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A pseudo-interior of $\lambda I$

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Abstract

We show that the subspace $\lambda_{comp}^R$ of $\lambda R$ is homeomorphic to the pseudo-boundary $B(Q) = \{x \in Q \mid \exists i \in \mathbb{N} : |x_i| = 1\}$ of the Hilbert cube $Q$. This answers a question of A. Verbeek raised in [9].

1. Introduction

If $X$ is a topological space, then the superextension $\lambda X$ of $X$ denotes the space of all maximal linked systems consisting of closed subsets of $X$ (a system is called linked if every two of its members meet; a maximal linked system or mls is a linked system not properly contained in another linked system) topologized by taking $\{[M \in \lambda X \mid G \in M] \mid G = G^- \subset X\}$ as a closed subbase (De Groot [4]). In case $(X, d)$ is a compact metric space, then $\lambda X$ also is compact metric (Verbeek [9]) and the topology of $\lambda X$ also can be described by the metric

$$\tilde{d}(\mathcal{M}, \mathcal{N}) = \sup_{S \in \mathcal{M}} \min_{T \in \mathcal{N}} d_H(S, T);$$

here $d_H(S, T)$ denotes the Hausdorff distance of $S$ and $T$ defined by $\inf\{\varepsilon > 0 \mid S \subset U_\varepsilon(T) \text{ and } T \subset U_\varepsilon(S)\}$, where as usual $U_\varepsilon(T)$ denotes the $\varepsilon$-neighborhood of $T$ (Verbeek [9]). Reflecting on this metric, one sees that there must be a connection between $\lambda X$ and the hyperspace of all nonvoid closed subsets $2^X$ of $X$. The hyperspace $2^X$ is homeomorphic to the Hilbert cube $Q$ if and only if $X$ is a non-degenerate Peano continuum (Curtis & Schori [3]) and it was con-

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jectured by Verbeek [9] that $\lambda X$ is homeomorphic to $Q$ if and only if $X$ is a nondegenerate metrizable continuum. Earlier, De Groot conjectured that $\lambda I$ is homeomorphic to the Hilbert cube, where $I$ denotes the real number interval $[-1, 1]$. This was shown to be true in [7]. If $X$ is a noncompact metrizable space then $\lambda X$ is not metrizable, although it contains some interesting dense metrizable subspaces such as $\lambda_{\text{comp}} X$ (Verbeek [9]). This subspace of $\lambda X$ consists of all maximal linked systems which have a compact defining set, where an mls $M$ is said to be defined on a set $M$ if

$$\text{for all } S \in M \text{ there exists an } S' \in M \text{ such that } S' \subseteq S \cap M.$$ 

It is obvious that $\lambda_{\text{comp}} X$ equals $\lambda X$ in case $X$ is compact, for then $X$ is a compact defining set for all $M \in \lambda X$. In case $X$ is noncompact there are many maximal linked systems which do not have a compact defining set, for example in case $X = \mathbb{R}$, the real line, $|\lambda_{\text{comp}} \mathbb{R}| = c$ while $|\lambda \mathbb{R}| = 2^c$. Verbeek [9] showed that $\lambda_{\text{comp}} \mathbb{R}$ is a dense, metrizable, contractible, separable, locally connected, strongly infinite dimensional subspace of $\lambda \mathbb{R}$ which is in no point locally compact; he conjectured that $\lambda_{\text{comp}} \mathbb{R}$ is homeomorphic to $l_2$, the separable Hilbert space. We will show that this is not true. In fact we will show that $\lambda_{\text{comp}} \mathbb{R}$ is homeomorphic to the pseudo-boundary $B(Q) = \{x \in Q \mid \exists i \in \mathbb{N} : |x_i| = 1\}$ of the Hilbert cube $Q$. As $\lambda_{\text{comp}} \mathbb{R}$ is homeomorphic to $\lambda_{\text{comp}} (-1, 1)$, which can be identified with the subspace of $\lambda I$ consisting of all maximal linked systems with a compact defining set in $(-1, 1)$ (Verbeek [9]), we can work in $\lambda I = Q$. We will show that $\lambda_{\text{comp}} (-1, 1)$ is a capset of $\lambda I$ (for definitions see section 3) so that $\lambda I \setminus \lambda_{\text{comp}} (-1, 1)$ is a pseudo-interior for $\lambda I$ and hence is homeomorphic to $l_2$ (Anderson [2]).

This paper is organised as follows: in the second section we give a retraction property of superextensions, which is needed to prove that $\lambda_{\text{comp}} (-1, 1)$ is a capset of $\lambda I$. The third section shows that $\lambda_{\text{comp}} (-1, 1)$ is a capset of $\lambda I$ using a lemma of Kroonenberg [6].

### 2. A retraction property of superextensions

All topological spaces under discussion are assumed to be normal $T_1$; linked system will always mean linked system consisting of closed subsets of the topological space under consideration. If $G$ is a closed subset of the topological space $X$, then we define $G^+$ as $G^+ = \{M \in \lambda X \mid G \in M\}$; $\lambda X$ is topologized by taking $\{G^+ \mid G$ is closed in $X\}$ as a closed subbase. This subbase has the property that each
linked subsystem of it has a nonvoid intersection so that by Alexander's subbase lemma, \(\lambda X\) always is compact. Moreover \(X\) can be embedded in it by means of the natural embedding \(i(x) = \{G \subseteq X \mid G\text{ is closed and } x \in G\}\). We will always identify \(X\) and \(\vec{i}[X]\). Every linked system is contained in at least one maximal linked system by Zorn's lemma. A linked system \(\mathcal{M}\) is called a pre-mls if it is contained in precisely one mls; this mls is then denoted by \(\overline{\mathcal{M}}\) and we say that \(\mathcal{M}\) is a pre-mls for \(\overline{\mathcal{M}}\). Obviously \(\mathcal{M}\) is a pre-mls iff for all closed sets \(S_0\) and \(S_1\) such that \(\mathcal{M} \cup \{S_i\}\) is linked \((i = 0, 1)\) we have \(S_0 \cap S_1 \neq \emptyset\). If \(S\) is a closed subset of the compact metric space \((X, d)\) then for each \(\epsilon > 0\) we define

\[B_\epsilon(S) = \{x \in X \mid d(x, S) \leq \epsilon\}.
\]

**Lemma 2.1:** Let \((X, d)\) be a compact metric space and let \(\mathcal{M}\) be a pre-mls for \(\overline{\mathcal{M}} \subseteq \lambda X\). Then for each \(\mathcal{N} \subseteq \lambda X\) we have that \(\overline{\alpha}(\mathcal{M}, \mathcal{N}) = \inf\{a \geq 0 \mid \forall S \in \mathcal{M} : B_a(S) \subseteq \mathcal{N}\}\).

**Proof:** Verbeek [9] proved the following

\[
\overline{\alpha}(\mathcal{M}, \mathcal{N}) = \min\{a \geq 0 \mid \forall S \in \mathcal{M} : B_a(S) \subseteq \mathcal{N}\text{ and } \forall T \in \mathcal{N} : B_a(T) \subseteq \mathcal{M}\}
= \min\{a \geq 0 \mid \forall S \in \mathcal{M} : B_a(S) \subseteq \mathcal{N}\}
\]

and therefore \(\inf\{a \geq 0 \mid \forall S \in \mathcal{M} : B_a(S) \subseteq \mathcal{N}\} \leq \overline{\alpha}(\mathcal{M}, \mathcal{N})\). Let us assume that \(\inf\{a \geq 0 \mid \forall S \in \mathcal{M} : B_a(S) \subseteq \mathcal{N}\} < \overline{\alpha}(\mathcal{M}, \mathcal{N})\). Then there exists an \(a_0\) such that \(0 \leq a_0 < \overline{\alpha}(\mathcal{M}, \mathcal{N})\) with the property that for all \(S \in \mathcal{M}\) we have that \(B_{a_0}(S) \subseteq \mathcal{N}\) while there exists a \(T \in \mathcal{N}\) such that \(B_{a_0}(T) \notin \mathcal{M}\). As \(\mathcal{M}\) is a pre-mls for \(\overline{\mathcal{M}}\) there is an \(M \in \mathcal{M}\) such that \(B_{a_0}(T) \cap M = \emptyset\). However \(B_{a_0}(M) \subseteq \mathcal{N}\), so that \(B_{a_0}(M) \cap T \neq \emptyset\). Now, as \(X\) is compact, this is a contradiction. \(\square\)

The distance between two maps \(f, g : X \to Y\), where \((Y, d)\) is compact metric, is defined by \(d(f, g) = \sup_{x \in X} d(f(x), g(x))\). The identity mapping on \(X\) is denoted by \(id_X\).

**Theorem 2.2:** Let \(X\) be a topological space and let \(\mathcal{M}\) be a linked system in \(X\). Then \(\cap\{M^+ \mid M \in \mathcal{M}\}\) is a retract of \(\lambda X\). Moreover, if \((X, d)\) is compact metric then the retraction map \(r\) can be chosen in such a way that \(\overline{d}(r, id_{\lambda X}) \leq \sup_{M \in \mathcal{M}} d_H(X, M)\).

**Proof:** Let \(\mathcal{M}\) be a linked system in \(X\). Notice that \(\cap\{M^+ \mid M \in \mathcal{M}\} \neq \emptyset\). Choose \(\mathcal{N} \subseteq \lambda X\) and define \(P\mathcal{N} = \{N \in \mathcal{N} \mid \{N\} \cup \mathcal{M}\text{ is linked}\}\). \(\mathcal{N}\).

(a) \(P\mathcal{N}\) is a pre-mls.
It is obvious that \( PN \) is linked; so assume to the contrary that it were not a pre-mls. Then there exist closed sets \( S_i \) such that \( PN \cup \{S_i\} \) is linked \((i = 0, 1)\) but \( S_0 \cap S_1 = \emptyset \). The normality of \( X \) implies that there exist closed sets \( G_i \) \((i = 0, 1)\) such that \( S_0 \cap G_1 = \emptyset = G_0 \cap S_1 \) and \( G_0 \cup G_1 = X \). Now, as \( N \) is a maximal linked system one of the sets \( G_i \) must belong to \( N \) (if \( G_i \notin N \) \((i = 0, 1)\) then there exist \( M_i \in N \) such that \( M_i \cap G_1 = \emptyset \) \((i = 0, 1)\) so that \( M_0 \cap M_1 = \emptyset \) contradicting the linkedness of \( N \)) so that we may assume that \( G_0 \in N \). Now, \( S_0 \subseteq G_0 \) implies that \( N \cup \{G_0\} \) is linked and consequently \( G_0 \in PN \). This is a contradiction since \( G_0 \cap S_1 = \emptyset \).

(b) Define \( r: \lambda X \rightarrow \lambda X \) by \( r(N) = PN \). Then \( r \) is continuous.

Let \( G \) be a closed set of \( X \) and assume that \( r^{-1}(G^+) \neq \emptyset \). We will show that \( r^{-1}(G^+) \) is closed in \( \lambda X \). Choose \( N \notin r^{-1}(G^+) \). Then \( r(N) \notin G^+ \) and consequently \( r(N) \cup \{G\} \) is not linked; therefore \( PN \cup \{G\} \) is not linked. Choose \( N \in PN \) so that \( N \cap G = \emptyset \). Now, if \( N \in M \), then \( r^{-1}(G^+) \) is void, which is a contradiction. Therefore \( N \notin N \). Choose closed sets \( S_i \) \((i = 0, 1)\) such that \( S_0 \cap N = \emptyset = G \cap S_1 \) and \( S_0 \cup S_1 = X \). Then \( N \in \lambda X \setminus S_0^\circ \subseteq S_1^\circ \), while moreover \( \lambda X \setminus S_0^\circ \cap r^{-1}(G^+) = \emptyset \). For assume to the contrary that there exists a \( x \in (\lambda X \setminus S_0^\circ) \cap r^{-1}(G^+) \). Then \( S_1 \in \xi \) and \( M \cup \{N\} \) is linked implies that \( M \cup \{S_i\} \) is linked and consequently \( S_1 \in P \xi \subseteq r(\xi) \). This is a contradiction, since \( G \in r(\xi) \) and \( S_1 \cap G = \emptyset \).

(c) \( r(\lambda X) = \cap \{M^+ \mid M \in M\} \) and \( r \) is a retraction.

Choose \( N \in \lambda X \). Then \( M \in PN \subseteq r(N) \) so that \( r(N) \in \cap \{M^+ \mid M \in M\} \). Moreover if \( N \in \cap \{M^+ \mid M \in M\} \) then \( PN = N \) and therefore \( r(N) = N \).

(d) If \((X, d)\) is compact metric, then \( \bar{d}(r, id_\lambda x) = \sup_{M \in \mu} d_H(X, M) \).

Let \( a = \sup_{M \in \mu} d_H(X, M) \) and choose \( N \in \lambda X \). Take \( N \in PN \) and consider \( B_\delta(N) \). If \( N \in N \) then also \( B_\delta(N) \in N \); if \( N \notin N \) then \( N \in M \) and therefore \( B_\delta(N) = X \) which also is an element of \( N \). It now follows that

\[
\bar{d}(N, r(N)) = \inf \{a \geq 0 \mid \forall S \in PN : B_\delta(S) \subseteq N\}
\]

(lemma 2.2)

\[
\leq \sup_{M \in \mu} d_H(X, M).
\]

If \( Y \) is a closed subset of \( X \), then \( \lambda Y \) can be embedded in \( \lambda X \) by the natural embedding \( j_{\lambda Y} \) defined by

\[
j_{\lambda Y}(M) := \{G \subseteq X \mid G \text{ is closed and } G \cap Y \in M\}
\]

(Verbeek [9]). It should be noticed that \( j_{\lambda Y}(M) \) is indeed a maximal linked system. We will always identify \( \lambda Y \) and \( j_{\lambda Y}(\lambda Y) \).
LEMMA 2.3: Let \( Y \) be a closed subset of \( X \). Then \( \mathcal{M} \in \lambda X \) is an element of \( \lambda Y \) if and only if \( \{ M \cap Y \mid M \in \mathcal{M} \} \) is linked.

PROOF: If \( \mathcal{M} \in \lambda Y \), then \( \{ M \cap Y \mid M \in \mathcal{M} \} \) is a maximal linked system in \( Y \) and if \( \{ M \cap Y \mid M \in \mathcal{M} \} \) is linked, then it is easy to see that it is also maximal linked (in \( Y \)) and that \( j_{YX}(\{ M \cap Y \mid M \in \mathcal{M} \}) = \mathcal{M} \).

The importance of Theorem 2.2 now is demonstrated in the proof of the following theorem.

THEOREM 2.4: Let \( (X, d) \) be a compact connected metric space and let \( Y \) be a nonempty closed proper subset of \( X \). Then for each \( \epsilon > 0 \) there exists a continuous map \( f : \lambda X \to \lambda X \setminus \lambda Y \) such that \( d(f, id_{\lambda X}) < \epsilon \).

PROOF: Choose \( \epsilon > 0 \) and choose two disjoint finite sets \( G_0 \) and \( G_1 \) such that \( d_{H}(G_i, X) < \epsilon \) \((i = 0, 1)\). Let \( p \in X \setminus Y \) and define \( F_i = G_i \cup \{ p \} \). Let \( f_{\epsilon} \) be the retraction of \( \lambda X \) onto \( F_0^+ \cap F_1^+ \) as defined in Theorem 2.2. Then \( \tilde{d}(f_{\epsilon}, id_{\lambda X}) \leq \max\{ d_{H}(F_0, X), d_{H}(F_1, X) \} < \epsilon \) and moreover \( f_{\epsilon}(\lambda X) \cap \lambda Y = \emptyset \). For take \( N \in f_{\epsilon}(\lambda X) \); then \( F_i \in N \) \((i = 0, 1)\) and \( (F_0 \cap Y) \cap (F_1 \cap Y) = \emptyset \) and consequently, by Lemma 2.3, \( \forall \in \lambda Y \).

3. A Pseudo-interior of \( \lambda I \)

By the Hilbert cube \( Q \) we mean the countable infinite product of intervals \([-1, 1]^\omega\) with the product topology. The topology is generated by the metric

\[
d(x, y) = \sum_{i=1}^{\infty} 2^{-i} |x_i - y_i|.
\]

A closed subset \( A \) of \( Q \) is called a Z-set (Anderson [1]) if for each \( \epsilon > 0 \) there exists a continuous map \( f : Q \to Q \setminus A \) such that \( d(f, id_Q) < \epsilon \). In addition, a subset \( M \) of \( Q \) is called a capset for \( Q \) (Anderson [2]) if \( M \) can be written as \( M = \bigcup_{i=1}^{\infty} M_i \), where each \( M_i \) is a Z-set in \( Q \), \( M_i \subseteq M_{i+1} \) \((i \in \mathbb{N})\) and such that the following absorption property holds: for each \( \epsilon > 0 \) and \( i \in \mathbb{N} \) and every Z-set \( K \subset Q \) there exists a \( j > i \) and an embedding \( h : K \to M_j \) such that \( h \mid K \cap M_i = id_{K \cap M_i} \) and \( d(h, id_K) < \epsilon \). It is known that every capset of \( Q \) is equivalent to \( B(Q) = \{ x \in Q \mid \exists i \in \mathbb{N} : |x_i| = 1 \} \), the pseudo-boundary of \( Q \), under an autohomeomorphism of \( Q \) [2]). The complement of a capset is called a pseudo-interior of \( Q \) and is homeomorphic to \( l_2 \), the separable Hilbert space ([2]). We will show that \( \lambda_{\text{comp}}(-1, 1) \) is a capset of \( \lambda I \),
using the fact that $\lambda I = Q$ ([7]). It then follows that $\lambda I \setminus \lambda_{\text{comp}}(-1, 1)$ is a pseudo-interior for $\lambda I$. In [6] an alternative characterization of capsets is given and we will make use of that characterization.

**Lemma 3.1 ([6]):** Suppose $M$ is a $\sigma$-compact subset of $Q$ such that

(i) For every $\epsilon > 0$, there exists a map $h : Q \to Q \setminus M$ such that $d(h, \text{id}_Q) < \epsilon$.

(ii) $M$ contains a family of compact subsets $M_1 \subseteq M_2 \subseteq \cdots$ such that each $M_i$ is a copy of $Q$ and $M_i$ is a $Z$-set in $M_{i+1}$ ($i \in \mathbb{N}$), and such that for each $\epsilon > 0$ there exists an integer $i \in \mathbb{N}$ and a map $h : Q \to M_i$ with $d(h, \text{id}_Q) < \epsilon$.

Then $M$ is a capset for $Q$.

First we will show that $\lambda_{\text{comp}}(-1, 1)$ is $\sigma$-compact.

**Lemma 3.2:** $\lambda_{\text{comp}}(-1, 1) = \bigcup_{n=2}^{\infty} \lambda [-1 + 1/n, 1 - 1/n]$.

**Proof:** Choose $\mathcal{M} \in \lambda_{\text{comp}}(-1, 1)$ and let $M \subseteq (-1, 1)$ be a compact defining set for $\mathcal{M}$. Then choose $n_0 \geq 2$ such that $M \subseteq [-1 + 1/n_0, 1 - 1/n_0]$; from Lemma 2.3 it now follows that $\mathcal{M} \subseteq \lambda[-1 + 1/n_0, 1 - 1/n_0]$.

Moreover, if $\mathcal{M} \in \lambda[-1 + 1/n, 1 - 1/n]$ then for all $M \in \mathcal{M}$ we have that also $M \cap [-1 + 1/n, 1 - 1/n]$ belongs to $\mathcal{M}$, showing that $[-1 + 1/n, 1 - 1/n]$ is a defining set for $\mathcal{M}$. For assume to the contrary that for some $M \in \mathcal{M}$ it would be true that $M \cap [-1 + 1/n, 1 - 1/n] \notin \mathcal{M}$; then there would exist an $M_0 \in \mathcal{M}$ such that $M_0 \cap [-1 + 1/n, 1 - 1/n] \cap M = \emptyset$, contradicting the linkedness of $\{M \cap [-1 + 1/n, 1 - 1/n] \mid M \in \mathcal{M}\}$ (Lemma 2.3).

**Lemma 3.3:** For each $\epsilon > 0$ there exists a map $f_\epsilon : \lambda I \to \lambda I \setminus \lambda_{\text{comp}}(-1, 1)$ such that $\tilde{d}(f_\epsilon, \text{id}_{\lambda I}) < \epsilon$.

**Proof:** Choose $\epsilon > 0$. For each $n \geq 2$, let $F_{n,0}$ and $F_{n,1}$ be finite subsets of $I$ such that

(i) $d_H(I, F_{n,i}) < \frac{\epsilon}{4} (i = 0, 1)$

(ii) $F_{n,0} \cap F_{n,1} \cap [-1 + 1/n, 1 - 1/n] = \emptyset$

(iii) $(-1, 1) \subset F_{n,0} \cap F_{n,1}$,

and let $f_\epsilon$ be the retraction map, given by Theorem 2.2, of $\lambda I$ onto $\bigcap_{n=2}^{\infty} (F_{n,0}^+ \cap F_{n,1}^+)$. Then $\tilde{d}(f_\epsilon, \text{id}_{\lambda I}) \leq \sup\{d_H(I, F_{n,i}) \mid n \geq 2, \ i = 0, 1\} \leq \frac{\epsilon}{4} < \epsilon$, while moreover the image of $\lambda I$ is disjoint from $\lambda_{\text{comp}}(-1, 1)$.
For choose \( N \in f_2(\lambda I) \) and \( n \geq 2 \); then \( F_{n,i} \in N \) \((i = 0, 1)\) and \( F_{n,0} \cap F_{n,1} \cap [-1 + 1/n, 1 - 1/n] = \emptyset \). Therefore \( N \) is not an element of \( \lambda [-1 + 1/n, 1 - 1/n] \) by Lemma 2.3. Consequently \( N \not\subset \lambda_{\text{comp}}(-1, 1) \) (Lemma 3.2). \( \square \)

**THEOREM 3.4:** \( \lambda_{\text{comp}}(-1, 1) \) is a capset for \( \lambda I \).

**PROOF:** Choose \( \varepsilon > 0 \) and let \( n \geq 2 \) such that \( 1/n < \varepsilon \). Define a retraction \( r: [-1, 1] \rightarrow [-1 + 1/n, 1 - 1/n] \) by

\[
r(x) = \begin{cases} 
-1 + 1/n & \text{if } -1 \leq x \leq -1 + 1/n \\
x & \text{if } -1 + 1/n \leq x \leq 1 - 1/n \\
1 - 1/n & \text{if } 1 - 1/n \leq x \leq 1 
\end{cases}
\]

This map can be extended to a map \( \bar{r}: \lambda I \rightarrow \lambda [-1 + 1/n, 1 - 1/n] \) in the following manner

\[
\bar{r}(M) = \{G \subset [-1 + 1/n, 1 - 1/n] \mid G \text{ is closed and } r^{-1}(G) \in M\}
\]

(Verbeek [9]). Let \( j: \lambda [-1 + 1/n, 1 - 1/n] \rightarrow \lambda I \) be the natural embedding defined by \( j(M) = M = \{G \subset I \mid G \text{ is closed and } G \cap [-1 + 1/n, 1 - 1/n] \in M\} \). The composition \( g = j \circ \bar{r}: \lambda I \rightarrow \lambda I \) can be described by

\[
g(M) = (G \subset I \mid G \text{ is closed and } r^{-1}(G \cap [-1 + 1/n, 1 - 1/n]) \in M) .\]

We will show that \( g \) moves the points less than \( \varepsilon \). It is clear that \( g(\lambda I) = \lambda [-1 + 1/n, 1 - 1/n] \). Choose \( M \in \lambda I \) and assume that \( \bar{d}(M, g(M)) > 1/n \). Then there exists an \( M \in M \) such that \( B_{1/n}(M) \notin g(M) \) (Lemma 2.1). Consequently there exists a \( G \in g(M) \) such that \( r^{-1}(G \cap [-1 + 1/n, 1 - 1/n]) \in M \) and \( B_{1/n}(M) \cap G = \emptyset \). Now take a \( p \in M \cap r^{-1}(G \cap [-1 + 1/n, 1 - 1/n]) \). Then \( d(r(p), p) \leq 1/n \) and hence \( r(p) \in G \cap [-1 + 1/n, 1 - 1/n] \cap B_{1/n}(M) \subset G \cap B_{1/n}(M) \), which is a contradiction. It now follows that \( \bar{d}(g, id_M) \leq 1/n < \varepsilon \).

It is obvious that \( \lambda [-1 + 1/n, 1 - 1/n] \subset \lambda [-1 + 1/n + 1, 1 - 1/n + 1] \) \((n \geq 2)\), so that by Theorem 2.4, Lemma 3.2, Lemma 3.3 and the fact that \( \lambda [-1 + 1/n, 1 - 1/n] = \lambda I = Q \) the family \( \{\lambda [-1 + 1/n, 1 - 1/n] \mid n \geq 2\} \) satisfies all conditions of Lemma 3.1. Therefore \( \lambda_{\text{comp}}(-1, 1) \) is a capset for \( \lambda I \). \( \square \)

**COROLLARY 3.5:** \( \lambda_{\text{comp}}^{\mathbb{R}} \) is homeomorphic to \( B(Q) = \{x \in Q \mid \exists i \in \mathbb{N} : |x_i| = 1\} \). \( \lambda I \setminus \lambda_{\text{comp}}(-1, 1) \) is homeomorphic to \( l_2 \).

The space \( \lambda^{\mathbb{R}} \) now turns out to be a very strange space. It is a connected, locally connected (super)compact Hausdorff space of cardinality \( 2^c \) and weight \( c \), which possesses a dense subset.
homeomorphic to $B(Q)$. The closure of $\mathbb{R}$ in $\lambda\mathbb{R}$ is $\beta\mathbb{R}$, its Čech-Stone compactification (Verbeek [9]).

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