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POLYNOMIAL IDENTITIES AND RADICALS

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Introduction

There are a number of parallels between the theories of varieties and Kurosh–Amitsur radical and semi-simple classes of rings. In [12] we obtained necessary and sufficient conditions (when the universal class $\mathcal{W}$ for radical theory is a variety) for varieties to be radical classes and for varieties to be semi-simple classes. When $\mathcal{W}$ is the class of associative rings, such varieties can be described quite explicitly ([13], [23]).

In this paper we explore the associative case a little further. We are principally concerned with two questions.

(1) For which radical classes $\mathcal{R}$ do there exist polynomial identities satisfied by all rings in $\mathcal{R}$?

(2) How close can a non-radical variety come to being a radical class? In examining (1) we find that with essentially one exception, if a radical class contains nilpotent rings, there is no polynomial identity satisfied by all its members. On the other hand, using the standard identities, we are able to exhibit many radical classes of hereditarily idempotent rings (as defined by Andrunakievich [2]) whose members satisfy fixed identities. Specifically, if $\mathcal{R}$ is a radical class of hereditarily idempotent rings, so is

$$\{A \in \mathcal{R} \mid A \text{ satisfies the standard identity of degree } n\}$$

for each n. This is a consequence of our characterization of the lower radical class defined by the variety of rings satisfying the standard identity of degree n as the class

$$\{A \mid A/\mathcal{B}(A) \text{ satisfies the standard identity of degree } n\}.$$

We shall use the following notation throughout the paper:

$\mathcal{N}$: Nil radical class.
$\mathcal{B}$: Baer lower (= prime) radical class.
σ_n: Standard identity of degree n.
$\mathcal{C}^{(n)}$: Variety defined by σ_n.
$A \lhd B$: A is an ideal of B.
All rings are associative unless otherwise indicated.

1. Radical classes with nilpotent members

The proof of the following result is similar to that of an analogous result for Δ-matrix algebras over a commutative ring Δ with identity (see e.g. [20] Theorem 5.1, p. 22).

LEMMA 1.1: *Let R be a ring such that the ring $R(n \times n)$ of $n \times n$ matrices over R satisfies a multilinear homogeneous identity β of degree $k < 2n$. Then there exists a non-zero integer m such that $mR^k = 0$.*

PROOF: Let β be an identity of the form

$$\sum_{\theta \in \Theta} n_\theta x_{\theta(1)} \dots x_{\theta(k)} = 0,$$

where Θ is a set of permutations of $\{1, 2, \dots, k\}$ containing the identity, ι. It may clearly be assumed that $n_\iota \neq 0$. For $a \in R$, let $[a]_{ij}$ denote the matrix whose (i, j) entry is a and whose others are 0. If $k = 2l$ is even, consider the matrices

$$[a_1]_{11}, [a_2]_{12}, [a_3]_{22}, [a_4]_{23}, \dots, [a_{k-1}]_{l,l}, [a_k]_{l,l+1}$$

and if $k = 2l - 1$ is odd, consider

$$[a_1]_{11}, [a_2]_{12}, [a_3]_{22}, [a_4]_{23}, \dots, [a_{k-2}]_{l-1,l-1}[a_{k-1}]_{l-1,l}, [a_k]_{l,l},$$

where $a_1, a_2, \dots, a_k$ are any elements of R. We have

$$[a_1]_{11}[a_2]_{12} \dots [a_k]_{l,l+1} = [a_1 a_2 \dots a_k]_{1,l+1},$$
$$[a_1]_{11}[a_2]_{12} \dots [a_k]_{l,l} = [a_1 a_2 \dots a_k]_{1,l}$$

respectively, and the products in any other order are zero. Hence

$$[n_\iota a_1 a_2 \dots a_k]_{1r} = n_\iota [a_1]_{11}[a_2]_{12} \dots [a_k]_{l,r} = 0.$$

where $r = l$ or $l+1$, as appropriate. It follows that $n_\iota R^k = 0$.

Let P be a non-empty set of primes, $\mathscr{D}_P$ the class of rings whose additive groups are direct sums of divisible p-groups with $p \in P$. The classes $\mathscr{D}_P$ are radical and provide the sole occurrence of exceptional properties in a number of instances [10], [11], [18]; they appear in a similar role in our first theorem.

THEOREM 1.2: *Every ring in every $\mathscr{D}_P$ satisfies the identity $xy = 0$. On the other hand, if a radical class $\mathscr{R}$ contains a nilpotent ring whose additive group is not torsion divisible, there is no polynomial identity satisfied by every ring in $\mathscr{R}$.*

PROOF: The first assertion is well known, so suppose $\mathscr{R}$ contains a nilpotent ring R whose additive group is not torsion divisible (and thus $R \neq 0$). Then $R/pR \in \mathscr{R}$ for all primes p, so $\mathscr{R}$ contains a non-zero nilpotent ring A whose additive group is either torsion-free divisible or a reduced p-group, for some prime p. It then follows from Theorem 2.5 of [11] that $\mathscr{R}$ contains the zeroring on the additive group of A and hence ([9], Theorem 4.2) either (i) all $\mathscr{B}$-rings with divisible additive groups or (ii) all $\mathscr{B}$-rings whose additive groups are p-groups.

If there is a polynomial identity β of degree k, satisfied by every ring in $\mathscr{R}$, β may be assumed to be homogeneous and multilinear ([19], Theorem 3.8: the proof is valid for Z-algebras). Let B be a $\mathscr{B}$-ring whose additive group is torsion-free divisible or a p-group. Then these properties are shared by the ring $B(n \times n)$ of $n \times n$ matrices over B, for each n; in particular $B(n \times n)$ satisfies β when $2n < k$. But then Lemma 1.1 says that $mB^k = 0$ for some non-zero integer m. If B has torsion-free divisible additive group and is nilpotent of index $> k$, this is clearly impossible. On the other hand, the Zassenhaus algebra C ([6], pp. 19–20) over the ring of integers mod. p^r, where $p^r > m$, is in $\mathscr{B}$ and is additively a p-group, but $mC^k = mC \neq 0$.

2. A theorem of Osborn and some consequences

The following result can be obtained from Theorem 11.15 of [19] and its proof.

THEOREM 2.1: *The following conditions are equivalent for a ring A with* $\mathcal{N}(A) = 0$.

(i) *Every non-zero homomorphic image of A has a non-zero ideal which satisfies a multilinear identity which possibly depends on the ideal, but which has degree* $\leq$ *a fixed positive integer d.*
(ii) *A satisfies the standard identity of degree* $2[d/2]$.

It is indicated in [19] that the condition $\mathcal{N}(A) = 0$ in Theorem 2.1 can be replaced by the requirement that A be semiprime, this result being attributed to Martindale. A weaker form of the theorem for semiprime rings is adequate for our purposes, and we give a more elementary proof of this.

THEOREM 2.2: *Let A be a semiprime ring such that every non-zero homomorphic image of A has a non-zero ideal satisfying* σ_n. *Then A satisfies* σ_n.

PROOF: Since every ring which satisfies σ_{n-1} must satisfy σ_n, it is enough that we prove that $\mathcal{N}(A) = 0$ and invoke Theorem 2.1.

Now A itself has a non-zero ideal I satisfying σ_n. Then $I \cap \mathcal{N}(A)$ is a nil ideal and satisfies σ_n, so $I \cap \mathcal{N}(A) \subseteq \mathcal{B}(A) = 0$ ([17], Theorem 4). Let

$$\mathcal{J} = \{J \mid J \lhd A,\ J \text{ satisfies } \sigma_n \text{ and } J \cap \mathcal{N}(A) = 0\}.$$

By Zorn's Lemma, $\mathcal{J}$ has a maximal element $M \neq 0$. If $M \neq A$, then A/M has a non-zero ideal $\bar{K} = K/M$ which satisfies σ_n. Now $(K/M) \cap [(\mathcal{N}(A) + M)/M]$ is a nil ring satisfying σ_n and hence belongs to $\mathcal{B}$. But then

$$(K/M) \cap ([\mathcal{N}(A) + M]/M) \lhd [\mathcal{N}(A) + M]/M \cong \mathcal{N}(A)/\mathcal{N}(A) \cap M \cong \mathcal{N}(A),$$

so $(K/M) \cap ([\mathcal{N}(A) + M]/M)$ is isomorphic to an ideal of $\mathcal{B}(\mathcal{N}(A)) = \mathcal{N}(A) \cap \mathcal{B}(A) = 0$. Thus we have $K \cap \mathcal{N}(A) \subseteq M$, whence $K \cap \mathcal{N}(A) \subseteq M \cap \mathcal{N}(A) = 0$. By maximality of M, K does not satisfy σ_n; on the other hand, M and K/M do, while $\mathcal{N}(K) = K \cap \mathcal{N}(A) = 0$. Now let L be an ideal of K, $L \neq K$. If $M \subseteq L$, then K/L, as a homomorphic image of K/M, satisfies σ_n, while if $M \not\subseteq L$, we have $0 \neq (M + L)/L \lhd K/L$ and $(M + L)/L \cong M/M \cap L$ satisfies σ_n. By Theorem 2.1, K satisfies σ_n, violating the maximality of M in $\mathcal{J}$ (if $M \neq A$). Hence $M = A$ and $\mathcal{N}(A) = 0$.

COROLLARY 2.3: *If a semiprime ring A has an ideal I such that I and A/I satisfy σ_n, then A satisfies σ_n.*

PROOF: Let J be an ideal of A, $J \neq A$. If $I \subseteq J$, then $A/J \cong (A/I)/(J/I)$ satisfies σ_n, while if $I \not\subseteq J$, we have $0 \neq (I+J)/J \lhd A/J$ and $(I+J)/J \cong I/I \cap J$ satisfies σ_n. The result now follows from Theorem 2.2.

3. The lower radical class defined by $\mathscr{C}^{(n)}$

Those varieties which are also radical classes are completely known [12], [13]. They are the only varieties which contain no nilpotent rings. As we saw in §1, the presence of nilpotent rings in a radical class generally means there will be no polynomial identity satisfied by all rings in the class. In particular, if $\mathscr{V}$ is a variety with some nilpotent members, there is no proper variety containing the lower radical class $L(\mathscr{V})$ defined by $\mathscr{V}$, so $L(\mathscr{V})$ is considerable larger than $\mathscr{V}$. We can thus ask, informally: how many rings in $L(\mathscr{V})\backslash\mathscr{V}$ are put there by the nilpotent members of $\mathscr{V}$? This provides us with a sort of measure of the extent by which $\mathscr{V}$ fails to be a radical class. We shall, in this section, answer the question for the varieties $\mathscr{C}^{(n)}$ defined by the standard identities σ_n. There are some similarities between this work and that of Freidman [7], [8].

For each n let $\mathscr{R}_{(n)}$ denote the class of rings A for which there is a series

$$0 = A_0 \lhd A_1 \lhd A_2 \lhd \cdots \lhd A_m = A \qquad (*)$$

where each $A_{i+1}/A_i \in \mathscr{B} \cup \mathscr{C}^{(n)}$. We demonstrate a few closure properties of $\mathscr{R}_{(n)}$.

PROPOSITION 3.1: *$\mathscr{R}_{(n)}$ is hereditary.*

PROOF: Let $A \in \mathscr{R}_{(n)}$ have a series (*) and let I be an ideal of A. Then we have a series

$$0 = A_0 \cap I \lhd A_1 \cap I \lhd \cdots \lhd A_m \cap I = I$$

with $(A_{i+1} \cap I)/(A_1 \cap I) = (A_{i+1} \cap I)/[A_i \cap (A_{i+1} \cap I)]$

$$\cong [(A_{i+1} \cap I) + A_i]/A_i \lhd A_{i+1}/A_i$$

so $I \in \mathscr{R}_{(n)}$.

PROPOSITION 3.2: *$\mathcal{R}_{(n)}$ is homomorphically closed.*

PROOF: If A has a series (*) and $I \lhd A$, we have, for each i,

$$\begin{aligned}(A_{i+1}+I)/(A_i+I) &= (A_{i+1}+A_i+I)/(A_i+I)\\ &\cong A_{i+1}/A_{i+1}\cap(A_i+I)\end{aligned}$$

where the last ring is a homomorphic image of A_{i+1}/A_i and so belongs to $\mathcal{B}\cup\mathcal{C}^{(n)}$. Hence the factors of

$$0=(A_0+I)/I \lhd (A_1+I)/I \lhd \cdots \lhd (A_m+I)/I = A/I$$

are in $\mathcal{B}\cup\mathcal{C}^{(n)}$, so $A/I\in\mathcal{R}_{(n)}$.

PROPOSITION 3.3: *$\mathcal{R}_{(n)}$ is closed under extensions.*

PROOF: Let A be a ring with an ideal I such that I and $A/I\in\mathcal{R}_{(n)}$. Then there are series

$$0=I_0\lhd I_1\lhd\cdots\lhd I_m=I$$

and

$$0=B_0/I\lhd B_1/I\lhd\cdots\lhd B_k/I=A/I$$

with factors in $\mathcal{B}\cup\mathcal{C}^{(n)}$. The same is true of the factors of

$$0=I_0\lhd I_1\lhd\cdots\lhd I_m=I=B_0\lhd B_1\lhd\cdots\lhd B_k=A,$$

so $A\in\mathcal{R}_{(n)}$.

THEOREM 3.4: *A ring A belongs to $\mathcal{R}_{(n)}$ if and only if $A/\mathcal{B}(A)\in\mathcal{C}^{(n)}$.*

The proof of this theorem requires

LEMMA 3.5: *Let R be a semiprime ring with an ideal $J\in\mathcal{C}^{(n)}$ such that $R/J\in\mathcal{B}$. Then $R\in\mathcal{C}^{(n)}$.*

PROOF: Let $K\neq R$ be an ideal of R. If $J\not\subseteq K$, then $0\neq (J+K)/K\lhd R/K$ and $(J+K)/K\cong J/J\cap K\in\mathcal{C}^{(n)}$. If $J\subseteq K$, then $R/K\in\mathcal{B}$ so we have $L^2=0\neq L$ for some $L\lhd R/K$. But then $L\in\mathcal{C}^{(n)}$. By Theorem 2.2, R is in $\mathcal{C}^{(n)}$.

Proof of Theorem 3.4: Let A be in $\mathcal{R}_{(n)}$. Then $A/\mathcal{B}(A) \in \mathcal{R}_{(n)}$ so there is a series

$$0 = A_0/\mathcal{B}(A) \lhd A_1/\mathcal{B}(A) \lhd \cdots \lhd A_m/\mathcal{B}(A) = A/\mathcal{B}(A)$$

with factors in $\mathcal{B} \cup \mathcal{C}^{(n)}$. Since $A/\mathcal{B}(A)$ is semiprime, so is each $A_i/\mathcal{B}(A)$, so $A_1/\mathcal{B}(A)$ belongs to $\mathcal{C}^{(n)}$. Now $[A_2/\mathcal{B}(A)]/[A_1/\mathcal{B}(A)] \in \mathcal{B}$ or $\mathcal{C}^{(n)}$; if the former, then $A_2/\mathcal{B}(A) \in \mathcal{C}^{(n)}$ by Lemma 3.5, if the latter, then $A_2/\mathcal{B}(A) \in \mathcal{C}^{(n)}$ by Corollary 2.3. In any case, $A_2/\mathcal{B}(A)$ is in $\mathcal{C}^{(n)}$. Repetitions of this procedure lead to the conclusion that $A/\mathcal{B}(A) \in \mathcal{C}^{(n)}$. The converse is clear.

Theorem 3.6: *$\mathcal{R}_{(n)}$ is a radical class for each n.*

Proof: Let R be a ring, $\{I_\lambda \mid \lambda \in \Lambda\}$ a chain of ideals of R belonging to $\mathcal{R}_{(n)}$, $I = \cup_{\lambda \in \Lambda} I_\lambda$. Then

$$I/\mathcal{B}(I) = \bigcup_{\lambda \in \Lambda} [I_\lambda + \mathcal{B}(I)]/\mathcal{B}(I),$$

where $[I_\lambda + \mathcal{B}(I)]/\mathcal{B}(I) \cong I_\lambda/I_\lambda \cap \mathcal{B}(I) = I_\lambda/\mathcal{B}(I_\lambda) \in \mathcal{C}^{(n)}$. Hence $I/\mathcal{B}(I) \in \mathcal{C}^{(n)}$, i.e. $I \in \mathcal{R}_{(n)}$. By Propositions 3.1, 3.2 and 3.3 and the characterization of radical classes in [1], $\mathcal{R}_{(n)}$ is a hereditary radical class.

Recall the Kurosh lower radical construction (as modified in [24]): Let $\mathcal{M}$ be a non-empty homomorphically closed class of rings. Define $\mathcal{M}_1 = \mathcal{M}$,

$$\mathcal{M}_{n+1} = \{A \mid A/I \text{ has a non-zero ideal in } \mathcal{M}_n \text{ when } A/I \neq 0\}$$

for $n = 1, 2, \ldots, \mathcal{M}_\omega = \cup_n \mathcal{M}_n$. Then $\mathcal{M}_\omega$ is the lower radical class defined by $\mathcal{M}$.

Corollary 3.7: *$L(\mathcal{C}^{(n)}) = \mathcal{R}_{(n)} = \mathcal{C}_2^{(n)}$, for each n.*

Proof: Let A be a non-zero ring in $\mathcal{R}_{(n)}$. Then $A/\mathcal{B}(A) \in \mathcal{C}^{(n)}$, so $A \in \mathcal{C}^{(n)}$ if $\mathcal{B}(A) = 0$, while A has an ideal I with $I^2 = 0 \neq I$ otherwise, and such an I is in $\mathcal{C}^{(n)}$. Thus A has a non-zero ideal in $\mathcal{C}^{(n)}$ and the same goes for any non-zero homomorphic image of A, so $A \in \mathcal{C}_2^{(n)}$. Hence $\mathcal{R}_{(n)} \subseteq \mathcal{C}_2^{(n)} \subseteq L(\mathcal{C}^{(n)})$. Conversely, if $B \in \mathcal{C}^{(n)}$, then $B/\mathcal{B}(B) \in \mathcal{C}^{(n)}$, so $B \in \mathcal{R}_{(n)}$. Since $\mathcal{R}_{(n)}$ is a radical class, we have $L(\mathcal{C}^{(n)}) \subseteq \mathcal{R}_{(n)}$.

By the theorem in [4], a hereditary homomorphically closed class

$\mathcal{M}$ has lower radical class $\mathcal{M}_3$. There are varieties $\mathcal{M}$ with $L(\mathcal{M}) = \mathcal{M} = \mathcal{M}_1$ and we have just seen examples where $L(\mathcal{M}) = \mathcal{M}_2 \neq \mathcal{M}_1$. It would be interesting to know whether three steps are ever necessary in the lower radical construction over a variety.

Before proceeding, we take note of a couple of results resembling Theorem 3.6. Freidman [7] has shown that if $\mathcal{A}$ is a local hereditary radical class (e.g. the Jacobson or Levitzki radical class) then so is

$$\{A \mid A/\mathcal{A}(A) \text{ is commutative}\}.$$

P.N. Stewart has shown that for any special radical class $\mathcal{U}$ containing the generalized nil radical class,

$$\{A \mid A/\mathcal{U}(A) \text{ is commutative}\}$$

is a special radical class (private communication). For alternative rings, Freidman [8] has shown, in effect, that

$$\{A \mid A/\mathcal{G}(A) \text{ is associative}\}$$

is a hereditary local radical class, where $\mathcal{G}$ is Smiley's non-associative generalization [22] of the Brown–McCoy radical class.

We now consider some further consequences of Theorem 3.4. Let $\mathfrak{X}, \mathcal{Y}$ be classes of rings. We denote by $\mathfrak{X} \circ \mathcal{Y}$ the class of rings A which have ideals $I \in \mathfrak{X}$ with $A/I \in \mathcal{Y}$.

PROPOSITION 3.8: $\mathcal{C}^{(n)} \circ \mathcal{B} \subseteq \mathcal{B} \circ \mathcal{C}^{(n)}$.

For a ring A, let $A(\sigma_n) = \bigcap \{I \mid I \lhd A \text{ and } A/I \in \mathcal{C}^{(n)}\}$. ($A(\sigma_2)$ is usually called the commutator of A.) Clearly $A(\sigma_n) \subseteq \mathcal{B}(A)$ if and only if $A/\mathcal{B}(A) \in \mathcal{C}^{(n)}$.

PROPOSITION 3.9: *Let A be a ring with a series*

$$0 = A_0 \lhd A_1 \lhd \cdots \lhd A_m = A$$

such that $A_{i+1}/A_i \in \mathcal{C}^{(n)}$ for each i. Then $A(\sigma_n) \subseteq \mathcal{B}(A)$.

It has been shown by Jennings ([14], Theorem 5.8) that $A(\sigma_2)$ is nilpotent in this case. Freidman ([7], p. 230) has proved that if A has a transfinite ascending series with commutative factors then $A(\sigma_2)$ is locally nilpotent.

The results of §1 indicate that in order to find more examples of radical classes whose members satisfy a polynomial identity we have to exclude nilpotent rings from consideration. We shall denote by $\mathscr{H}$ the class of *hereditarily idempotent* rings, the rings whose ideals are all idempotent or, equivalently, the rings whose homomorphic images are all semiprime (see [2]; also [5]). $\mathscr{H}$ is a hereditary radical class ([2], Theorem 2).

THEOREM 3.10: *$\mathscr{H} \cap \mathscr{C}^{(n)}$ is a hereditary radical class for each n.*

PROOF: $\mathscr{H} \cap \mathscr{C}^{(n)} = \mathscr{H} \cap \mathscr{R}_{(n)}$ and intersections of radical classes are radical classes [15].

COROLLARY 3.11: *If $\mathscr{U}$ is a radical subclass of $\mathscr{H}$, then $\mathscr{U} \cap \mathscr{C}^{(n)}$ is a radical class.*

Taking account of Theorem 1.2, we get

COROLLARY 3.12: *If $\mathscr{R}$ is a hereditary radical class, then $\mathscr{R} \cap \mathscr{C}^{(n)}$ is a radical class if and only if $\mathscr{R} \subseteq \mathscr{H}$.*

Armendariz and Fisher [3] showed that a ring (not necessarily with an identity element) in $\mathscr{H}$ which satisfies a polynomial identity is regular, so all the radical classes induced by σ_n in Theorem 3.10 and Corollaries 3.11 and 3.12 consist of regular rings.

4. Semi-simple classes, restricted attainability

A variety is a radical class (in the associative case) precisely when it is a semi-simple class, and this is so if and only if the identities of the variety are attainable. In this section we briefly consider these concepts and their connections with the standard identities.

The following result is a companion piece to Corollary 3.11.

PROPOSITION 4.1: *Let $\mathscr{S}$ be a semi-simple class containing only semiprime rings. Then $\mathscr{S} \cap \mathscr{C}^{(n)}$ is a semi-simple class. If $\mathscr{M} \subseteq \mathscr{S}$ and every ring in $\mathscr{S}$ is a subdirect product of rings in $\mathscr{M}$, then every ring in $\mathscr{S} \cap \mathscr{C}^{(n)}$ is a subdirect product of rings in $\mathscr{M} \cap \mathscr{C}^{(n)}$.*

PROOF: By Theorem 1 of [16] or Theorem 1 of [21] we need to show that $\mathscr{S} \cap \mathscr{C}^{(n)}$ is hereditary and closed under subdirect products and extensions. $\mathscr{S}$ has these properties and $\mathscr{C}^{(n)}$ has the first two, while Corollary 2.3 implies that the class of semiprime rings satisfying σ_n is closed under extensions. The last assertion follows from the fact that $\mathscr{C}^{(n)}$ is homomorphically closed.

COROLLARY 4.2: *If $\mathscr{S}$ corresponds to a special radical class, so does $\mathscr{S} \cap \mathscr{C}^{(n)}$ for each n.*

Tamura [25] introduced the notion of *attainability* of identities in universal algebra. For our purposes, the concept is defined as follows. Let Γ be a set of identities and for each ring A let

$$A(\Gamma) = \cap\{I \lhd A | A/I \text{ satisfies } \alpha \forall \alpha \in \Gamma\}.$$

Then Γ is attainable if $A(\Gamma)(\Gamma) = A(\Gamma)$ for all rings A. More generally, we say that Γ is attainable on a class $\mathscr{K}$ of rings if $A(\Gamma)(\Gamma) = A(\Gamma)$ for all $A \in \mathscr{K}$. The attainable identities for associative rings are described in [13].

We conclude by demonstrating the partial attainability of the standard identities.

PROPOSITION 4.2: *σ_n is attainable on the class $\mathscr{H}$ of hereditarily idempotent rings.*

PROOF: Let $\mathscr{P}$ denote the class of semiprime rings, $\mathscr{U}$ the radical class corresponding to the semi-simple class $\mathscr{P} \cap \mathscr{C}^{(n)}$. Then for $A \in \mathscr{H}$ we have

$$\begin{aligned} A(\sigma_n) &= \cap\{I \lhd A | A/I \in \mathscr{C}^{(n)}\} \\ &= \cap\{I \lhd A | A/I \in \mathscr{P} \cap \mathscr{C}^{(n)}\} \\ &= \mathscr{U}(A). \end{aligned}$$

Since $A(\sigma_n) \in \mathscr{H}$, we then have $A(\sigma_n)(\sigma_n) = \mathscr{U}(\mathscr{U}(A)) = \mathscr{U}(A) = A(\sigma_n)$.

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