X. If $\chi(X) > \chi_{\min} + 1$, we will use 3.1 to show that X has the homotopy type of the sum $Y \vee \overset{t}{\vee} S^i$ of the minimal root Y with a bouquet of $t = \chi(X) - \chi_{\min}$ i-spheres S^i. If π is a finite abelian group and $\chi(X) > \chi_{\min}$, a similar argument (using 3.2) will show that $X \simeq Y \vee \overset{t}{\vee} S^i$.

First, we will identify $\pi_1 X$ with $\pi = \pi_1 Y$ via an arbitrary isomorphism $\theta: \pi \to \pi_1 X$. The i-type $\mathbb{T}(X)$ is isomorphic to the i-type $(\pi, {}_\theta\pi_i X, k')$ via the isomorphism $(\theta, id): (\pi, {}_\theta\pi_i X, k') \to (\pi_1 X, \pi_i X, kX)$. Notice that $id: {}_\theta\pi_i X \to \pi_i X$ is a θ-isomorphism. k' is the image of kX under the isomorphism

$$\theta^*: H^{i+1}(\pi_1 X, \pi_i X) \longrightarrow H^{i+1}(\pi, {}_\theta\pi_i X)$$

(θ^* is an isomorphism by [6, page 108]). Now consider the i-type $\mathbb{T}(Y) = (\pi, \pi_i, kY)$. It follows from Schanuel's lemma that

$${}_\theta\pi_i X \oplus (Z\pi)^\ell \cong \pi_i \oplus (Z\pi)^j \quad (\pi_i = \pi_i Y)$$

as π-modules, with $t = j - \ell \geq 2$. Because $\pi_i \oplus Z\pi^2$ has the cancellation property (if π is finite *abelian*, one uses that $\pi_i \oplus Z\pi$ has the cancellation property) we have an isomorphism $\beta\colon {}_\theta\pi_i X \cong \pi_i \oplus (Z\pi)^t$. Thus

$$\mathbb{T}(X) \underset{(\theta,\, id)}{\cong} (\pi, {}_\theta\pi_i X, k') \underset{(id,\, \beta)}{\cong} (\pi, \pi_i \oplus (Z\pi)^t, k''),$$

where $k'' = \beta_*(k')$ with $\beta_*\colon H^{i+1}(\pi, {}_\theta\pi_i X) \xrightarrow{\cong} H^{i+1}(\pi; \pi_i \oplus (Z\pi)^t)$ induced by β.

By theorem 3.5 of [3, page 264], we must have k'', kY members of $\ker \partial$, a multiplicative subgroup of Z_n^*. Hence by proposition 3.1, there is an isomorphism

$$\alpha\colon \pi_i \oplus (Z\pi)^t \longrightarrow \pi_i \oplus (Z\pi)^t \quad (t \geq 2)$$

of degree $kY/k'' \in \ker \partial$. This yields an isomorphism of i-types carrying $k'' \mapsto kY$:

$$(id, \alpha)\colon (\pi, \pi_i \oplus (Z\pi)^t, k'') \cong (\pi, \pi_i \oplus (Z\pi)^t, kY).$$

This last i-type is just $\mathbb{T}(Y \vee \overset{t}{\vee} S^i)$. Thus $\mathbb{T}(X)$ is isomorphic to $\mathbb{T}(Y \vee \overset{t}{\vee} S^i)$ and hence

$$X \simeq Y \vee \overset{t}{\vee} S^i. \quad \square$$

5. Spaces dominated by 2-complexes

As an application of proposition 3.1, we (almost) extend C. T. C. Wall's theorem concerning spaces dominated by finite 2-complexes [14, theorem F, page 66] to all finite (abelian) groups. The extension to finite cyclic groups has been given in [2, corollary 5.3, page 242] and, independently, in [1, theorem 4, page 261].

DEFINITION: An algebraic two-type $\mathbb{T} = (\pi, \pi_2, k)$ is *finitely 2-realizable* iff there is a $(\pi, 2)$-complex X such that $\mathbb{T}(X) \cong \mathbb{T}$.

Let π be a finite group of order n. We say that the two-type $\mathbb{T} = (\pi, \pi_2, k)$ is *finitely chain2-realizable* if there exists a free partial resolution of the trivial π-module Z of finite type realizing k; i.e., there exists an exact sequence of π-modules:

$$\mathscr{C}(\mathbb{T})\colon 0 \longrightarrow \pi_2 \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow Z \longrightarrow 0$$

where each C_i $(i = 0, 1, 2)$ is a finitely generated free π-module, such that comparison with the standard bar resolution gives $k \in H^3(\pi, \pi_2) \cong Z_n$ (see [3], page 256). This means that $k \in \ker \partial$. The Euler character $\chi(\mathscr{C})$ is given by $\operatorname{rank}_\pi C_2 - \operatorname{rank}_\pi C_1 + \operatorname{rank}_\pi C_0$. By Schanuel's lemma, $\chi(\mathscr{C})$ depends only on $\mathbb{T}$; hence we will denote it by $\chi(\mathbb{T})$, the Euler characteristic of the (finitely chain 2-realizable) two-type $\mathbb{T}$. If A is a finite abelian group, we will show that $\chi(\mathbb{T})$, if defined, is greater than or equal to $\chi_{\min}(A, 2)$.

Let $K(n)$ denote the CW-complex which is a $K(Z_n, 1)$ and has one cell in each dimension. Now let $A = Z_{\tau_1} \times \cdots \times Z_{\tau_s}$ $(\tau_j \mid \tau_{j+1}, j = 1, 2, \ldots, s-1)$ and consider the Eilenberg-MacLane space $K_A = \Pi_{j=1}^s K(\tau_j)$.

PROPOSITION 5.1: *In the tree $HT(A, i)$ for $i \geq 2$, the i-skeleton K_A^i of K_A is a minimal root.*

PROOF: We will show that $\chi_{\min}(A, i) = (-1)^i \chi(K_A^i)$. Because $K(n)$ has one cell in each dimension, the number $\sigma_\ell(s)$ of ℓ-cells in K_A $(\ell \geq 0)$ is precisely the numbers of ways one may choose an ordered s-tuple $(a_1, \ldots, a_s)$ (allowing repetitions) from the set $\{0, 1, \ldots, \ell\}$ such that $\Sigma_{j=1}^s a_j = \ell$.

Let p be any prime dividing τ_1. Then, considering Z_p as a trivial Z_{τ_j}-module $(j = 1, \ldots, s)$, we have

$$H_\ell(Z_{\tau_j}, Z_p) \cong Z_p$$

for all $l \geq 0$. By the Kunneth theorem

$$H_\ell(A, Z_p) \cong \bigoplus_{\substack{0 \leq a_j \leq \ell \\ \sum_{j=1}^s a_j = \ell}} (Z_p)_{(a_1, \ldots, a_s)}.$$

Thus, the dimension of $H_\ell(A; Z_p)$ as a Z_p-module $\equiv h_\ell(A; Z_p) = \sigma_\ell(K_A)$. Define $\mu_i(A)$ to be the minimum of the directed Euler characteristics of truncated, finitely generated free resolutions of length i,

$$0 \longrightarrow A_i \longrightarrow C_i \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow Z \longrightarrow 0$$

(each C_i is finitely generated, free A-module, Z is the trivial A-module) [11, page 193].

Theorem 1.2 of [11] says that

$$\mu_i(A) \geq \sum_{\ell=0}^{i} (-1)^{i-\ell} h_\ell(A, Z_p) = \sum_{\ell=0}^{i} (-1)^{i-\ell} \sigma_\ell(s) = (-1)^i \chi(K_A^i).$$

But $\mu_i(A) \leq \chi_{\min}(A, i) \leq (-1)^i \chi(K_A^i)$ by definition. Therefore K_A^i is a minimal root and $\mu_i(A) = \chi_{\min}(A, i) = \chi(K_A^i)$. $\square$

COROLLARY 5.2: *For any finitely chain 2-realizable two type* $\mathbb{T} = (A, \pi_2, k)$, *with A a finite abelian group*, $\chi(\mathbb{T}) \geq \chi_{\min}(A, 2)$.

PROOF: By 5.1, $\chi_{\min}(A, 2) = \mu_2(A) \leq \chi(\mathbb{T})$. □

However, for any arbitrary finite group π, it is not known if there is a two type $\mathbb{T}$ such that

$$\chi(\mathbb{T}) < \chi_{\min}(\pi, 2).$$

This would occur, for example, if $HT(\pi, 2)$ has a *minimal* root X such that $\pi_2 X \cong M \oplus Z\pi$. This two type $\mathbb{T}$ would then be finitely chain 2-realizable, but *not* 2-realizable. Does this ever happen?

Recall that a π-module M has the *cancellation property* $(CP) \Leftrightarrow$ for any M' such that $M' \oplus (Z\pi)^i \cong M \oplus (Z\pi)^j$ $(i \leq j)$ we have $M' \cong M \oplus (Z\pi)^{j-i}$. For any $(\pi, 2)$-complex X, the module $\pi_2 X \oplus Z\pi$ has the cancellation property [4, §4].

THEOREM 5.3: *Let A be a finite abelian group and let* $\mathbb{T} = (A, \pi_2, k)$ *be finitely chain 2-realizable. If* $\chi(\mathbb{T}) > \chi_{\min}$, *then* $\mathbb{T} = (A, \pi_2, k)$ *is finitely 2-realizable; if* $\chi(\mathbb{T}) = \chi_{\min}$, *then* $\mathbb{T} \oplus ZA = (A, \pi_2 \oplus ZA, k)$ *is finitely 2-realizable.*

PROOF: Let $\mathbb{T}$ be realizable as

$$\mathscr{C}(\mathbb{T}): 0 \longrightarrow \pi_2 \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow Z \longrightarrow 0$$

with each C_i a finitely generated, free A-module. By Schanuel's lemma [10, section 1, page 269] $\pi_2(K_A^2) \oplus (ZA)^i \cong \pi_2 \oplus (ZA)^j$ $(i \geq j)$. If $i > j$, then $\pi_2(K_A^2) \oplus (ZA)^t \cong \pi_2$ $(t = i - j = \chi(\mathbb{T}) - \chi_{\min})$ [3, proposition 5.1, page 267]. Thus $(A, \pi_2, k) \cong (A, \pi_2(K_A^2) \oplus (ZA)^t, k') \cong (A, \pi_2(K_A^2) \oplus (ZA)^t, kK_A^2)$ by proposition 3.2 because k', kK_A^2 are members of $\ker \partial$. Hence, $\mathbb{T} \cong \mathbb{T}(K_A^2 \vee \overset{t}{\vee} S^2)$. A similar argument shows the result if $\chi(\mathbb{T}) = \chi_{\min}$. □

For an arbitrary finite group π, the following holds:

(a) If $\chi(\mathbb{T}) \geq \chi_{\min}(\pi, 2) + 2$, then $\mathbb{T}$ is 2-realizable

(b) If $\chi(\mathbb{T}) < \chi_{\min}(\pi, 2) + 2$, then $\mathbb{T} \oplus (Z\pi)^j$ is 2-realizable, where $j = \chi_{\min} + 2 - \chi(\mathbb{T})$.

COROLLARY 5.4: *Let X be a connected CW-complex having finite abelian fundamental group A and suppose that X is dominated by a finite two-dimensional complex. Let* $\mathbb{T}(X)$ *denote the algebraic two-type of X. Suppose the Wall invariant* $Wa_2[X]$ *of X vanishes. If*

$\chi(\mathbb{T}) > \chi_{\min}(A, 2)$, then X has the homotopy type of an $(A, 2)$-complex. If $\chi(\mathbb{T}) = \chi_{\min}$, then $X \vee S^2$ has the homotopy type of a $(\pi, 2)$-complex.

PROOF: Because the Wall invariant of X is zero, X has the homotopy type of a finite 3-complex [14, theorem F, page 66] Y. Furthermore, $Wa_2[X] = Wa_2[Y] = 0$ implies that $\mathbb{T}(X) \cong \mathbb{T}(Y)$ is chain 2-realizable by a free chain complex $\mathscr{C}$ of finite type. If $\chi(\mathbb{T}(X)) = \chi(\mathscr{C}) > \chi_{\min}$, then $\mathbb{T}(X)$ is realizable as a 2-complex; if $\chi(\mathbb{T}(X)) = \chi(\mathscr{C}) = \chi_{\min}$, then $\mathbb{T}(X \vee S^2)$ is realizable as a 2-complex. It then follows from theorem 1.1 of [2, page 230] that X or $X \vee S^2$ have the homotopy type of a finite two-complex. $\square$

A similar conclusion holds for π an arbitrary finite group: If $\chi(\mathbb{T}) > \chi_{\min}(\pi, 2) + 1$ ($\chi(\mathbb{T}) \leq \chi_{\min}(\pi, 2) + 1$) then X ($X \vee jS^2$) has the homotopy type of a finite 2-complex (as before, $j = \chi_{\min} + 2 - \chi(\mathbb{T})$).

We formalize the notions involved in the proofs of the main theorems, 5.3 and 5.4 in the following fashion.

Let π be a finite group and X be a (π, i)-complex. We say that $\operatorname{Aut} \pi_i X$ *is transitive on k-invariants* iff for each $k \in \ker \partial \subset H^{i+1}(\pi, \pi_i X)$ there is a θ-automorphism $\alpha: \pi_i X \to \pi_i X$, $\theta \in \operatorname{Aut} \pi$, such that $\alpha_*(1) = \theta^*(k)$. Recall that a θ-homomorphism α has the property that $\alpha(x \cdot y) = \theta(x)\alpha(y)$ ($x \in \pi$, $y \in \pi_i X$). With this definition, it is clear that proposition 3.2 simply says that $\operatorname{Aut} \pi_i(K_A^i \vee S^i)$ is transitive on k-invariants. Similarly, 3.1 says that $\operatorname{Aut} \pi_i(X \vee 2S^i)$ is transitive on k-invariants, for any minimal root X of $HT(\pi, i)$.

Consider the function

$$s: \{0, 1\} \times \{0, 1, 2\} \times Z \to \{0, 1, 2\}$$

given by

$$s(\epsilon, \ell, \delta) = \begin{cases} 0 & \text{if } \ell > 0 \text{ and } \delta \geq \ell, \text{ or } \ell = 0 \text{ and } \delta > 0. \\ \ell - \delta & \text{if } \ell > 0 \text{ and } \delta < \ell \\ \epsilon & \text{if } \ell = 0 \text{ and } \delta = 0. \end{cases}$$

Note that $s(0, 0, \delta) = 0$ for all $\delta \geq 0$. Then consider the following five statements about a finite group π and a *minimal* root X of $HT(\pi, 2)$.

$Tr(\ell, X)$: For some ℓ, $0 \leq \ell \leq 2$, $\operatorname{Aut}(\pi_2 X \oplus (Z\pi)^\ell)$ is transitive on k-invariants.

$CP(\epsilon, X)$: $\pi_2 X \oplus (Z\pi)^\epsilon$ has the cancellation property ($\epsilon = 0, 1$).

$Ht(\ell, \epsilon)$: The essential height of $HT(\pi, 2) \leq \max(\epsilon, \ell)$ ($\epsilon = 0, 1, 0 \leq \ell \leq 2$).

$\mathscr{R}(\ell, \epsilon)$: Let $\mathbb{T} = (\pi, \pi_2, k)$ be any finitely chain 2-realizable 2-type

such that $\chi(\mathbb{T})-\chi_{\min}(\pi, 2)=\delta$. Then $\mathbb{T}\oplus(Z\pi)^s=(\pi, \pi_2\oplus(Z\pi)^s, k)$ is finitely 2-realizable, where $s=s(\epsilon, \ell, \delta)$.

$\mathcal{D}(\ell, \epsilon)$: Let Y be a connected complex with fundamental group π which is dominated by a finite 2-complex. Let the Wall invariant of Y vanish and $\delta=\chi(\mathbb{T}(Y))-\chi_{\min}$. Then $Y\vee\overset{s}{\vee}S^2$ has the homotopy type of a $(\pi, 2)$-complex where $s=s(\epsilon, \ell, \delta)$.

The following theorem has a proof similar to those of 5.3, 5.4 and theorems 1 and 2.

THEOREM 5.5: *Let π be a finite group and X be a minimal $(\pi, 2)$-complex. If we assume $Tr(X, \ell)$* and *$CP(X, \epsilon)$, then $Ht(\epsilon, \ell)$, $\mathcal{R}(\epsilon, \ell)$ and $\mathcal{D}(\epsilon, \ell)$ are true.* □

EXAMPLE 1: If $\pi=A$ is a finite abelian group and $X=K_A^2$, then $\epsilon=\ell=1$ and 5.5 yields 5.3, 5.4 and theorem 2.

EXAMPLE 2: If $\pi=Z_n$, $X=K(n)^2$ (see 5.1), then $\pi_2X\cong I$, the augmentation ideal in $Z(Z_n)$. By [3, proposition 5.3, page 267] I has CP, hence $\epsilon=0$. By proposition 4.1 of [3, page 265] I is transitive on k-invariants, hence $\ell=0$. Thus we recover the theorem of [3] that the height of $HT(Z_n, 2)$ is zero ($Ht(0,0)$) and theorem 5.2 ($\mathcal{R}(0,0)$) and corollary 5.3 ($\mathcal{D}(0,0)$) of [2].

EXAMPLE 3: Let $\pi=D_{2n}$, the dihedral group of order $2n$, with n odd. Let X be the cellular model associated with the efficient presentation $\mathcal{P}=\{x, y\colon y^2, yxyx^{-n+1}\}$ of D. D is a periodic group of minimal free period 4 and π_2X is transitive on k-invariants by [3, proposition 4.1] and has the cancellation property because D satisfies the Eichler condition (see [12, page 178] and [3, page 278]). Hence $\epsilon=\ell=0$. Thus *$HT(D, 2)$ has height* 0, *any finitely chain 2-realizable 2-type (D, D_2, k) is finitely 2-realizable* and *any complex Y with $\pi_1Y\cong D$ which is dominated by a finite 2-complex has the homotopy type of a $(D, 2)$-complex iff the Wall invariant vanishes.*

Note that the above statement is true for any group π satisfying Eichler's condition and having a $(\pi, 2)$-complex X such that $\pi_2X\cong Z\pi/(N)$ (hence, π must be a periodic group with period 4).

EXAMPLE 4: Let G be the group of order $4n$ with efficient presentation $\mathcal{P}=\{a, b: a^n=b^2, ba=a^{-1}b\}$. G is periodic of period 4 and if X is the cellular model associated with $\mathcal{P}$, then $\pi_2X\cong ZG/(N)$. If n is odd, then G satisfies the Eichler condition and $\epsilon=\ell=0$ (see example