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A PSEUDO-INTERIOR OF λI*

J. van Mill

Abstract

We show that the subspace $\lambda_{\text{comp}}\mathbb{R}$ of $\lambda\mathbb{R}$ is homeomorphic to the pseudo-boundary $B(Q) = \{x \in Q \mid \exists i \in \mathbb{N} : |x_i| = 1\}$ of the Hilbert cube Q. This answers a question of A. Verbeek raised in [9].

1. Introduction

If X is a topological space, then the *superextension* λX of X denotes the space of all maximal linked systems consisting of closed subsets of X (a system is called *linked* if every two of its members meet; a *maximal linked system* or *mls* is a linked system not properly contained in another linked system) topologized by taking $\{\{\mathcal{M} \in \lambda X \mid G \in \mathcal{M}\} \mid G = G^{-} \subset X\}$ as a closed subbase (De Groot [4]). In case (X, d) is a compact metric space, then λX also is compact metric (Verbeek [9]) and the topology of λX also can be described by the metric

$$\bar{d}(\mathcal{M}, \mathcal{N}) = \sup_{S \in \mathcal{M}} \min_{T \in \mathcal{N}} d_H(S, T);$$

here $d_H(S, T)$ denotes the Hausdorff distance of S and T defined by $\inf\{\epsilon > 0 \mid S \subset U_\epsilon(T) \text{ and } T \subset U_\epsilon(S)\}$, where as usual $U_\epsilon(T)$ denotes the ϵ-neighborhood of T (Verbeek [9]). Reflecting on this metric, one sees that there must be a connection between λX and the hyperspace of all nonvoid closed subsets 2^X of X. The hyperspace 2^X is homeomorphic to the Hilbert cube Q if and only if X is a non-degenerate Peano continuum (Curtis & Schori [3]) and it was con-

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jectured by Verbeek [9] that λX is homeomorphic to Q if and only if X is a nondegenerate metrizable continuum. Earlier, De Groot conjectured that λI is homeomorphic to the Hilbert cube, where I denotes the real number interval $[-1, 1]$. This was shown to be true in [7]. If X is a noncompact metrizable space then λX is not metrizable, although it contains some interesting dense metrizable subspaces such as $\lambda_{\text{comp}}X$ (Verbeek [9]). This subspace of λX consists of all maximal linked systems which have a compact defining set, where an *mls* $\mathcal{M}$ is said to be defined on a set M if

$$\text{for all } S \in \mathcal{M} \text{ there exists an } S' \in \mathcal{M} \text{ such that } S' \subset S \cap M.$$

It is obvious that $\lambda_{\text{comp}}X$ equals λX in case X is compact, for then X is a compact defining set for all $\mathcal{M} \in \lambda X$. In case X is noncompact there are many maximal linked systems which do not have a compact defining set, for example in case $X = \mathbb{R}$, the real line, $|\lambda_{\text{comp}}\mathbb{R}| = c$ while $|\lambda\mathbb{R}| = 2^c$. Verbeek [9] showed that $\lambda_{\text{comp}}\mathbb{R}$ is a dense, metrizable, contractible, separable, locally connected, strongly infinite dimensional subspace of $\lambda\mathbb{R}$ which is in no point locally compact; he conjectured that $\lambda_{\text{comp}}\mathbb{R}$ is homeomorphic to l_2, the separable Hilbert space. We will show that this is *not* true. In fact we will show that $\lambda_{\text{comp}}\mathbb{R}$ is homeomorphic to the pseudo-boundary $B(Q) = \{x \in Q \mid \exists i \in \mathbb{N} : |x_i| = 1\}$ of the Hilbert cube Q. As $\lambda_{\text{comp}}\mathbb{R}$ is homeomorphic to $\lambda_{\text{comp}}(-1, 1)$, which can be identified with the subspace of λI consisting of all maximal linked systems with a compact defining set in $(-1, 1)$ (Verbeek [9]), we can work in $\lambda I \simeq Q$. We will show that $\lambda_{\text{comp}}(-1, 1)$ is a capset of λI (for definitions see section 3) so that $\lambda I \backslash \lambda_{\text{comp}}(-1, 1)$ is a pseudo-interior for λI and hence is homeomorphic to l_2 (Anderson [2]).

This paper is organised as follows: in the second section we give a retraction property of superextensions, which is needed to prove that $\lambda_{\text{comp}}(-1, 1)$ is a capset of λI. The third section shows that $\lambda_{\text{comp}}(-1, 1)$ is a capset of λI using a lemma of Kroonenberg [6].

2. A retraction property of superextensions

All topological spaces under discussion are assumed to be normal T_1; linked system will always mean linked system consisting of closed subsets of the topological space under consideration. If G is a closed subset of the topological space X, then we define G^+ as $G^+ = \{\mathcal{M} \in \lambda X \mid G \in \mathcal{M}\}$; λX is topologized by taking $\{G^+ \mid G \text{ is closed in } X\}$ as a closed subbase. This subbase has the property that each linked subsystem of it has a nonvoid intersection so that by Alexander's subbase lemma, λX always is compact. Moreover X can be embedded in it by means of the natural embedding $\underline{i}(x) = \{G \subset X \mid G$ is closed and $x \in G\}$. We will always identify X and $\underline{i}[X]$. Every linked system is contained in at least one maximal linked system by Zorn's lemma. A linked system $\mathcal{M}$ is called a *pre-mls* if it is contained in precisely one mls; this mls is then denoted by $\underline{\mathcal{M}}$ and we say that $\mathcal{M}$ is a pre-mls for $\underline{\mathcal{M}}$. Obviously *$\mathcal{M}$ is a pre-mls iff for all closed sets S_0 and S_1 such that $\mathcal{M} \cup \{S_i\}$ is linked* ($i = 0, 1$) *we have* $S_0 \cap S_1 \neq \emptyset$. If S is a closed subset of the compact metric space (X, d) then for each $\epsilon > 0$ we define

$$B_\epsilon(S) = \{x \in X \mid d(x, S) \leq \epsilon\}.$$

LEMMA 2.1: *Let (X, d) be a compact metric space and let $\mathcal{M}$ be a pre-mls for $\underline{\mathcal{M}} \in \lambda X$. Then for each $\mathcal{N} \in \lambda X$ we have that* $\bar{d}(\underline{\mathcal{M}}, \mathcal{N}) = \inf\{a \geq 0 \mid \forall S \in \mathcal{M} : B_a(S) \in \mathcal{N}\}$.

PROOF: Verbeek [9] proved the following

$$\begin{aligned}\bar{d}(\underline{\mathcal{M}}, \mathcal{N}) &= \min\{a \geq 0 \mid \forall S \in \underline{\mathcal{M}} : B_a(S) \in \mathcal{N} \text{ and } \forall T \in \mathcal{N} : B_a(T) \in \underline{\mathcal{M}}\} \\ &= \min\{a \geq 0 \mid \forall S \in \underline{\mathcal{M}} : B_a(S) \in \mathcal{N}\}\end{aligned}$$

and therefore $\inf\{a \geq 0 \mid \forall S \in \mathcal{M} : B_a(S) \in \mathcal{N}\} \leq \bar{d}(\underline{\mathcal{M}}, \mathcal{N})$. Let us assume that $\inf\{a \geq 0 \mid \forall S \in \mathcal{M} : B_a(S) \in \mathcal{N}\} < \bar{d}(\underline{\mathcal{M}}, \mathcal{N})$. Then there exists an a_0 such that $0 \leq a_0 < \bar{d}(\underline{\mathcal{M}}, \mathcal{N})$ with the property that for all $S \in \mathcal{M}$ we have that $B_{a_0}(S) \in \mathcal{N}$ while there exists a $T \in \mathcal{N}$ such that $B_{a_0}(T) \notin \underline{\mathcal{M}}$. As $\mathcal{M}$ is a pre-mls for $\underline{\mathcal{M}}$ there is an $M \in \mathcal{M}$ such that $B_{a_0}(T) \cap M = \emptyset$. However $B_{a_0}(M) \in \mathcal{N}$, so that $B_{a_0}(M) \cap T \neq \emptyset$. Now, as X is compact, this is a contradiction. □

The distance between two maps f and $g : X \to Y$, where (Y, d) is compact metric, is defined by $d(f, g) = \sup_{x \in X} d(f(x), g(x))$. The identity mapping on X is denoted by id_X.

THEOREM 2.2: *Let X be a toplogical space and let $\mathcal{M}$ be a linked system in X. Then $\cap\{M^+ \mid M \in \mathcal{M}\}$ is a retract of λX. Moreover, if (X, d) is compact metric then the retraction map r can be chosen in such a way that* $\bar{d}(r, id_{\lambda X}) \leq \sup_{M \in \mathcal{M}} d_H(X, M)$.

PROOF: Let $\mathcal{M}$ be a linked system in X. Notice that $\cap\{M^+ \mid M \in \mathcal{M}\} \neq \emptyset$. Choose $\mathcal{N} \in \lambda X$ and define $P\mathcal{N} = \{N \in \mathcal{N} \mid \{N\} \cup \mathcal{M}$ is linked$\} \cup \mathcal{M}$.

(a) $P\mathcal{N}$ is a pre-mls.

It is obvious that $P\mathcal{N}$ is linked; so assume to the contrary that it were not a pre-mls. Then there exist closed sets S_i such that $P\mathcal{N} \cup \{S_i\}$ is linked ($i = 0, 1$) but $S_0 \cap S_1 = \emptyset$. The normality of X implies that there exist closed sets G_i ($i = 0, 1$) such that $S_0 \cap G_1 = \emptyset = G_0 \cap S_1$ and $G_0 \cup G_1 = X$. Now, as $\mathcal{N}$ is a maximal linked system one of the sets G_i must belong to $\mathcal{N}$(if $G_i \notin \mathcal{N}$ ($i = 0, 1$) then there exist $M_i \in \mathcal{N}$ such that $M_i \cap G_i = \emptyset$ ($i = 0, 1$) so that $M_0 \cap M_1 = \emptyset$ contradicting the linkedness of $\mathcal{N}$) so that we may assume that $G_0 \in \mathcal{N}$. Now, $S_0 \subset G_0$ implies that $\mathcal{M} \cup \{G_0\}$ is linked and consequently $G_0 \in P\mathcal{N}$. This is a contradiction since $G_0 \cap S_1 = \emptyset$.

(b) Define $r : \lambda X \to \lambda X$ by $r(\mathcal{N}) = \underline{P\mathcal{N}}$. Then r is continuous.

Let G be a closed set of X and assume that $r^{-1}(G^+) \neq \emptyset$. We will show that $r^{-1}(G^+)$ is closed in λX. Choose $\mathcal{N} \notin r^{-1}(G^+)$. Then $r(\mathcal{N}) \notin G^+$ and consequently $r(\mathcal{N}) \cup \{G\}$ is not linked; therefore $P\mathcal{N} \cup \{G\}$ is not linked. Choose $N \in P\mathcal{N}$ so that $N \cap G = \emptyset$. Now, if $N \in \mathcal{M}$, then $r^{-1}(G^+)$ is void, which is a contradiction. Therefore $N \in \mathcal{N}$. Choose closed sets S_i ($i = 0, 1$) such that $S_0 \cap N = \emptyset = G \cap S_1$ and $S_0 \cup S_1 = X$. Then $\mathcal{N} \in \lambda X \backslash S_0^+ \subset S_1^+$, while moreover $(\lambda X \backslash S_0^+) \cap r^{-1}(G^+) = \emptyset$. For assume to the contrary that there exists a $\xi \in (\lambda X \backslash S_0^+) \cap r^{-1}(G^+)$. Then $S_1 \in \xi$ and $\mathcal{M} \cup \{N\}$ is linked implies that $\mathcal{M} \cup \{S_1\}$ is linked and consequently $S_1 \in P\xi \subset r(\xi)$. This is a contradiction, since $G \in r(\xi)$ and $S_1 \cap G = \emptyset$.

(c) $r(\lambda X) = \cap \{M^+ \mid M \in \mathcal{M}\}$ and r is a retraction.

Choose $\mathcal{N} \in \lambda X$. Then $\mathcal{M} \subset P\mathcal{N} \subset r(\mathcal{N})$ so that $r(\mathcal{N}) \in \cap \{M^+ \mid M \in \mathcal{M}\}$. Moreover if $\mathcal{N} \in \cap \{M^+ \mid M \in \mathcal{M}\}$ then $P\mathcal{N} = \mathcal{N}$ and therefore $r(\mathcal{N}) = \mathcal{N}$.

(d) If (X, d) is compact metric, then $\bar{d}(r, id_{\lambda X}) \leq \sup_{M \in \mathcal{M}} d_H(X, M)$.

Let $a = \sup_{M \in \mathcal{M}} d_H(X, M)$ and choose $\mathcal{N} \in \lambda X$. Take $N \in P\mathcal{N}$ and consider $B_a(N)$. If $N \in \mathcal{N}$ then also $B_a(N) \in \mathcal{N}$; if $N \notin \mathcal{N}$ then $N \in \mathcal{M}$ and therefore $B_a(N) = X$ which also is an element of $\mathcal{N}$. It now follows that

$$\bar{d}(\mathcal{N}, r(\mathcal{N})) = \inf \{a \geq 0 \mid \forall S \in P\mathcal{N} : B_a(S) \in \mathcal{N}\} \qquad \text{(lemma 2.2)}$$

$$\leq \sup_{M \in \mathcal{M}} d_H(X, M). \square$$

If Y is a closed subset of X, then λY can be embedded in λX by the natural embedding j_{YX} defined by

$$j_{YX}(\mathcal{M}) := \{G \subset X \mid G \text{ is closed and } G \cap Y \in \mathcal{M}\}$$

(Verbeek [9]). It should be noticed that $j_{YX}(\mathcal{M})$ is indeed a *maximal* linked system. We will always identify λY and $j_{YX}(\lambda Y)$.

LEMMA 2.3: *Let Y be a closed subset of X. Then $\mathcal{M} \in \lambda X$ is an element of λY if and only if $\{M \cap Y \mid M \in \mathcal{M}\}$ is linked.*

PROOF: If $\mathcal{M} \in \lambda Y$, then $\{M \cap Y \mid M \in \mathcal{M}\}$ is a maximal linked system in Y and if $\{M \cap Y \mid M \in \mathcal{M}\}$ is linked, then it is easy to see that it is also maximal linked (in Y) and that $j_{YX}(\{M \cap Y \mid M \in \mathcal{M}\}) = \mathcal{M}$.$\square$

The importance of Theorem 2.2 now is demonstrated in the proof of the following theorem.

THEOREM 2.4: *Let (X, d) be a compact connected metric space and let Y be a nonempty closed proper subset of X. Then for each $\epsilon > 0$ there exists a continuous map $f_\epsilon : \lambda X \to \lambda X \backslash \lambda Y$ such that $\bar{d}(f_\epsilon, id_{\lambda X}) < \epsilon$.*

PROOF: Choose $\epsilon > 0$ and choose two disjoint finite sets G_0 and G_1 such that $d_H(G_i, X) < \epsilon$ $(i = 0, 1)$. Let $p \in X \backslash Y$ and define $F_i = G_i \cup \{p\}$. Let f_ϵ be the retraction of λX onto $F_0^+ \cap F_1^+$ as defined in Theorem 2.2. Then $\bar{d}(f_\epsilon, id_{\lambda X}) \le \max\{d_H(F_0, X),\ d_H(F_1, X)\} < \epsilon$ and moreover $f_\epsilon(\lambda X) \cap \lambda Y = \emptyset$. For take $\mathcal{N} \in f_\epsilon(\lambda X)$; then $F_i \in \mathcal{N}$ $(i = 0, 1)$ and $(F_0 \cap Y) \cap (F_1 \cap Y) = \emptyset$ and consequently, by Lemma 2.3, $\mathcal{N} \notin \lambda Y$.■

3. A Pseudo-interior of λI

By the *Hilbert cube Q* we mean the countable infinite product of intervals $[-1, 1]^\infty$ with the product topology. The topology is generated by the metric

$$d(x, y) = \sum_{i=1}^{\infty} 2^{-i} \, |x_i - y_i|.$$

A closed subset A of Q is called a *Z-set* (Anderson [1]) if for each $\epsilon > 0$ there exists a continuous map $f : Q \to Q \backslash A$ such that $d(f, id_Q) < \epsilon$. In addition, a subset M of Q is called a *capset* for Q (Anderson [2]) if M can be written as $M = \bigcup_{i=1}^{\infty} M_i$, where each M_i is a Z-set in Q, $M_i \subset M_{i+1}$ $(i \in \mathbb{N})$ and such that the following absorption property holds: for each $\epsilon > 0$ and $i \in \mathbb{N}$ and every Z-set $K \subset Q$ there exists a $j > i$ and an embedding $h : K \to M_j$ such that $h \mid K \cap M_i = id_{K \cap M_i}$ and $d(h, id_K) < \epsilon$. It is known that *every capset of Q is equivalent to $B(Q) = \{x \in Q \mid \exists i \in \mathbb{N} : |x_i| = 1\}$, the pseudo-boundary of Q, under an autohomeomorphism of Q* [2]). The complement of a capset is called a *pseudo-interior* of Q and is homeomorphic to l_2, the separable Hilbert space ([2]). We will show that $\lambda_{\mathrm{comp}}(-1, 1)$ is a capset of λI, using the fact that $\lambda I \simeq Q$ ([7]). It then follows that $\lambda I \backslash \lambda_{\text{comp}}(-1, 1)$ is a pseudo-interior for λI. In [6] an alternative characterization of capsets is given and we will make use of that characterization.

LEMMA 3.1 ([6]): *Suppose M is a σ-compact subset of Q such that*
(i) *For every $\epsilon > 0$, there exists a map $h: Q \to Q \backslash M$ such that $d(h, id_Q) < \epsilon$.*
(ii) *M contains a family of compact subsets $M_1 \subset M_2 \subset \cdots$ such that each M_i is a copy of Q and M_i is a Z-set in M_{i+1} ($i \in \mathbb{N}$), and such that for each $\epsilon > 0$ there exists an integer $i \in \mathbb{N}$ and a map $h: Q \to M_i$ with $d(h, id_Q) < \epsilon$.*
Then M is a capset for Q.

First we will show that $\lambda_{\text{comp}}(-1, 1)$ is σ-compact.

LEMMA 3.2: $\lambda_{\text{comp}}(-1, 1) = \bigcup_{n=2}^{\infty} \lambda[-1 + 1/n, 1 - 1/n]$.

PROOF: Choose $\mathcal{M} \in \lambda_{\text{comp}}(-1, 1)$ and let $M \subset (-1, 1)$ be a compact defining set for $\mathcal{M}$. Then choose $n_0 \geq 2$ such that $M \subset [-1 + 1/n_0, 1 - 1/n_0]$; from Lemma 2.3 it now follows that $\mathcal{M} \in \lambda[-1 + 1/n_0, 1 - 1/n_0]$.

Moreover, if $\mathcal{M} \in \lambda[-1 + 1/n, 1 - 1/n]$ then for all $M \in \mathcal{M}$ we have that also $M \cap [-1 + 1/n, 1 - 1/n]$ belongs to $\mathcal{M}$, showing that $[-1 + 1/n, 1 - 1/n]$ is a defining set for $\mathcal{M}$. For assume to the contrary that for some $M \in \mathcal{M}$ it were true that $M \cap [-1 + 1/n, 1 - 1/n] \notin \mathcal{M}$; then there would exist an $M_0 \in \mathcal{M}$ such that $M_0 \cap [-1 + 1/n, 1 - 1/n] \cap M = \emptyset$, contradicting the linkedness of $\{M \cap [-1 + 1/n, 1 - 1/n] \mid M \in \mathcal{M}\}$ Lemma 2.3).■

LEMMA 3.3: *For each $\epsilon > 0$ there exists a map $f_\epsilon : \lambda I \to \lambda I \backslash \lambda_{\text{comp}}(-1, 1)$ such that $\bar{d}(f_\epsilon, id_{\lambda I}) < \epsilon$.*

PROOF: Choose $\epsilon > 0$. For each $n \geq 2$, let $F_{n,0}$ and $F_{n,1}$ be finite subsets of I such that
(i) $d_H(I, F_{n,i}) < \frac{1}{2}\epsilon$ $(i = 0, 1)$
(ii) $F_{n,0} \cap F_{n,1} \cap [-1 + 1/n, 1 - 1/n] = \emptyset$
(iii) $\{-1, 1) \subset F_{n,0} \cap F_{n,1}$,

and let f_ϵ be the retraction map, given by Theorem 2.2, of λI onto $\bigcap_{n=2}^{\infty} (F_{n,0}^+ \cap F_{n,1}^+)$. Then $\bar{d}(f_\epsilon, id_{\lambda I}) \leq \sup\{d_H(I, F_{n,i}) \mid n \geq 2,\ i = 0, 1\} \leq \frac{1}{2}\epsilon < \epsilon$, while moreover the image of λI is disjoint from $\lambda_{\text{comp}}(-1, 1)$.

For choose $\mathcal{N} \in f_\epsilon(\lambda I)$ and $n \geq 2$; then $F_{n,i} \in \mathcal{N}$ $(i = 0, 1)$ and $F_{n,0} \cap F_{n,1} \cap [-1 + 1/n, \ 1 - 1/n] = \emptyset$. Therefore $\mathcal{N}$ is not an element of $\lambda[-1 + 1/n, \ 1 - 1/n]$ by Lemma 2.3. Consequently $\mathcal{N} \notin \lambda_{\text{comp}}(-1, 1)$ (Lemma 3.2).□

THEOREM 3.4: *$\lambda_{\text{comp}}(-1, 1)$ is a capset for λI.*

PROOF: Choose $\epsilon > 0$ and let $n \geq 2$ such that $1/n < \epsilon$. Define a retraction $r : [-1, 1] \to [-1 + 1/n, \ 1 - 1/n]$ by

$$r(x) = \begin{cases} -1 + 1/n & \text{if } -1 \leq x \leq -1 + 1/n \\ x & \text{if } -1 + 1/n \leq x \leq 1 - 1/n \\ 1 - 1/n & \text{if } 1 - 1/n \leq x \leq 1 \end{cases}$$

This map can be extended to a map $\bar{r} : \lambda I \to \lambda[-1 + 1/n, \ 1 - 1/n]$ in the following manner

$$\bar{r}(\mathcal{M}) = \{G \subset [-1 + 1/n, 1 - 1/n] \mid G \text{ is closed and } r^{-1}(G) \in \mathcal{M}\}$$

(Verbeek [9]). Let $j : \lambda[-1 + 1/n, 1 - 1/n] \to \lambda I$ be the natural embedding defined by $j(\mathcal{M}) = \underline{\mathcal{M}} = \{G \subset I \mid G$ is closed and $G \cap [-1 + 1/n, 1 - 1/n] \in \mathcal{M}\}$. The composition $g = j \circ \bar{r} : \lambda I \to \lambda I$ can be described by

$$g(\mathcal{M}) = \{G \subset I \mid G \text{ is closed and } r^{-1}(G \cap [-1 + 1/n, 1 - 1/n]) \in \mathcal{M}\}.$$

We will show that g moves the points less than ϵ. It is clear that $g(\lambda I) = \lambda[-1 + 1/n, \ 1 - 1/n]$. Choose $\mathcal{M} \in \lambda I$ and assume that $\bar{d}(\mathcal{M}, g(\mathcal{M})) > 1/n$. Then there exists an $M \in \mathcal{M}$ such that $B_{1/n}(M) \notin g(\mathcal{M})$ (Lemma 2.1). Consequently there exists a $G \in g(\mathcal{M})$ such that $r^{-1}(G \cap [-1 + 1/n, \ 1 - 1/n]) \in \mathcal{M}$ and $B_{1/n}(M) \cap G = \emptyset$. Now take a $p \in M \cap r^{-1}(G \cap [-1 + 1/n, \ 1 - 1/n])$. Then $d(r(p), p) \leq 1/n$ and hence $r(p) \in G \cap [-1 + 1/n, \ 1 - 1/n] \cap B_{1/n}(M) \subset G \cap B_{1/n}(M)$, which is a contradiction. It now follows that $\bar{d}(g, id_{\lambda I}) \leq 1/n < \epsilon$.

It is obvious that $\lambda[-1 + 1/n, \ 1 - 1/n] \subset \lambda[-1 + 1/n + 1, \ 1 - 1/n + 1]$ $(n \geq 2)$, so that by Theorem 2.4, Lemma 3.2, Lemma 3.3 and the fact that $\lambda[-1 + 1/n, \ 1 - 1/n] \simeq \lambda I \simeq Q$ the family $\{\lambda[-1 + 1/n, \ 1 - 1/n] \mid n \geq 2\}$ satisfies all conditions of Lemma 3.1. Therefore $\lambda_{\text{comp}}(-1, 1)$ is a capset for λI.□

COROLLARY 3.5: *$\lambda_{\text{comp}}\mathbb{R}$ is homeomorphic to $B(Q) = \{x \in Q \mid \exists i \in \mathbb{N} : |x_i| = 1\}$. $\lambda I \backslash \lambda_{\text{comp}}(-1, 1)$ is homeomorphic to l_2.*

The space $\lambda\mathbb{R}$ now turns out to be a very strange space. It is a connected, locally connected (super)compact Hausdorff space of cardinality 2^c and weight c, which possesses a dense subset homeomorphic to $B(Q)$. The closure of $\mathbb{R}$ in $\lambda\mathbb{R}$ is $\beta\mathbb{R}$, its Čech-Stone compactification (Verbeek [9]).

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