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A GEOMETRIC CHARACTERIZATION OF THE RADON–NIKODYM PROPERTY IN BANACH SPACES

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Abstract

It is shown that a Banach space E has the Radon–Nikodym property (R.N.P.) if and only if every nonempty weakly-closed bounded subset of E has an extreme point.

Notations

E, $\| \|$ is a real Banach space with dual E'. For sets $A \subset E$, let $c(A)$ and $\bar{c}(A)$ denote the convex hull and closed convex hull, respectively. If $x \in E$ and $\epsilon > 0$, then $B(x, \epsilon) = \{y \in E; \|x - y\| < \epsilon\}$. A subset A of E is said to be dentable if for every $\epsilon > 0$ there exists a point $x \in A$ such that $x \notin \bar{c}(A \backslash B(x, \epsilon))$.

Suppose that C is a nonempty, bounded, closed and convex subset of E. Let $M(C) = \sup\{\|x\|; x \in C\}$. If $f \in E'$, let $M(f, C) = \sup\{f(x); x \in C\}$, and for each $\alpha > 0$, let $S(f, \alpha, C) = \{x \in C; f(x) \geq M(f, C) - \alpha\}$. Such a set is called a slice of C.

LEMMA 1: *Let C and C_1 be nonempty, bounded, closed and convex subsets of E, such that $C_1 \subset C$ and $C_1 \neq C$. Then there exist $x \in C$, $f \in E'$ and $\alpha > 0$ with $f(x) = M(f, C) > M(f, C_1) + \alpha$.*

PROOF: Without restriction, we can assume $M(C) \leq 1$. Take $x_1 \in C \backslash C_1$. By the separation theorem we have $f_1 \in E'$ and $\alpha_1 > 0$ with $f_1(x_1) > M(f_1, C_1) + \alpha_1$.

Let $\alpha = \alpha_1/3$. Using a result of Bishop and Phelps (see [1]), we

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obtain $x \in C$ and $f \in E'$ such that $f(x) = M(f, C)$ and $\|f - f_1\| < \alpha$.

Therefore $f(x) \geq f(x_1) > f_1(x_1) - \alpha > M(f_1, C_1) + 2\alpha > M(f, C_1) + \alpha$.

LEMMA 2: *Let C be a nonempty, bounded, closed and convex subset of E. If for every $\epsilon > 0$, there exist convex and closed subsets C_1 and C_2 of C, such that $C = \bar{c}(C_1 \cup C_2)$, $C_1 \neq C$ and* diam $C_2 \leq \epsilon$, *then C is dentable.*

PROOF: Take $\epsilon > 0$ and let C_1, C_2 be convex and closed subsets of C, such that $C = \bar{c}(C_1 \cup C_2)$, $C_1 \neq C$ and diam $C_2 \leq \epsilon/2$. By Lemma 1, there exist $x \in C$, $f \in E'$ and $\alpha > 0$ with $f(x) = M(f, C) > M(f, C_1) + \alpha$.

Let $d = \text{diam}\, C$ and consider the set

$$Q = \left\{\lambda y_1 + (1-\lambda)y_2;\ y_1 \in C_1,\ y_2 \in C_2 \text{ and } \lambda \in \left[\frac{\epsilon}{12d}, 1\right]\right\}.$$

It follows immediately that $\bar{Q}$ is a closed, convex subset of C and $x \notin \bar{Q}$. Suppose $z_1, z_2 \in C\backslash\bar{Q}$. We find z'_1, z'_2 such that $z'_i \in c(C_1 \cup C_2)$, $z'_i \notin Q$ and $\|z_i - z'_i\| < \epsilon/6$ $(i = 1, 2)$. There exist $y^i_1 \in C_1$, $y^i_2 \in C_2$ and $\lambda_i \in [0, \epsilon/12d]$, with $z'_i = \lambda_i y^i_1 + (1-\lambda_i)y^i_2$ $(i = 1, 2)$. We obtain:

$$\|z_1 - z_2\| < \|z'_1 - z'_2\| + \frac{\epsilon}{3} \leq \|y^1_2 - y^2_2\| + \lambda_1\|y^1_1 - y^1_2\| + \lambda_2\|y^2_1 - y^2_2\| + \frac{\epsilon}{3} \leq \epsilon.$$

This implies that $C\backslash\bar{Q} \subset B(x, \epsilon)$ and therefore $\bar{c}(C\backslash B(x, \epsilon)) \subset \bar{Q}$. Because $x \notin \bar{Q}$, we have that $x \notin \bar{c}(C\backslash B(x, \epsilon))$, which proves the lemma.

THEOREM 3: *If the Banach space E hasn't the RNP, there exists a nonempty, bounded and weakly-closed subset of E without extreme points.*

PROOF: If E hasn't the RNP, there is a closed and separable subspace of E, which hasn't the RNP (see [4]). Therefore we can assume E separable.

Let C be a non-dentable, convex, closed and bounded subset of E. By Lemma 2, there exists $\epsilon > 0$, such that if $C = \bar{c}(C_1 \cup C_2)$, where C_1, C_2 are closed, convex and diam $C_2 \leq \epsilon$, then $C = C_1$. Suppose $C = \cup_{p \in \mathbb{N}^*} B_p$, where B_p is the intersection of C and a closed ball with radius $\epsilon/2$. By induction on $p \in \mathbb{N}^*$, we construct sequences $(N_p)_p$, $(V_p)_p$ and $(\alpha_p)_p$, where N_p is a finite subset of $\mathbb{N}^p$, $V_p = \{(x_\omega, \lambda_\omega, f_\omega);\ \omega \in N_p\}$ a subset of $C \times [0, 1] \times E'$ and $\alpha_p > 0$, with the following properties:

(1) N_p is the projection of N_{p+1} on the p first co-ordinates ($p \in \mathbb{N}^*$).
(2) $\Sigma_{(\omega,i)\in N_{p+1}} \lambda_{(\omega,i)} = 1$ ($p \in \mathbb{N}^*$, $\omega \in N_p$).
(3) $\|x_\omega - \Sigma_{(\omega,i)\in N_{p+1}} \lambda_{(\omega,i)} x_{(\omega,i)}\| < (1/2^{p+1})$ ($p \in \mathbb{N}^*$, $\omega \in N_p$).
(4) $f_\omega(x_\omega) = M(f_\omega, C)$ ($p \in \mathbb{N}^*$, $\omega \in N_p$).
(5) $S(f_{(\omega,i)}, \alpha_{p+1}, C) \subset S(f_\omega, \alpha_p, C)$ ($p \in \mathbb{N}^*$, $(\omega, i) \in N_{p+1}$).
(6) $S(f_\omega, \alpha_p, C) \cap B_p = \emptyset$ ($p \in \mathbb{N}^*$, $\omega \in N_p$).
(In (2) and (3), i is the summation index).

CONSTRUCTION:
(1) Take $N_1 = \{1\}$ and $\lambda_1 = 1$. Applying Lemma 1, we find $x_1 \in C$, $f_1 \in E'$ and $\alpha_1 > 0$ such that $f_1(x_1) = M(f_1, C)$ and $S(f_1, \alpha_1, C) \cap B_1 = \emptyset$.
(2) Suppose we found N_p, V_p and α_p.
Take $\omega \in N_p$.
Let $S = \{x \in C;\ \exists f \in E'$ such that $f(x) = M(f, C)$
$> \sup f((C \backslash S(f_\omega, \alpha_p, C)) \cup B_{p+1})\}$

By lemma 1, we obtain easily

$$C = \bar{c}((C \backslash S(f_\omega, \alpha_p, C)) \cup B_{p+1} \cup S).$$

Because diam $B_{p+1} \le \epsilon$, this implies

$$x_\omega \in C = \bar{c}((C \backslash S(f_\omega, \alpha_p, C)) \cup S)$$

Thus there are sequences $(a_m)_m$ in $C \backslash S(f_\omega, \alpha_p, C)$, $(b_m)_m$ in $c(S)$ and $(t_m)_m$ in $[0, 1]$, with $x_\omega = \lim_{m\to\infty} (t_m a_m + (1 - t_m) b_m)$.

Because $f_\omega(t_m a_m + (1 - t_m) b_m) \le M(f_\omega, C) - t_m \alpha_p$, it follows that $\lim_{m\to\infty} t_m = 0$ and thus $x_\omega = \lim_{m\to\infty} b_m \in \bar{c}(S)$.

Take $m_\omega \in \mathbb{N}^*$, $x_{(\omega,i)} \in S$, $\lambda_{(\omega,i)} \in [0, 1]$, $f_{(\omega,i)} \in E'$ $(1 \le i \le m_\omega)$ *and* $\beta_\omega > 0$, such that:

(1) $\Sigma_{i=1}^{m_\omega} \lambda_{(\omega,i)} = 1$.
(2) $\|x_\omega - \Sigma_{i=1}^{m_\omega} \lambda_{(\omega,i)} x_{(\omega,i)}\| < (1/2^{p+1})$.
(3) $f_{(\omega,i)}(x_{(\omega,i)}) = M(f_{(\omega,i)}, C)$ $(1 \le i \le m_\omega)$.
(4) $S(f_{(\omega,i)}, \beta_\omega, C) \subset S(f_\omega, \alpha_p, C)$ $(1 \le i \le m)$.
(5) $S(f_{(\omega,i)}, \beta_\omega, C) \cap B_{p+1} = \emptyset$ $(1 \le i \le m_\omega)$.

Finally, let

$$N_{p+1} = \{(\omega, i);\ \omega \in N_p \text{ and } 1 \le i \le m_\omega\}$$
$$V_{p+1} = \{(x_{(\omega,i)}, \lambda_{(\omega,i)}, f_{(\omega,i)};\ (\omega, i) \in N_{p+1}\}$$
$$\alpha_{p+1} = \min\{\beta_\omega;\ \omega \in N_p\}.$$

We verify that this completes the construction. Now, for every $p \in \mathbb{N}^*$ and $\omega \in N_p$, we define

$$y_\omega = \lim_{\nu\to\infty} \sum \lambda_{(\omega,i_1)} \ldots \lambda_{(\omega,i_1,\ldots,i_\nu)} x_{(\omega,i_1,\ldots,i_\nu)},$$

where for each $\nu \in \mathbb{N}^*$ the summation happens over all integers $i_1, \ldots, i_\nu$ satisfying $(\omega, i_1, \ldots, i_\nu) \in N_{p+\nu}$. It is clear that these limits exist. Furthermore, we have for each $p \in \mathbb{N}^*$ and $\omega \in N_p$:

(1) $y_\omega = \Sigma_{(\omega,i)\in N_{p+1}} \lambda_{(\omega,i)} y_{(\omega,i)}$.

(2) $y_\omega \in S(f_\omega, \alpha_p, C)$.

(In (1) is i the summation index).

We will show that $R = \{y_\omega; p \in \mathbb{N}^* \text{ and } \omega \in N_p\}$ is the required set.

If $z \in C$, there exists $n \in \mathbb{N}^*$ such that $z \in B_n$. By construction $U = \cap_{\omega \in N_n} (E \backslash S(f_\omega, \alpha_n, C))$ is a weak neighborhood of z and $U \cap R$ is finite. Hence R is weakly closed and we also remark that R is discreet in its weak topology. It remains to show that R hasn't extreme points. Take $p \in \mathbb{N}^*$ and $\omega \in N_p$.

Then there is some $n \in \mathbb{N}^*$ with $y_\omega \in B_n$. Clearly, $n > p$. Since $y_\omega \in c(\mathrm{U}_{\Omega \in N_n}(S(f_\Omega, \alpha_n, C) \cap R))$, and for each $\Omega \in N_n$ we have $S(f_\Omega, \alpha_n, C) \cap B_n = \emptyset$, y_ω is not an extreme point of R.

This completes the proof of the theorem.

COROLLARY 4: *A Banach space E has the RNP if and only if every bounded, closed and convex subset C* of E contains an extreme point of its weak*-closure $\tilde{C}$ in E''.

PROOF: The necessity is a consequence of the work of Phelps (see [5]).

If now E does not possess the RNP, there exists a bounded, weakly closed subset R of E without extreme points. Clearly $C = \bar{c}(R)$ does not contain an extreme point of its weak*-closure.

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