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A GEOMETRIC CHARACTERIZATION OF THE RADON–NIKODYM PROPERTY IN BANACH SPACES

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Abstract

It is shown that a Banach space E has the Radon–Nikodym property (R.N.P.) if and only if every nonempty weakly-closed bounded subset of E has an extreme point.

Notations

$E, \|\cdot\|$ is a real Banach space with dual E' . For sets $A \subset E$, let $c(A)$ and $\bar{c}(A)$ denote the convex hull and closed convex hull, respectively. If $x \in E$ and $\epsilon > 0$, then $B(x, \epsilon) = \{y \in E; \|x - y\| < \epsilon\}$. A subset A of E is said to be dentable if for every $\epsilon > 0$ there exists a point $x \in A$ such that $x \notin \bar{c}(A \setminus B(x, \epsilon))$.

Suppose that C is a nonempty, bounded, closed and convex subset of E . Let $M(C) = \sup\{\|x\|; x \in C\}$. If $f \in E'$, let $M(f, C) = \sup\{f(x); x \in C\}$, and for each $\alpha > 0$, let $S(f, \alpha, C) = \{x \in C; f(x) \geq M(f, C) - \alpha\}$. Such a set is called a slice of C .

LEMMA 1: *Let C and C_1 be nonempty, bounded, closed and convex subsets of E , such that $C_1 \subset C$ and $C_1 \neq C$. Then there exist $x \in C$, $f \in E'$ and $\alpha > 0$ with $f(x) = M(f, C) > M(f, C_1) + \alpha$.*

PROOF: Without restriction, we can assume $M(C) \leq 1$. Take $x_1 \in C \setminus C_1$. By the separation theorem we have $f_1 \in E'$ and $\alpha_1 > 0$ with $f_1(x_1) > M(f_1, C_1) + \alpha_1$.

Let $\alpha = \alpha_1/3$. Using a result of Bishop and Phelps (see [1]), we

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obtain $x \in C$ and $f \in E'$ such that $f(x) = M(f, C)$ and $\|f - f_i\| < \alpha$.

Therefore $f(x) \geq f(x_1) > f_1(x_1) - \alpha > M(f_1, C_1) + 2\alpha > M(f, C_1) + \alpha$.

LEMMA 2: *Let C be a nonempty, bounded, closed and convex subset of E . If for every $\epsilon > 0$, there exist convex and closed subsets C_1 and C_2 of C , such that $C = \bar{c}(C_1 \cup C_2)$, $C_1 \neq C$ and $\text{diam } C_2 \leq \epsilon$, then C is dentable.*

PROOF: Take $\epsilon > 0$ and let C_1, C_2 be convex and closed subsets of C , such that $C = \bar{c}(C_1 \cup C_2)$, $C_1 \neq C$ and $\text{diam } C_2 \leq \epsilon/2$. By Lemma 1, there exist $x \in C$, $f \in E'$ and $\alpha > 0$ with $f(x) = M(f, C) > M(f, C_1) + \alpha$.

Let $d = \text{diam } C$ and consider the set

$$Q = \left\{ \lambda y_1 + (1 - \lambda) y_2; y_1 \in C_1, y_2 \in C_2 \text{ and } \lambda \in \left[\frac{\epsilon}{12d}, 1 \right] \right\}.$$

It follows immediately that \bar{Q} is a closed, convex subset of C and $x \notin \bar{Q}$. Suppose $z_1, z_2 \in C \setminus \bar{Q}$. We find z'_1, z'_2 such that $z'_i \in c(C_1 \cup C_2)$, $z'_i \notin Q$ and $\|z_i - z'_i\| < \epsilon/6$ ($i = 1, 2$). There exist $y_1^i \in C_1, y_2^i \in C_2$ and $\lambda_i \in [0, \epsilon/12d]$, with $z'_i = \lambda_i y_1^i + (1 - \lambda_i) y_2^i$ ($i = 1, 2$). We obtain:

$$\|z_1 - z_2\| < \|z'_1 - z'_2\| + \frac{\epsilon}{3} \leq \|y_2^1 - y_2^2\| + \lambda_1 \|y_1^1 - y_1^2\| + \lambda_2 \|y_1^2 - y_2^2\| + \frac{\epsilon}{3} \leq \epsilon.$$

This implies that $C \setminus \bar{Q} \subset B(x, \epsilon)$ and therefore $\bar{c}(C \setminus B(x, \epsilon)) \subset \bar{Q}$. Because $x \notin \bar{Q}$, we have that $x \notin \bar{c}(C \setminus B(x, \epsilon))$, which proves the lemma.

THEOREM 3: *If the Banach space E hasn't the RNP, there exists a nonempty, bounded and weakly-closed subset of E without extreme points.*

PROOF: If E hasn't the RNP, there is a closed and separable subspace of E , which hasn't the RNP (see [4]). Therefore we can assume E separable.

Let C be a non-dentable, convex, closed and bounded subset of E . By Lemma 2, there exists $\epsilon > 0$, such that if $C = \bar{c}(C_1 \cup C_2)$, where C_1, C_2 are closed, convex and $\text{diam } C_2 \leq \epsilon$, then $C = C_1$. Suppose $C = \bigcup_{p \in \mathbb{N}^*} B_p$, where B_p is the intersection of C and a closed ball with radius $\epsilon/2$. By induction on $p \in \mathbb{N}^*$, we construct sequences $(N_p)_p$, $(V_p)_p$ and $(\alpha_p)_p$, where N_p is a finite subset of \mathbb{N}^p , $V_p = \{(x_\omega, \lambda_\omega, f_\omega); \omega \in N_p\}$ a subset of $C \times [0, 1] \times E'$ and $\alpha_p > 0$, with the following properties:

- (1) N_p is the projection of N_{p+1} on the p first co-ordinates ($p \in \mathbb{N}^*$).
 - (2) $\sum_{(\omega,i) \in N_{p+1}} \lambda_{(\omega,i)} = 1$ ($p \in \mathbb{N}^*$, $\omega \in N_p$).
 - (3) $\|x_\omega - \sum_{(\omega,i) \in N_{p+1}} \lambda_{(\omega,i)} x_{(\omega,i)}\| < (1/2^{p+1})$ ($p \in \mathbb{N}^*$, $\omega \in N_p$).
 - (4) $f_\omega(x_\omega) = M(f_\omega, C)$ ($p \in \mathbb{N}^*$, $\omega \in N_p$).
 - (5) $S(f_{(\omega,i)}, \alpha_{p+1}, C) \subset S(f_\omega, \alpha_p, C)$ ($p \in \mathbb{N}^*$, $(\omega, i) \in N_{p+1}$).
 - (6) $S(f_\omega, \alpha_p, C) \cap B_p = \emptyset$ ($p \in \mathbb{N}^*$, $\omega \in N_p$).
- (In (2) and (3), i is the summation index).

CONSTRUCTION:

(1) Take $N_1 = \{1\}$ and $\lambda_1 = 1$. Applying Lemma 1, we find $x_1 \in C$, $f_1 \in E'$ and $\alpha_1 > 0$ such that $f_1(x_1) = M(f_1, C)$ and $S(f_1, \alpha_1, C) \cap B_1 = \emptyset$.

(2) Suppose we found N_p , V_p and α_p .

Take $\omega \in N_p$.

Let $S = \{x \in C; \exists f \in E' \text{ such that } f(x) = M(f, C)$

$$> \sup f((C \setminus S(f_\omega, \alpha_p, C)) \cup B_{p+1})\}$$

By lemma 1, we obtain easily

$$C = \bar{c}((C \setminus S(f_\omega, \alpha_p, C)) \cup B_{p+1} \cup S).$$

Because $\text{diam } B_{p+1} \leq \epsilon$, this implies

$$x_\omega \in C = \bar{c}((C \setminus S(f_\omega, \alpha_p, C)) \cup S)$$

Thus there are sequences $(a_m)_m$ in $C \setminus S(f_\omega, \alpha_p, C)$, $(b_m)_m$ in $c(S)$ and $(t_m)_m$ in $[0, 1]$, with $x_\omega = \lim_{m \rightarrow \infty} (t_m a_m + (1 - t_m) b_m)$.

Because $f_\omega(t_m a_m + (1 - t_m) b_m) \leq M(f_\omega, C) - t_m \alpha_p$, it follows that $\lim_{m \rightarrow \infty} t_m = 0$ and thus $x_\omega = \lim_{m \rightarrow \infty} b_m \in \bar{c}(S)$.

Take $m_\omega \in \mathbb{N}^*$, $x_{(\omega,i)} \in S$, $\lambda_{(\omega,i)} \in [0, 1]$, $f_{(\omega,i)} \in E'$ ($1 \leq i \leq m_\omega$) and $\beta_\omega > 0$, such that:

- (1) $\sum_{i=1}^{m_\omega} \lambda_{(\omega,i)} = 1$.
- (2) $\|x_\omega - \sum_{i=1}^{m_\omega} \lambda_{(\omega,i)} x_{(\omega,i)}\| < (1/2^{p+1})$.
- (3) $f_{(\omega,i)}(x_{(\omega,i)}) = M(f_{(\omega,i)}, C)$ ($1 \leq i \leq m_\omega$).
- (4) $S(f_{(\omega,i)}, \beta_\omega, C) \subset S(f_\omega, \alpha_p, C)$ ($1 \leq i \leq m_\omega$).
- (5) $S(f_{(\omega,i)}, \beta_\omega, C) \cap B_{p+1} = \emptyset$ ($1 \leq i \leq m_\omega$).

Finally, let

$$N_{p+1} = \{(\omega, i); \omega \in N_p \text{ and } 1 \leq i \leq m_\omega\}$$

$$V_{p+1} = \{(x_{(\omega,i)}, \lambda_{(\omega,i)}, f_{(\omega,i)}); (\omega, i) \in N_{p+1}\}$$

$$\alpha_{p+1} = \min\{\beta_\omega; \omega \in N_p\}.$$

We verify that this completes the construction. Now, for every $p \in \mathbb{N}^*$ and $\omega \in N_p$, we define

$$y_\omega = \lim_{\nu \rightarrow \infty} \sum \lambda_{(\omega,i_1)} \dots \lambda_{(\omega,i_1, \dots, i_\nu)} x_{(\omega,i_1, \dots, i_\nu)},$$

where for each $\nu \in \mathbb{N}^*$ the summation happens over all integers i_1, \dots, i_ν satisfying $(\omega, i_1, \dots, i_\nu) \in N_{p+\nu}$. It is clear that these limits exist. Furthermore, we have for each $p \in \mathbb{N}^*$ and $\omega \in N_p$:

- (1) $y_\omega = \sum_{(\omega, i) \in N_{p+1}} \lambda_{(\omega, i)} y_{(\omega, i)}$.
- (2) $y_\omega \in S(f_\omega, \alpha_p, C)$.

(In (1) is i the summation index).

We will show that $R = \{y_\omega; p \in \mathbb{N}^* \text{ and } \omega \in N_p\}$ is the required set.

If $z \in C$, there exists $n \in \mathbb{N}^*$ such that $z \in B_n$. By construction $U = \bigcap_{\omega \in N_n} (E \setminus S(f_\omega, \alpha_n, C))$ is a weak neighborhood of z and $U \cap R$ is finite. Hence R is weakly closed and we also remark that R is discreet in its weak topology. It remains to show that R hasn't extreme points. Take $p \in \mathbb{N}^*$ and $\omega \in N_p$.

Then there is some $n \in \mathbb{N}^*$ with $y_\omega \in B_n$. Clearly, $n > p$. Since $y_\omega \in c(U_{\Omega \in N_n} (S(f_\Omega, \alpha_n, C) \cap R))$, and for each $\Omega \in N_n$ we have $S(f_\Omega, \alpha_n, C) \cap B_n = \emptyset$, y_ω is not an extreme point of R .

This completes the proof of the theorem.

COROLLARY 4: *A Banach space E has the RNP if and only if every bounded, closed and convex subset C of E contains an extreme point of its weak*-closure \tilde{C} in E ".*

PROOF: The necessity is a consequence of the work of Phelps (see [5]).

If now E does not possess the RNP, there exists a bounded, weakly closed subset R of E without extreme points. Clearly $C = \bar{c}(R)$ does not contain an extreme point of its weak*-closure.

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