COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 36, nº 2 (1978), p. 203-207
<http://www.numdam.org/item?id=CM_1978__36_2_203_0>

COMPOSITIO MATHEMATICA, Vol. 36, Fasc. 2, 1978, pag. 203–208
Sijthoff & Noordhoff International Publishers–Alphen aan den Rijn
Printed in the Netherlands

SOME COUNTEREXAMPLES FOR INFINITE DIMENSIONAL LIE ALGEBRAS*

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The purpose of this paper is to provide counterexamples which show that two important theorems from group theory do not have valid Lie theoretic analogues. These theorems have recently been proved by J.S. Wilson [5, pp. 19–21] and are very useful in the study of the chain conditions on normal subgroups of infinite groups. The theorems state that the maximal and the minimal conditions on normal subgroups are inherited by any subgroup of finite index. The analogous theorems for Lie algegras would say that the chain conditions on ideals of a Lie algebra are inherited by any ideal of finite co-dimension. The examples presented in this paper indicate that, in general, this is not the case.

The product of two elements of a Lie algebra will be denoted as $[x, y]$ and the expression $[x, {}_n y]$ is defined inductively by: $[x, {}_1 y] = [x, y]$ and $[x, {}_n y] = [[x, {}_{n-1} y], y]$. We will say that a Lie algebra has the property Min-$\lhd$, or is in the class Min-$\lhd$, if every collection of ideals contains a minimal ideal; that is, the algebra satisfies the minimal condition on ideals. Similarly, we will use the notation Max-$\lhd$ when referring to the maximal condition on ideals.

The first example is to show that Min-$\lhd$ is not inherited by ideals of finite co-dimension. Let I be the infinite dimensional abelian Lie algebra with basis $\{x_i\}$ for $i \geq 1$ over any field. Define a derivation, δ, on I by $x_i\delta = x_{i-1}$ for $i > 1$ and $x_1\delta = 0$. Now let D be the split extension of I by the one-dimensional algebra $\langle\delta\rangle$; that is $D = \langle\delta\rangle \oplus I$. Then $I \lhd D$, D/I is finite dimensional, and I does not satisfy Min-$\lhd$ since it is infinite dimensional abelian. D, however, does satisfy the chain condition. Let H be an ideal of D. Then either H is contained in I or there is a $\delta_1 \in H$ with $\delta_1 \equiv \delta (\text{mod}\, I)$. But then we have $H \geq [\delta_1, I] = I$, and $H = D$. Thus the only proper ideals of D are contained in

* The material in this paper is taken from the author's Ph.D. thesis, The University of Michigan, 1974.

I. Suppose H is an ideal of D which is contained in I and let x be a non-zero element of H. If $x = \sum_{i=1}^{n} \alpha_i x_i$ with $\alpha_n \neq 0$, then $[x, {}_{n-1}\delta] = \alpha_n x_1 \in H$ and so $x_1 \in H$. Likewise $[x, {}_{n-2}\delta] = \alpha_{n-1}x_1 + \alpha_n x_2 \in H$ so we can write $x_2 = \alpha_n^{-1}([x, {}_{n-2}\delta] - \alpha_{n-1}x_1) \in H$. Continuing in this way we find that $x_1, x_2, \ldots, x_n \in H$. This means that all the elements of H can be written in terms of a finite number of the x_i and H is finite dimensional or else we can obtain $x_i \in H$ for arbitrarily large i and conclude $H = I$. Consequently, the only ideals of D are O, I, D, or are of finite dimension so that $D \in \text{Min-}\lhd$.

A similar example can be used in the case of Max-$\lhd$. Let k be any field and consider $k[x]$, the polynomial ring in one indeterminate, as an abelian Lie algebra over k. Define a derivation, δ, on $k[x]$ by $x^i\delta = x^{i+1}$; that is, just multiplication by x in the polynomial ring. The subspaces which are invariant under δ correspond exactly to the ideals of $k[x]$ which is known to be Noetherian. Now take P to be the split extension of $k[x]$ by $\langle\delta\rangle$. The ideals of P are then O, P, H, or $\langle\delta_1\rangle + H$ where H is an ideal of the ring $k[x]$ and $\delta_1 \equiv \delta (\text{mod } k[x])$. We can do better than this when considering ideals of the form $J = \langle\delta_1\rangle + H$. Since $1 \in k[x]$, $[1, \delta_1] = x \in J$ and we have that (x), the ideal of $k[x]$ generated by x, is contained in H. This means the only ideals of this form are $\delta_1 + k[x]$ or $\delta_1 + (x)$. Now P must satisfy Max-$\lhd$ while $k[x] \lhd P$, $P/k[x]$ is finite dimensional and $k[x]$ does not satisfy Max-$\lhd$ (as an abelian Lie algebra).

It is thus relatively easy to see that the two chain conditions, independently, are not inherited by ideals of finite co-dimension. It might still be possible that imposing the two conditions simultaneously would lead to at least one of the conditions being inherited by the proper type of ideals. But this too is false.

C.W. Curtis [2, pp. 954–5] presents an example, due to Jacobson, of a two-dimensional soluble Lie algebra with an infinite dimensional irreducible representation over a field of characteristic 0. Consider the representation space as an abelian Lie algebra, I, and the soluble algebra, S, as an algebra of derivations on I. Let L be the split extension of I by S. Now L satisfies both chain conditions, $I \lhd L$, L/I is finite dimensional, but I satisfies neither chain condition.

A counterexample for chacteristic p, analogous to the above, does not exist, for any irreducible representation of a finite dimensional algebra over a field of characteristic p must also be finite dimensional. This result is buried in the proof of a theorem by Curtis [2, p. 952] but a more direct proof has been supplied by J.E. McLaughlin and can be found in the author's doctoral thesis ([3, pp. 38–40].

We can, however, construct an example of a Lie algebra, S, over a field, k, of characteristic $p > 0$ (for each prime p) which satisfies both Max-$\lhd$ and Min-$\lhd$ but which has an ideal, I, of finite co-dimension which satisfies neither chain condition.

Let L be an arbitrary Lie algebra over a field k of characteristic $p > 0$. Let ϕ_i: $L \to L_i$, $i \geq 0$, be Lie isomorphisms. Set $I = L_0 \oplus L_1 \oplus \ldots$ and let $x_i = x\phi_i$ for all $i \geq 0$ and all $x \in L$. Each L_i is a vector space over k and so their sum, I, is also. We now define a multiplication on I as follows:

$$[x_i, y_j] = \lambda_{i,j}[x, y]_{i+j} \text{ where } \lambda_{i,j} = \binom{i+j}{i} \pmod p.$$

Under this multiplication I is a Lie algebra. First note that

$$\lambda_{i,j} = \binom{i+j}{i} = \binom{i+j}{j} = \lambda_{j,i}$$

so that

$$[x_i, y_j] = \lambda_{i,j}[x, y]_{i+j} = -\lambda_{i,j}[y, x]_{i+j} = -\lambda_{j,i}[y, x]_{j+i} = -[y_j, x_i].$$

To check the Jacobi identity we note the following:

$$\lambda_{i,j}\lambda_{i+j,k} = \frac{(i+j)!}{i!j!}\frac{(i+j+k)!}{(i+j)!k!} = \frac{(i+j+k)!}{i!j!k!}$$

and likewise

$$\lambda_{i,k}\lambda_{i+k,j} = \frac{(i+j+k)!}{i!j!k!} \qquad \lambda_{j,k}\lambda_{j+k,i} = \frac{(i+j+k)!}{i!j!k!}.$$

Since these are all equal, we call their common value $\lambda_{i,j,k}$. Now

$$\begin{aligned}
&[x_i, y_j, z_k] + [y_j, z_k, x_i] + [z_k, x_i, y_k] \\
&\quad = \lambda_{i,j}\lambda_{i+j,k}[x, y, z]_{i+j+k} + \lambda_{j,k}\lambda_{j+k,i}[y, z, x]_{i+j+k} + \lambda_{k,i}\lambda_{k+i,j}[z, x, y]_{i+j+k} \\
&\quad = \lambda_{i,j,k}([x, y, z] + [y, z, x] + [z, x, y])_{i+j+k} = 0.
\end{aligned}$$

Thus I is a Lie algebra over k.

We define a derivation, δ, on I by $x_i\delta = x_{i-1}$ for $i > 0$ and $x_0\delta = 0$. To check that δ is a derivation we have that

$$[x_i, y_j]\delta = \lambda_{i,j}[x, y]_{i+j-1}$$

and also

$$[x_i\delta, y_j] + [x_i, y_j\delta] = \lambda_{i-1,j}[x, y]_{i+j-1} + \lambda_{i,j-1}[x, y]_{i+j-1}$$
$$= (\lambda_{i-1,j} + \lambda_{i,j-1})\,[x, y]_{i+j-1}.$$

Hence δ will be a derivation as long as $\lambda_{i,j} = \lambda_{i-1,j} + \lambda_{i,j-1}$ which is an easily verified identity.

Now let S be the split extension of I by the one-dimensional algebra $\langle\delta\rangle$. Consider the case when our original algebra, L, is simple, non-abelian, and look at the ideals of S. Just as in the algebra D (the first example), the only proper ideals of S are contained in I. Suppose $K \lhd S$, $K \leq I$ and suppose that $x = \alpha_0 x_0^0 + \ldots + \alpha_n x_n^n \in K$ where $x_i^j = x^j\phi_i \in L_i$ and $\alpha_n x_n^n \neq 0$. Then $[x, {}_n\delta] = \alpha_n x_0^n \in K$ and so $x_0^n \in K$. Therefore $K \geq [x_0^n, L_0] = [x^n, L]_0 = L_0$ since L is simple non-abelian. But now $[L_0, L_i] \leq K$ for all i and since $\lambda_{0,i} = 1$ for all i, $[L_0, L_i] = [L, L]_i = L_i \leq K$ so $K = I$. Thus the only ideals of S are S, I, and 0 so that S obviously satisfies both Min-$\lhd$ and Max-$\lhd$.

S/I is finite dimensional but I satisfies neither chain condition. Since $[L_i, L_j] \leq L_{i+j}$, if we set $K_n = \Sigma_{i=n}^{\infty} L_i$ we have $K_n \lhd I$ and $K_{n+1} < K_n$ so that the K_n form an infinite strictly decreasing chain of ideals. To see that $I \notin$ Max-$\lhd$, let p be the characteristic of the field and set

$$J_n = \sum_{\substack{k=1 \\ k \neq ip^n,\ \text{all } i}}^{\infty} L_k \quad \text{for } n \geq 1.$$

Then $J_1 \leq J_2 \leq \ldots$ and the inclusions are proper since, in particular, $L_{p^n} \leq J_{n+1}$ while $L_{p^n} \not\leq J_n$. To check that the J_n are ideals of I, suppose $L_i \leq J_n$ and consider any L_j. If $i + j \neq mp^n$ for any m, then $[L_i, L_j] \leq L_{i+j} \leq J_n$. If $i + j = mp^n$, then, since $L_i \leq J_n$, $i \neq rp^n$ for any r. In this case we have that $[L_i, L_j] = \lambda_{i,j} L_{i+j}$ but $\lambda_{i,j} = \binom{i+j}{i} = \binom{mp^n}{i} \equiv 0 \pmod p$ so $[L_i, L_j] = 0 \leq J_n$. Thus the J_n are ideals and we have an infinite strictly increasing chain of ideals of I so that $I \notin$ Max-$\lhd$ and our example is complete.

The questions of whether the chain conditions are inherited can be extended to subalgebras which are somewhat more general than ideals. We say that a subalgebra, H, of a Lie algebra, L, is a subideal if there is a finite descending chain of subalgebras from L to H where each subalgebra is an ideal in the previous subalgebra. An α-step subideal is a subideal with such a chain of length α. We use the notation Min-si and Min-$\lhd^{\alpha}$ to denote the minimum condition on subideals and α-step subideals respectively. Since any subideal of a subideal is itself a subideal of the whole algebra, Min-si and Max-si will be inherited by any ideal. For Min-$\lhd^{\alpha}$, $\alpha > 1$, the question is completely answered. For the case of a Lie algebra over a field of characteristic 0, I.N. Stewart has shown [4, p. 93] that Min-$\lhd^2$ implies Min-si so that for $\alpha > 1$, Min-$\lhd^{\alpha}$ will be inherited by any ideal since Min-si is. At the same time, Stewart shows that for characteristic p, Min-$\lhd^3$ implies Min-si so that Min-$\lhd^{\alpha}$ is inherited by all ideals for $\alpha > 2$. The case $\alpha = 2$ is answered in the negative by an example of Amayo and Stewart [1, pp. 16–19]. Their example was constructed to show that Min-$\lhd^2$ does not imply Min-si in the case of characteristic p and it does contain an ideal of finite co-dimension which does not satisfy Min-$\lhd^2$ while the whole algebra does. Their example also illustrates that Max-$\lhd^2$ is not inherited in the characteristic p case but the remaining questions for Max-$\lhd^{\alpha}$ are still unanswered.

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(Oblatum 17–XII–1976)

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