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REPRESENTATION AND DUALITY OF MULTIPLICATION OPERATORS ON ARCHIMEDEAN RIESZ SPACES

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1. Introduction and definitions

We shall be concerned with three classes of linear operators on Archimedean Riesz spaces. A positive linear operator T on the Riesz space E is a *positive orthomorphism* if whenever $x, y \in E$ and $x \wedge y = 0$ then $x \wedge Ty = 0$. An *orthomorphism* is the difference of two positive orthomorphisms. $P(E)$ will denote the vector space of all orthomorphisms on E, and $P(E)_+$ the cone of positive orthomorphisms.

The *stabiliser* of E is the vector space of all linear operators on E which leave every ideal invariant. We denote this space by $S(E)$, and its positive cone by $S(E)_+$. A linear operator T on E lies in $S(E)$ if and only if for each $x \in E_+$ there is a non-negative real number λ_x such that $-\lambda_x x \leq Tx \leq \lambda_x x$. $Z(E)$ is the subspace of $S(E)$ consisting of those T for which there is a non-negative real λ with $-\lambda x \leq Tx \leq \lambda x$ for all $x \in E_+$. $Z(E)$ is the *ideal centre* of E.

$Z(E)$ appears to have been introduced, for Archimedean ordered vector spaces, by Buck [4] and has received quite a lot of attention recently, especially for ordered topological vector spaces. $P(E)$ was studied, for Archimedean lattice groups, in [3] and [5], where it was shown that if E is represented by Bernau's representation [1] then the elements of $P(E)$ may be described by pointwise multiplication by an extended real valued continuous function. The proofs given there do not lend themselves to application to other representations. In section 3 we shall have need of representing elements of $P(E)$ in this way for other representations of E, where now E is an Archimedean Riesz space. Most of section 2 is denoted to proving that this can be done. Zaanen [9] has already dealt with a number of special cases.

$S(E) \cap P(E)$ and $Z(E) \cap P(E)$ were studied briefly in [3]. We shall see below that in fact $S(E) \subset P(E)$. This is not completely obvious as we did not specify that elements of $S(E)$ were differences of positive elements of $S(E)$. It appears to be an open question whether every linear operator T on E such that $|x| \wedge |Ty| = 0$ whenever $x, y \in E$ with $|x| \wedge |y| = 0$ must lie in $P(E)$.

If E and F are Riesz spaces we denote by $L^{\sim}(E, F)$ the vector space of all differences of positive linear operators from E into F. In particular we write $L^{\sim}(E)$ for $L^{\sim}(E, E)$ and $E^{\sim}$ for $L^{\sim}(E, \mathbb{R})$. $E^{\times}$ will denote the space of *normal integrals* on E, i.e. those $f \in E^{\sim}$ such that $f(x_\gamma) \to 0$ whenever (x_γ) is a net in E directed downward to 0.

If $T \in L^{\sim}(E)$ the formula

$$(T^{\sim} f)(x) = f(Tx) \quad (x \in E, f \in E^{\sim})$$

defines $T^{\sim} \in L^{\sim}(E^{\sim})$. Section 3 is devoted to a study of the duality theory for elements of $P(E)$, $S(E)$ and $Z(E)$. In order to obtain a satisfactory theory it is necessary to assume that $E^{\sim}$ separates the points of E, and hence that E is Archimedean. The situation for $P(E)$ and $Z(E)$ is quite straightforward, but that for $S(E)$ is rather more complicated. The final section contains some examples.

The author is grateful to the referee for the suggestion that he include a proof of Theorem 2.3 and also for supplying the present proof of Theorem 2.5. This replaced a proof that leaned heavily on work published elsewhere by several authors.

2. Representations

If S is a topological space $C^{\infty}(S)$ will denote the set of all continuous extended real valued functions on S which are finite on a dense subset of S. If $f, g \in C^{\infty}(S)$ and $\lambda \in \mathbb{R}\backslash\{0\}$ then $\lambda f : s \mapsto \lambda . f(s)$ and $f \vee g : s \mapsto f(s) \vee g(s)$ lie in $C^{\infty}(S)$. There may or may not be $h \in C^{\infty}(S)$ with $h(s) = f(s) + g(s)$ whenever the sum is defined (we shall say that sums of the form $\infty + (-\infty)$ and products of the form $0.(\pm\infty)$ are not defined). If such an h does exist we denote it by $f + g$.

If E is an Archimedean Riesz space and S a topological space the map $x \mapsto x^{\wedge} : E \to C^{\infty}(S)$ is a *representation* of E if

(1) $E^{\wedge} = \{x^{\wedge} : x \in E\}$ is a vector space and a sublattice of $C^{\infty}(S)$.

(2) $x \mapsto x^{\wedge}$ is a Riesz space isomorphism of E onto $E^{\wedge}$.

The representation is *admissible* if

(3) For each $s \in S$ there is $x \in E$ with $0 < x^{\wedge}(s) < \infty$.

There are always many representations of an Archimedean Riesz space. One of the most useful is that of Bernau [1] which is admissible. We shall make use of a representation for the pair $(E, E^{\sim})$ in the case that $E^{\sim}$ separates the points of E. By an *admissible functional representation* for such a pair we mean a pair of admissible representations of E and $E^{\sim}$ in the same $C^{\infty}(S)$ (where S is extremally disconnected, locally compact and Hausdorff), such that $E^{\sim\wedge}$ is an ideal in $C^{\infty}(S)$ containing the characteristic functions of compact open sets, and which are related as follows. There is a Radon measure μ on S, for which nowhere dense sets are locally μ-negligible, such

$$f(x) = \int_S f^{\wedge} x^{\wedge} \, \mathrm{d}\mu \; (x \in E, f \in E^{\sim}).$$

The existence of such a representation is vital to the proofs in section 3. This will be deduced from the results of Fremlin in [8].

Recall that a topological space is *extremally disconnected* if the closure of every open set is open. A compact Hausdorff space which is extremally disconnected is called *Stonian*. A measure μ on the Stonian space S is *normal* if the linear functional $f \mapsto \int f \, \mathrm{d}\mu$ lies in $C(S)^{\times}$. Equivalently, if μ is positive, μ vanishes on all the nowhere dense Borel subsets of S. The Stonian space S is *Hyperstonian* if $C(S)^{\times}$ separates the points of $C(S)$.

LEMMA 2.1: *If (X, μ) is a positive finite measure algebra then there is a Hyperstonian space S and a strictly positive normal Radon measure ν on S such that $L^1(\mu)$ and $L^1(\nu)$ are linearly order isomorphic.*

$L^1(\mu)$ is a Banach lattice for its usual norm and order and its Banach dual is isometrically order isomorphic to $L^{\infty}(\mu)$. By [6] there is a Hyperstonian space S such that $L^{\infty}(\mu)$ is isometrically order isomorphic to $C(S)$ and $L^1(\mu)$ to $C(S)^{\times}$. The linear functional $f \mapsto \int f \, \mathrm{d}\mu$ on $L^{\infty}(\mu)$ is strictly positive and lies in $L^{\infty}(\mu)^{\times}$. If ν is the measure defining the corresponding element of $C(S)^{\times}$ then ν is strictly positive. An application of the Radon–Nikodym theorem now allows us to identify $C(S)^{\times}$ with $L^1(\nu)$, completing the proof.

The following lemma is essentially proved in [6].

LEMMA 2.2: *Let S be a Hyperstonian space and ν a strictly positive normal Radon measure on S. Every real valued measurable function* *on S coincides, except on a set of ν-measure* 0, *with a unique member of* $C^\infty(S)$.

If S is a locally compact Hausdorff space and μ a Radon measure on S then $\mathcal{M}(S, \mu)$ will denote the space of equivalence classes, under the relation of local μ-almost everywhere equality, of locally μ-integrable real valued functions on S.

THEOREM 2.3: *If E is a Riesz space and $E^\sim$ separates the points of E then there is an admissible functional representation of the pair* $(E, E^\sim)$.

If $x \in E$ then the functional $f \mapsto f(x)$ on $E^\sim$ lies in $E^{\sim\times}$. Thus $E^{\sim\times}$ separates the points of $E^\sim$ and $E^\sim$ is, in the terminology of [8], a preK space. By using Lemma 2.1 in place of Lemma 5 of [8] we can modify the proof of Theorem 6 of [8] to obtain a Riesz space isomorphism $f \mapsto f^\wedge$ of $E^\sim$ onto an ideal $E^{\sim\wedge}$ in $\mathcal{M}(S, \mu)$ which contains the characteristic function of every compact set, where S is a disjoint union of Hyperstonian spaces X_i and μ a strictly positive Radon measure on S the restriction of which to each X_i is normal. It is clear that S is locally compact and extremally disconnected and that nowhere dense subsets of S are locally μ-negligible.

By Theorem 7 of [8] there is a Riesz space isomorphism $X \mapsto X^\wedge$ of $(E^{\sim\wedge})^\times$ into $\mathcal{M}(S, \mu)$ such that

$$X(f^\wedge) = \int_S f^\wedge X^\wedge \, d\mu \quad (f^\wedge \in E^{\sim\wedge}).$$

If we combine the natural embedding of E into $E^{\sim\times}$, the identification of $E^{\sim\times}$ with $(E^{\sim\wedge})^\times$ and this map (all of which are lattice isomorphisms) we obtain a map $x \mapsto x^\wedge$ taking E into a sublattice $E^\wedge$ of $\mathcal{M}(S, \mu)$ such that

$$f(x) = \int_S f^\wedge x^\wedge \, d\mu \quad (x \in E, f \in E^\sim).$$

Lemma 2.2 enables us to choose the unique continuous representative from each equivalence class in $\mathcal{M}(S, \mu)$ so that our elements $x^\wedge$ and $f^\wedge$ may be assumed to lie in $C^\infty(S)$.

Finally, to ensure that $E^\wedge$ is admissible replace S by its open subset $S_0 = \{s \in S : \exists x \in E \text{ with } 0 < x^\wedge(s) < \infty\}$, which is extremally disconnected, and replace each $x^\wedge$ and $f^\wedge$ by their restrictions to S_0. The representations are still isomorphisms for given $x \in E$ and $f \in E^{\sim}$, $S \backslash S_0 \in x^{\wedge -1}(0) \cup x^{\wedge -1}(\infty) \cup x^{-1}(-\infty)$. The final two sets in this union are nowhere dense and thus locally μ-negligible so that $\int_{S \backslash S_0} f^{\wedge} x^{\wedge} \, d\mu = 0$.

The main result of this section is Theorem 2.5. For some representations this result is known (see [3], [5] and [9]). We shall need to know the following lemma.

LEMMA 2.4: *Let E be an Archimedean Riesz space, S a topological space and $x \to x^{\wedge}$ be an admissible representation of E in $C^{\infty}(S)$. Let $T \in P(E)_+$ and set $S_0 = \{s \in S : \exists y \in E_+$ with $0 < y^{\wedge}(s) < \infty$ and $(Ty)^{\wedge}(s) < \infty\}$. If $x \in E$, $s \in S_0$ and $x^{\wedge}(s) = 0$ then $(Tx)^{\wedge}(s) = 0$.*

We may, by considering x^+ and x^- separately, assume $x \in E_+$. Let $y \in E_+$ with $0 < y^{\wedge}(s) < \infty$ and $(Ty)^{\wedge}(s) < \infty$. Let $\epsilon > 0$ and set

$$x_\epsilon = x - (x \wedge \epsilon y), \; y_\epsilon = \epsilon y - (x \wedge \epsilon y)$$

so that $x_\epsilon, y_\epsilon \in E$, $(x_\epsilon)^{\wedge}(s) = x^{\wedge}(s) = 0$ and $0 < (y_\epsilon)^{\wedge}(s) = \epsilon y^{\wedge}(s) < \infty$. As $x_\epsilon \wedge y_\epsilon = 0$, $Tx_\epsilon \wedge y_\epsilon = 0$ so that $(Tx_\epsilon)^{\wedge}(s) = 0$. Now

$$\begin{aligned} 0 \le (Tx)^{\wedge}(s) &= (Tx_\epsilon)^{\wedge}(s) + (T(x \wedge \epsilon y))^{\wedge}(s) \\ &= (T(x \wedge \epsilon y))^{\wedge}(s) \\ &\le \epsilon (Ty)^{\wedge}(s). \end{aligned}$$

This holds for all $\epsilon > 0$ so $(Tx)^{\wedge}(s) = 0$.

THEOREM 2.5: *Let E be an Archimedean Riesz space, S a topological space and $x \to x^{\wedge}$ be an admissible representation of E in $C^{\infty}(S)$. If $T \in P(E)$ there is $q \in C^{\infty}(S)$ such that for each $x \in E$*

$$(Tx)^{\wedge}(s) = q(s) x^{\wedge}(s)$$

for all $s \in S$ for which the product is defined.

We first suppose $T \in P(E)_+$ and let S_0 be the corresponding subset of S defined as in Lemma 2.4. If $s \in S_0$ choose $y \in E_+$ with $0 < y^{\wedge}(s) < \infty$ and $(Ty)^{\wedge}(s) < \infty$. Let $q(s) = (Ty)^{\wedge}(s)/y^{\wedge}(s)$. If $x \in E$ with $|x^{\wedge}(s)| < \infty$ then $x_1 = x - (x^{\wedge}(s)/y^{\wedge}(s))y$ satisfies $x_1^{\wedge}(s) = 0$. By the preceding lemma $(Tx_1)^{\wedge}(s) = 0$, so that

$$(Tx)^{\wedge}(s) = (x^{\wedge}(s)/y^{\wedge}(s))(Ty)^{\wedge}(s) = q(s) x^{\wedge}(s).$$

In particular if $0 < x^\wedge(s) < \infty$ then

$$\frac{(Tx)^\wedge(s)}{x^\wedge(s)} = q(s) = \frac{(Ty)^\wedge(s)}{y^\wedge(s)}$$

so that the definition of $q(s)$ does not depend on the choice of y. Clearly the function q on S_0 is finite valued and continuous.

S_0 is an open subset of S which is dense in S. The latter is because if $s \in S$ and U is any neighbourhood of s in S we can choose $y \in E_+$ with $0 < y^\wedge(s) < \infty$. If $(Ty)^\wedge(s) = \infty$ we can find $s_1 \in U$ with $0 < y^\wedge(s_1) < \infty$ (using the continuity of $y^\wedge$) and $(Ty)^\wedge(s_1) < \infty$ (as $(Ty)^\wedge$ is finite on a dense subset of S).

If we define q for $s_0 \in S \backslash S_0$ by $q(s_0) = \infty$ then q is continuous at s_0. For if we choose $y \in E_+$ with $0 < y^\wedge(s) < \infty$ we must have $(Ty)^\wedge(s) = \infty$. If A is a positive real we can find a neighbourhood V of s_0 on which $y^\wedge(s) < y^\wedge(s_0) + 1$ and $(Ty)^\wedge(s) > A/(y^\wedge(s_0) + 1)$. Hence for $s \in S_0 \cap V$, $q(s) > A$. As S_0 is open it follows that q is continuous on S, and that $q \in C^\infty(S)$ follows from the density of S_0.

It follows by continuity that, for each $x \in E$, $(Tx)^\wedge(s) = q(s)x^\wedge(s)$ holds for all $s \in S$ for which the product is defined.

Finally let $T = T_1 - T_2$ with $T_1, T_2 \in P(E)_+$ and let $q_1, q_2 \in C^\infty(S)$ be the corresponding functions. If $s \in S$, $x \in E$, if at least one of $q_1(s)$ and $q_2(s)$ is finite and if the product is defined then $(Tx)^\wedge(s) = (q_1(s) - q_2(s))x^\wedge(s)$. If $s_0 \in S$ and $q_1(s_0) = q_2(s_0) = \infty$ choose $y \in E_+$ with $0 < y^\wedge(s_0) < \infty$. In any neighbourhood of s_0 there is an s with $q_1(s) - q_2(s)$ defined and equal to $(Ty)^\wedge(s)/y^\wedge(s)$. As $(Ty)^\wedge/y^\wedge$ is continuous at s_0 we can define $q_1(s_0) - q_2(s_0)$ in such a way as to make the function $q = q_1 - q_2$ continuous at s_0. Clearly $q \in C^\infty(S)$ and $(Tx)^\wedge(s) = q(s)x^\wedge(s)$ wherever the product is defined.

Remark 1: The assumption that the representation be admissible, whilst it possibly may be weakened, cannot in general be omitted. For example, if $E = \{f \in C([0, 1]) : f(0) = 0\}$ then each bounded continuous function on $(0, 1]$ defines an orthomorphism of E, but need not have a continuous extension to $[0, 1]$.

Remark 2: If we are given any representation $x \to x^\wedge : E \to C^\infty(S)$ then $x \to x^\wedge|_T : E \to C^\infty(T)$, where $T = \{s \in S : \exists x \in E_+ \text{ with } 0 < x^\wedge(s) < \infty\}$, is an admissible representation of E.

Remark 3: If S is an extremally disconnected compact Hausdorff space the conclusion of the theorem holds for any representation

$x \to x^\wedge : E \to C^\infty(S)$. To see this, use the theorem to find q, defined on the open subset T of S (defined as in Remark 2), with the desired properties. Any extension $\bar{q}$ of q to an element of $C^\infty(S)$ will have the desired property.

REMARK 4: Similar results may be proved for representations as equivalence classes of functions measurable with respect to a Radon measure μ, by identifying these classes with $C^\infty(S)$ for a suitable extremally disconnected compact Hausdorff space S in a well known manner.

REMARK 5: Suppose that $x \mapsto x^\wedge : E \to C^\infty(S)$ is an admissible representation of E and that $q \in C^\infty(S)$ has the property that for all $x \in E$ there is $y_x \in E$ with $y_x{}^\wedge(s) = q(s)x^\wedge(s)$ wherever the product is defined. The map $x \mapsto y_x$ is well-defined and linear, and clearly if $x \wedge x' = 0$ then $y_x \wedge x' = 0$. $x \mapsto y_x$ lies in $P(E)$ as these remarks also apply to q^+ and q^- as, e.g.

$$q^+(s)x^\wedge(s) = [q(s)x^{+\wedge}(s)]^+ - [q(s)x^{-\wedge}(s)]^-$$

wherever all the products are defined.

REMARK 6: If $x \mapsto x^\wedge : E \to C^\infty(S)$ is an admissible representation and $T \in P(E)$, we shall denote the q of Theorem 2.5 by $T^\wedge$. It is clear that $T \in S(E)$ if and only if $T^\wedge$ is bounded on the support of $x^\wedge$ for each $x \in E$, and that $T \in Z(E)$ if and only if $T^\wedge$ is bounded.

The final result we need in this section is:

PROPOSITION 2.6: *For every Archimedean Riesz space E, $S(E) \subset P(E)$.*

Let $x \mapsto x^\wedge$ be an admissible representation of E in some $C^\infty(S)$, which always exists. If $s \in S$ the set $\{x \in E : x^\wedge(s) = 0\}$ is an ideal in E. Thus if $T \in S(E)$ and $x^\wedge(s) = 0$ we have $(Tx)^\wedge(s) = 0$. A simplified version of the proof of Theorem 2.5 now yields $q \in C^\infty(S)$ with

$$(Tx)^\wedge(s) = q(s)x^\wedge(s) \quad (x \in E)$$

for all $s \in S$ for which the product is defined. The result now follows from Remark 5.

In general the spaces $P(E)$, $S(E)$ and $Z(E)$ are all distinct (see section 4).

3. Duality

For the whole of this section E will denote a Riesz space whose order dual, $E^\sim$, separates its points, so that in particular E is Archimedean. We consider an element of $L^\sim(E)$ and ask when it, or one of its adjoints, lies in one of the spaces we have been considering. Our first result is very easily proved.

PROPOSITION 3.1: *If $T \in L^\sim(E)$ then $T \in Z(E)$ if and only if $T^\sim \in Z(E^\sim)$.*

$$\begin{aligned} T \in Z(E) &\Leftrightarrow \exists \lambda \geq 0 \text{ with } -\lambda x \leq Tx \leq \lambda x \quad (x \in E_+) \\ &\Leftrightarrow \exists \lambda \geq 0 \text{ with } -\lambda f(x) \leq f(Tx) = (T^\sim f)(x) \leq \lambda f(x) \\ &\qquad\qquad\qquad\qquad\qquad \text{for all } x \in E_+ \text{ and } f \in E_+^\sim \\ &\Leftrightarrow \exists \lambda \geq 0 \text{ with } -\lambda f \leq T^\sim f \leq \lambda f \quad (f \in E_+^\sim) \\ &\Leftrightarrow T^\sim \in Z(E^\sim). \end{aligned}$$

Before proving the next result we need a lemma.

LEMMA 3.2: *Let $(E, E^\sim)$ have an admissible functional representation in $C^\infty(S)$ and let μ be the corresponding measure. If $\phi \in C^\infty(S)_+$ and $|\int_S x^\wedge \phi \, d\mu| < \infty$ for all $x \in E$ then there is $f \in E_+^\sim$ with $\phi = f^\wedge$.*

Define $h \in E_+^\sim$ by $h(x) = \int_S x^\wedge \phi \, d\mu$ $(x \in E)$. If $g \in E_+^\sim$ and $g^\wedge \leq \phi$ then $g(x) = \int_S x^\wedge g^\wedge \, d\mu \leq \int_S x^\wedge \phi \, d\mu = h(x)$ for all $x \in E_+$. Thus $h \geq g$ and hence $h^\wedge \geq g^\wedge$. We know $\phi(s) = \sup\{\psi(s);\ \psi \in C^\infty(S), \operatorname{supp}(\psi)$ is compact, ψ is bounded$\} = \sup\{g^\wedge(s) : g \in E_+^\sim,\ g^\wedge \leq \phi\}$ for each $s \in S$, so that $h^\wedge \geq \phi$. As $E^{\sim\wedge}$ is an ideal in $C^\infty(S)$ we must have $\phi = f^\wedge$ for some $f \in E^\sim$ (in fact $f = h$).

THEOREM 3.3: *If $T \in L^\sim(E)$ then $T \in P(E)$ if and only if $T^\sim \in P(E^\sim)$.*

Suppose $T \in P(E)$. Represent $(E, E^\sim)$ admissibly in $C^\infty(S)$, with μ the corresponding measure. By Theorem 2.5 there is $T^\wedge \in C^\infty(S)$ with $(Tx)^\wedge(s) = T^\wedge(s)x^\wedge(s)$ for all $s \in S$ for which the product is defined. In particular this fails to hold on a nowhere dense set only, which is locally μ-negligible. As S is extremally disconnected and locally compact there is, for each $f \in E_+^\sim$, $\phi \in C^\infty(S)$ with $\phi(s) = T^\wedge(s)f^\wedge(s)$ except, again, on a locally μ-negligible set. Clearly we have, if $x \in E$,

$$\begin{aligned}\int_S \phi(s)x^\wedge(s)\,\mathrm{d}\mu(s) &= \int_S T^\wedge(s)f^\wedge(s)x^\wedge(s)\,\mathrm{d}\mu(s)\\ &= \int_S (Tx)^\wedge(s)f^\wedge(s)\,\mathrm{d}\mu(s)\\ &= f(Tx)\\ &= (T^\sim f)(x)\end{aligned}$$

which is finite. It follows from Lemma 3.2 that there is $g_f \in E_+^\sim$ with $(g_f)^\wedge = \phi$. We have now a linear map $f \mapsto g_f = g_{f^+} - g_{f^-}$ on $E^\sim$, and $(g_f)^\wedge(s) = T^\wedge(s)f^\wedge(s)$ whenever the product is defined. It follows from Remark 5 that $f \mapsto g_f$ lies in $P(E^\sim)$. But $g_f(x) = \int_S \phi(s)x^\wedge(s)\,\mathrm{d}\mu(s) = (T^\sim f)(x)$ for all $x \in E$, so $g_f = T^\sim f$ and $T^\sim \in P(E^\sim)$.

If $T^\sim \in P(E^\sim)$ then, as $E^{\sim\sim}$ separates the points of $E^\sim, T^{\sim\sim} \in P(E^{\sim\sim})$. Let $\pi : E \to E^{\sim\sim}$ be the natural injection, so that $\pi(Tx) = T^{\sim\sim}(\pi x)$ $(x \in E)$. We know π is a lattice homomorphism, so if $x, y \in E$ with $x \wedge y = 0$ we have $\pi x \wedge \pi y = 0$. Hence $0 = \pi x \wedge T^{\sim\sim}\pi y = \pi x \wedge \pi(Ty) = \pi(x \wedge Ty)$, so that $x \wedge Ty = 0$. Thus $T \in P(E)$.

Bigard [2], has shown that if $T \in P(E)$ then $T^\times \in P(E^\times)$, where $T^\times$ denotes the restriction of $T^\sim$ to $E^\times$.

It is not true that $T \in S(E)$ if and only if $T^\sim \in S(E^\sim)$, and an example to show this will be given in section 4. The positive result we do have requires us to go the second dual.

THEOREM 3.4: *If $T \in L^\sim(E)$ then $T \in S(E)$ if and only if $T^{\sim\sim} \in S(E^{\sim\sim})$.*

Again we let $\pi : E \to E^{\sim\sim}$ be the natural injection. This time we choose an admissible functional representation of the pair $(E^\sim, E^{\sim\sim})$ in $C^\infty(S)$, with μ the associated measure. We certainly have that $T^{\sim\sim} \in S(E^{\sim\sim})$ implies $T \in S(E)$, so we shall assume $T \in S(E)_+$ and prove that $T^{\sim\sim} \in S(E^{\sim\sim})_+$. As $S(E)$ is positively generated this will prove the result.

We know $T \in P(E)$ so $T^\sim \in P(E^\sim)$ and $T^{\sim\sim} \in P(E^{\sim\sim})$ by Theorem 3.3. There is thus a function $T^\wedge \in C^\infty(S)$ such that $T^\sim$ and $T^{\sim\sim}$ are represented in $C^\infty(S)$ by multiplication by $T^\wedge$ at all points of S for which the product is defined. We know that if $x \in E_+$ there is $\lambda_x \geq 0$ with $0 \leq \pi(Tx) = T^{\sim\sim}(\pi x) \leq \lambda_x(\pi x)$. Thus $T^\wedge$ is bounded on the support of $(\pi x)^\wedge$ for each $x \in E$. We must prove that $T^\wedge$ is bounded on the support of $X^\wedge$ for each $X \in E_+^{\sim\sim}$.

Let $B_n = \{s \in S : n-1 < T^\wedge(s) < n+1\}^-$, an open and closed subset of S since $T^\wedge$ is continuous and S is extremally disconnected. Set $A_n = B_n \cap \operatorname{supp}(X^\wedge)$ which is again open and closed. Let $P = \{n \in \mathbb{N} : A_n \neq \emptyset\}$. We claim P is a finite set.

If $n \in P$ choose $s_n \in A_n$ with $X^\wedge(s_n) > 0$ and $n-1 < T^\wedge(s_n) < n+1$. Choose $f_n \in E_+$ with $f_n^\wedge(s_n) > 0$, possible by the admissibility of the representation. $P(E^\sim)$ is a lattice so we may form

$$g_n^\wedge = [-|T^\wedge(s_n)I_E - T^\sim| + \alpha_n I_E]^+ \cdot f_n \geq 0,$$

where α_n is chosen so that $n+1-T^\wedge(s_n),\ T^\wedge(s_n)-(n-1) > \alpha_n > 0$. We have

$$g_n^\wedge(s) = [-|T^\wedge(s_n) - T^\wedge(s)| + \alpha_n]^+ f_n^\wedge(s)$$

whenever the product is defined. If $g_n^\wedge(s) > 0$ then $|T^\wedge(s_n) - T^\wedge(s)| < \alpha_n$ so that $n+1-T^\wedge(s) = [n+1-T^\wedge(s_n)] + [T^\wedge(s_n) - T^\wedge(s)] > 0$, and similarly $T^\wedge(s)-(n-1) > 0$. Thus $\operatorname{supp}(g_n^\wedge) \subset B_n$. Also $g_n^\wedge(s_n) = \alpha_n f_n^\wedge(s_n) > 0$.

Note that $X(g_n) > 0$. For let K be a non-empty compact open subset of S, containing s_n, with $X^\wedge \geq \beta\chi_K$ and $g_n \geq \gamma\chi_K$ for some $\beta, \gamma > 0$. $\chi_K = Y^\wedge$ for some $0 \neq Y \in E_+^{\sim\sim}$, and $\mu(K) > 0$ for else $Y(h) = \int_S h^\wedge Y^\wedge \, d\mu = \int_K h^\wedge \, d\mu = 0$ for all $h \in E^\sim$. Then $X(g_n) = \int_S X^\wedge g_n^\wedge \, d\mu \geq \beta \cdot \gamma \cdot \mu(K) > 0$.

If $x \in E$ then $\operatorname{supp}(\pi x)^\wedge \cap B_n = \emptyset$ for all but finitely many n, as $T^\wedge$ is bounded on $\operatorname{supp}(\pi x)^\wedge$. Thus $g_n(x) = (\pi x)(g_n) = \int_S (\pi x)^\wedge g_n^\wedge \, d\mu = 0$ for all but finitely many n, since $\operatorname{supp}(g_n^\wedge) \subset B_n$. Thus the series $\Sigma_{n\in P} X(g_n)^{-1} g_n(x)$ converges for all $x \in E$. We may thus define $h \in E_+^\sim$ by $h(x) = \Sigma_{n\in P} X(g_n)^{-1} g_n(x)$. If F is a finite subset of P, then for $x \in E_+$,

$$h(x) \geq \sum_{n\in F} X(g_n)^{-1} g_n(x),$$

so that

$$h \geq \sum_{n\in F} X(g_n)^{-1} g_n.$$

Thus

$$X(h) \geq \sum_{n\in F} X(g_n)^{-1} X(g_n) = |F|.$$

It follows that P must be a finite set.

Thus there is $m \in \mathbb{N}$ with

$$T^\wedge(\operatorname{supp}(X^\wedge)) \subset [0, m] \cup \{\infty\}.$$

As $T^\wedge \in C^\infty(S)$ continuity shows that $T^\wedge(\operatorname{supp}(X^\wedge)) \subset [0, m]$, completing the proof.

Finally we answer the question "when is it true that $T \in S(E)$ and $T^\sim \in S(E^\sim)$?"

THEOREM 3.5: *If $T \in L^\sim(E)$ the following are equivalent:*
(1) $T \in Z(E)$
(2) $T \in S(E)$ *and* $T^\sim \in S(E^\sim)$.

The proof is similar to that of Theorem 3.4, the proof of (1) $\Rightarrow$ (2) being obvious. Represent $(E, E^\sim)$ admissibly in $C^\infty(S)$ with μ the corresponding measure. By Theorem 2.5 there is $T^\wedge \in C^\infty(S)$ with $(Tx)^\wedge(s) = T^\wedge(s)x^\wedge(s)$ and (by the proof of Theorem 3.3) $(T^\sim f)^\wedge(s) = T^\wedge(s)f^\wedge(s)$ $(x \in E,\ f \in E^\sim)$ whenever the products are defined. We also know $T^\wedge$ is bounded on the sets $\operatorname{supp}(x^\wedge)$ $(x \in E)$ and $\operatorname{supp}(f^\wedge)$ $(f \in E^\sim)$.

Let $S_n = \{s \in S : n - 1 < T^\wedge(s) < n + 1\}^-$, an open and closed set, $P = \{n \in \mathbb{N} : S_n \neq \emptyset\}$. We need only show that P is a finite set. Let K_n be a non-empty compact open subset of S_n for each $n \in P$. For each such n there is $f_n \in E^\sim_+$ with $f^\wedge_n = \chi_{K_n}$. Define $F \in E^\sim_+$ by

$$F(x) = \sum f_n(x) = \sum_{n \in P} \int_K x^\wedge \, d\mu \quad (x \in E),$$

the sum converging as $\operatorname{supp}(x^\wedge)$ meets only finitely many S_n, and hence only finitely many K_n. If $x \in E_+$, $n \in P$ then $F(x) \geq f_n(x)$, so $F \geq f_n$ and $F^\wedge \geq \chi_{K_n}$. Thus $\operatorname{supp}(F^\wedge) \cap S_n \neq \emptyset$ for all $n \in P$. But $T^\wedge$ is bounded on $\operatorname{supp}(F^\wedge)$, so we must have P finite and the proof is complete.

4. Examples

Let Ω denote the space of all real sequences, $\Phi = \{(x_n) \in \Omega : x_n = 0$ for all but finitely many $n\}$, $m = \{(x_n) \in \Omega : \exists K \geq 0$ with $-K \leq x_n \leq K \forall n \in \mathbb{N}\}$. It is easy to verify that we have the following identifications:

$$Z(\Omega) = S(\Omega) = m, P(\Omega) = \Omega,$$
$$Z(\Phi) = m, S(\Phi) = P(\Phi) = \Omega.$$

This shows that we need have neither $Z(E) = S(E)$ nor $S(E) = P(E)$ in general. Also since $\Omega^{\sim}$ may be identified with Φ and $\Phi^{\sim}$ with Ω this shows that neither $S(E)$ nor $\{T \in L^{\sim}(E): T^{\sim} \in S(E^{\sim})\}$ need include the other.

An exception occurs when E is a Banach lattice. This will follow from the next proposition, special cases of which have been proved by Bigard ([2], Théorème 8) and Flösser ([7], Satz 1.15).

PROPOSITION 4.1: *Let E be a normed lattice and $T \in P(E)$. T is norm bounded with $\|T\| \le \lambda$ if and only if $-\lambda I_E \le T \le \lambda T_E$.*

Suppose $-\lambda I_E \le T \le \lambda I_E$ and $x \in E$. By using a representation of T as a multiplication operator it is clear that $|Tx| = |T(|x|)|$. Hence

$$-\lambda |x| \le T(|x|) \le \lambda |x|$$

implies

$$|Tx| = |T(|x|)| \le \lambda |x|,$$

so that

$$\|Tx\| \le \lambda \|x\|$$

and hence

$$\|T\| \le \lambda.$$

Suppose conversely that $\|T\| \le \lambda$, then $T^{\sim}$ leaves E^*, the normed dual of E, invariant and $\|T^{\sim}|_{E^*}\| \le \lambda$. Represent $(E, E^{\sim})$ admissibly in $C^{\infty}(S)$ and use Theorems 2.5 and 3.3 to find $T^{\sim\wedge} \in C^{\infty}(S)$ such that, for all $f \in E^{\sim}$, $(T^{\sim}f)^{\wedge}(s) = T^{\sim\wedge}(s)f^{\wedge}(s)$ whenever the product is defined. If we prove that $-\lambda I_{E^*} \le T^{\sim}|_{E^*} \le \lambda I_{E^*}$ the proof of Proposition 3.1 will show that $-\lambda I_E \le T \le \lambda I_E$.

We prove $T^{\sim}|_{E^*} \le \lambda I_{E^*}$, the proof that $-\lambda I_{E^*} \le T^{\sim}|_{E^*}$ being similar. If this fails we can find $g \in E^*_+$ with $T^{\sim}g \not\le \lambda g$. It follows that

$$A = \{s \in S: T^{\sim\wedge}(s) > \lambda + \epsilon, g^{\wedge}(s) > 0\}$$

is non-empty for some $\epsilon > 0$. Let B be a non-empty open and closed subset of A. As E^* is a lattice ideal in $E^\sim$, there is $h \in E^*$ with $h^\wedge = \chi_B \cdot g^\wedge$. It follows from the representation that

$$T^\sim h \geq (\lambda + \epsilon)h > 0,$$

and hence that $\|T^\sim h\| \geq (\lambda + \epsilon)\|h\|$, contradicting $\|T^\sim\| \leq \lambda$.

COROLLARY 4.2: *If E is a Banach lattice then $Z(E) = S(E) = P(E)$.*

This follows from the proposition as all positive linear operators on a Banach lattice, and hence their differences, are norm bounded.

Theorem 3.5 may be rephrased as $\{T \in L^\sim(E) : T^\sim \in S(E^\sim)\} \cap S(E) = Z(E)$. It is natural to ask what $\{T \in L^\sim(E) : T^\sim \in S(E^\sim)\} + S(E)$ is, especially as in many of the examples that first come to mind it is precisely $P(E)$. This is not the case in general as the following example shows.

EXAMPLE 4.3: *Let $E = \{(a_n) \in \Omega : (n^p a_n) \in l_1$ for all $p \in \mathbb{N}\}$. E is a Riesz space, $E^\sim$ separates the points of E and $\{T \in L^\sim(E) : T^\sim \in S(E^\sim)\} + S(E) \neq P(E)$.*

That E is a Riesz space is easy to check, and $E^\sim$ separates its points as the functionals $(a_n) \to a_m$ lie in $E^\sim$ for each $m \in \mathbb{N}$. If $(a_n) \in E$ then also $(na_n) \in E$, so $(a_n) \to (na_n)$ is an element of $P(E)$. This does not lie in $Z(E)$. We claim that $S(E) = \{T \in L^\sim(E) : T^\sim \in S(E^\sim)\} = Z(E)$, which will certainly prove the claim.

The sequence $(n^{-n})_{n=1}^\infty$ lies in E and has support the whole of $\mathbb{N}$. Thus if $T \in S(E)$ is represented by multiplication co-ordinatewise by the sequence (t_n) then $(t_n) \in m$ by Remark 6 and hence $T \in Z(E)$. Similarly if $T \in L^\sim(E)$ and $T^\sim \in S(E^\sim) \subset P(E^\sim)$ then $T \in P(E)$. Represent T as multiplication by the sequence (t_n). $F : (a_n) \to \Sigma_{n=1}^\infty a_n$ lies in $E^\sim$, so there is $\lambda \geq 0$ with $-\lambda F \leq T^\sim F \leq \lambda F$. If e_n is the sequence with 1 in its n'th position and zero elsewhere, which certainly lies in E, then

$$-\lambda = -\lambda F(e_n) \leq (T^\sim F)(e_n) = F(Te_n) = t_n \leq \lambda F(e_n) = \lambda.$$

Thus $(t_n) \in m$ and $T \in Z(E)$.

REFERENCES

[1] S.J. BERNAU: Unique representation of Archimedean lattice groups and normal Archimedean lattice rings. *Proc. London Math. Soc.* (3) *15* (1965) 599–631.
[2] A. BIGARD: Les orthomorphismes d'un espace réticulé Archimédien. *Indag. Math.* *34* (1972) 236–246.
[3] A. BIGARD and K. KEIMEL: Sur les endomorphismes conservant les polaires d'un groupe réticulé Archimédien. *Bull. Soc. Math. France 97* (1969) 381–398.
[4] R.C. BUCK: Multiplication operators. *Pacific J. Maths. 11* (1961) 95–104.
[5] P.F. CONRAD and J.E. DIEM: The ring of polar preserving endomorphisms of an abelian lattice ordered group. *Illinois J. Math. 15* (1971) 224–240.
[6] J. DIXMIER: Sur certains espaces considérés par M.H. Stone. *Summa Bras. Math.* *2* (1951) 151–182.
[7] H.O. FLÖSSER: Die Orthomorphismen einiger lokalkonvexer Vektorverbände. *Preprint.*
[8] D.H. FREMLIN: Abstract Köthe spaces II. *Proc. Cam. Phil. Soc. 63* (1967) 951–956.
[9] A.C. ZAANEN: Examples of Orthomorphisms. *J. Approx. Thy. 13* (1975) 192–204.

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