

COMPOSITIO MATHEMATICA

ROBERT E. JAMISON

**Some intersection and generation properties
of convex sets**

Compositio Mathematica, tome 35, n° 2 (1977), p. 147-161

http://www.numdam.org/item?id=CM_1977__35_2_147_0

© Foundation Compositio Mathematica, 1977, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (<http://http://www.compositio.nl/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

SOME INTERSECTION AND GENERATION PROPERTIES OF CONVEX SETS

Robert E. Jamison*

Introduction

In this paper it is shown that in a real vector space of at most countable dimension any convex set whose complement is also convex is the convex hull of a countable set. This result is valid over any ordered field whose order completion is first countable in the order topology. In an abstract setting of convexity it is also shown that the above generation property implies that any family of convex sets with empty intersection has a countable subfamily with empty intersection.

In a 1956 paper [10, p. 59], V. L. Klee proved the following intersection theorem for convex sets:

KLEE'S THEOREM: *Let V be a real vector space of countable dimension. If \mathcal{F} is a family of convex sets in V with empty intersection, then some countable subfamily of \mathcal{F} has empty intersection.*

Later in a lecture series [12] he observed that the proof in [10] was applicable to vector spaces over any ordered field with a countable dense subset. One objective of this paper is to extend Klee's result to the class of *completely sequential* ordered fields: every nonvoid subset contains an increasing cofinal sequence. This is the largest class of fields for which Klee's theorem is valid. It may not be a priori obvious that completely sequential is a strictly weaker requirement than order separable. However, we shall show that the real-valued function field generated by the functions $x \rightarrow x^r$, r a real number, is

* The author is grateful to the Department of Mathematical Sciences of Clemson University for their support during the preparation of this paper.

completely sequential but not order separable. (The “eventual” ordering is used here: $f \leq g$ iff $f(x) \leq g(x)$ for all sufficiently large x .)

Another aim of this paper is to prove the following relative of Klee’s theorem:

(A) THEOREM: *Let V be a vector space of countable dimension over a completely sequential ordered field. If H is a convex set in V whose complement $V \sim H$ is also convex, then H is the convex hull of a countable set.*

In fact, as we shall show, this result is formally stronger than the extension of Klee’s theorem. This will be demonstrated in the setting of general convexity developed in [6].

As Klee’s result can be interpreted as a countable version of Helly’s celebrated theorem [5] on intersections of convex sets and as Theorem A is a countable analogue of Carathéodory’s celebrated result on formation of convex hulls [1], the abstract connection between these two results can be regarded as a contribution to the understanding of the interplay between Helly and Carathéodory properties of abstract spaces. A somewhat old but still valuable survey of results in this direction is [2]. In the course of the present development, we shall also prove a general Helly type theorem for compact (in a combinatorial sense) families of convex sets. Other intersection theorems related to this and to Klee’s theorem above are discussed in [11].

The main result

In this paper F will denote a totally ordered field. (For a discussion of ordered fields and some interesting examples, see [4, Chapter 13].) If K is a subset of a vector space V over F , then K is *convex* if and only if

$$\lambda x + (1 - \lambda)y \in K$$

for every x and y in K and each scalar λ with $0 \leq \lambda \leq 1$. If both K and its complement $V \sim K$ are convex, then K is called a *hemispace*. (The open and the closed halfspaces determined by linear functionals are hemispaces. But there are many others.)

A totally ordered set T will be called *completely sequential* if every subset of T has a countable increasing cofinal subset and a countable decreasing cofinal subset. Let us note in passing that if T is completely sequential, then so is the order completion of T . In fact, T is completely sequential iff the order completion of T is first countable under the order topology. For an ordered field to be completely

sequential it is sufficient that every bounded set of positive numbers have a countable increasing cofinal subset. (It is surely necessary *but not sufficient* that 0 be the limit of a sequence of positive elements.) Note that if the ordered set T has a countable order dense subset, then T is completely sequential.

MAIN THEOREM: *Let V be a vector space over the ordered field F . Then the following are equivalent:*

- (i) *The dimension of V over F is countable and F is completely sequential.*
- (ii) *Every hemispace in V is the convex hull of a countable set.*
- (iii) *Every family of convex sets in V with empty intersection contains a countable subfamily with empty intersection.*

We first show that (i) implies (ii). From the general theory developed next, it will follow that (ii) implies (iii). As it is quite easy to see that (iii) implies (i), this will complete the proof.

One might wish to note that the Main Theorem is a result phrased in terms of the countably infinite cardinal. An analogous result is true for any infinite cardinal. In fact, the proof presented here requires only slight, but judicious, modification to be applicable to the general case. However, the presentation is restricted to the countable case for conceptual ease and concreteness.

Hemispace as convex hulls of countable sets

We begin with a proof of Theorem A ((i) implies (ii) of the Main Theorem). For this result it will be useful to know that if two linear functionals ϕ and ψ are bounded (either from above or below) on a nonempty hemispace H , then ϕ and ψ are linearly dependent. To see this, first note that by multiplying by -1 if need be, one may assume that ϕ and ψ are bounded below by numbers α and β , respectively. If ϕ and ψ are linearly independent, there is a point p in V such that $\phi(p) = -1$ and $\psi(p) = 1$. Let x be any point of H . Choose a scalar λ larger than both $\phi(x) - \alpha$ and $\psi(x) - \beta$; then set $v = x + \lambda p$ and $w = x - \lambda p$. Then $\phi(v) < \alpha$ and $\psi(w) < \beta$, so v and w are not in H . As x is the average of v and w , this contradicts the assumed convexity of $V \sim H$.

PROOF OF THEOREM A: Proceeding by induction, we shall first establish the result when V is finite dimensional. Let H be any nonempty hemispace in V . (The result is trivial if $H = \emptyset$.) If $\dim V =$

1, then V may be identified with F . Since F is completely sequential, there is a countable subset C of H which is both cofinal increasing and cofinal decreasing. Evidently, $H = \text{conv}(C)$.

Now suppose $\dim V = n > 1$. Since the linear functionals bounded on H are all linearly dependent, there is a linear functional ϕ that is not bounded on H . Thus for each λ in F , the set $H_\lambda = \{x \in H : \phi(x) = \lambda\}$ is nonempty. Since F is completely sequential, there is a countable subset I of F that is both cofinal increasing and cofinal decreasing. For any subset S of F with $I \subseteq S$, let

$$C(S) = \text{conv}(\bigcup \{H_\lambda : \lambda \in S\}).$$

Our first goal is to show that $H = C(S)$ for some countable subset S of F .

To this end, for each subset S of F , let

$$R(S) = \{\phi(x) \in F : x \in H \sim C(S)\}.$$

If S is countable and I is included in S , then $R(S)$ will be called a *residual subset* of F . We shall now show in a series of steps that \emptyset is residual.

(i) Each residual set is order convex in F .

Check. For $\alpha < \beta < \gamma$ with α and γ in the residual set $R(S)$, let x and z be points in $H \sim C(S)$ with $\phi(x) = \alpha$ and $\phi(z) = \gamma$. For some point y on the segment from x to z , one must have $\phi(y) = \beta$. Since the complement of H is convex, either the ray from y through x or the ray from y through z must belong to H . Suppose the former obtains. Since $\phi(x) < \phi(y)$ and I contains a decreasing cofinal subset of F , there is a point p on the ray from y through x with $\phi(p) = \lambda$ for some λ in I . As $I \subseteq S$, this implies that $p \in C(S)$. If y were in $C(S)$, the convexity of $C(S)$ would force $C(S)$ to contain x . The choice of x precludes this. Thus $y \in H \sim C(S)$, so $\beta = \phi(y)$ is in $R(S)$. The other alternative above is handled similarly. \checkmark

(ii) The countable intersection of residual sets contains a residual set.

Check. If $R(S_n)$ $n = 1, 2, \dots$ is a countable collection of residual sets, then $S = \bigcup_{n=1}^{\infty} S_n$ is countable and contains I since each S_n contains I . Thus $R(S)$ is residual, and it is easily checked that $R(S) \subseteq R(S_n)$ for all n . \checkmark

(iii) If \mathcal{F} is a decreasing chain of order convex subsets of F , then there is a countable subfamily \mathcal{G} of \mathcal{F} with $\bigcap \mathcal{G} = \bigcap \mathcal{F}$.

Check. For each set A in \mathcal{F} , define

$$U(A) = \{v \in F : v > \alpha \text{ for all } \alpha \text{ in } A\}$$

and

$$L(A) = \{\lambda \in F : \lambda < \alpha \text{ for all } \alpha \text{ in } A\}.$$

Now let

$$U = \bigcup \{U(A) : A \in \mathcal{F}\}$$

and

$$L = \bigcup \{L(A) : A \in \mathcal{F}\}.$$

Choose a cofinal decreasing sequence $v_1 \geq v_2 \geq \dots$ in U and a cofinal increasing sequence $\lambda_1 \leq \lambda_2 \leq \dots$ in L . For each n , select sets A_n and B_n in \mathcal{F} such that $v_n \in U(A_n)$ and $\lambda_n \in L(B_n)$. Let \mathcal{G} be the (obviously countable) collection of the A_n 's and B_n 's so chosen. Now if β is a number in F that is not in $\bigcap \mathcal{F}$, then $\beta \notin C$ for some set C in \mathcal{F} . Since C is order convex, either $\beta \in U(C)$ or $\beta \in L(C)$. In the first case, β is in U , so $v_n < \beta$ for some integer n . Whence β belongs to $U(A_n)$, so β cannot belong to A_n . Consequently β is not in $\bigcap \mathcal{G}$. The second case is handled similarly. It follows that $\bigcap \mathcal{G} = \bigcap \mathcal{F}$. \checkmark

(iv) There is a minimal residual set.

Check. Assertions (i), (ii) and (iii) imply that the intersection of any decreasing chain of residual sets contains a residual set. Zorn's lemma therefore applies. \checkmark

(v) A minimal residual set is empty.

Check. If $R(S)$ is a nonempty residual set, let λ be any number in $R(S)$, and take $S' = S \cup \lambda$. Then S' still includes I and is countable. Obviously, $\lambda \notin R(S')$, but $R(S') \subseteq R(S)$. Thus $R(S')$ is a residual set properly contained in $R(S)$. \checkmark

As a result of (iv) and (v) above, the empty set is residual. It follows that there is a countable subset S of F with $I \subseteq S$ such that $C(S) = H$. Now for each λ in F , H_λ is a hemisphere in the $(n-1)$ -

dimensional flat $\{x \in V : \phi(x) = \lambda\}$. The induction hypothesis then guarantees that H_λ is the convex hull of some countable C_λ . The set $C = \bigcup \{C_\lambda : \lambda \in S\}$ is countable and for each λ in S , one obtains $H_\lambda \subseteq \text{conv}(C)$. Thus $H = C(S) = \text{conv}(C)$. This completes the induction and proves the theorem when V is finite dimensional.

Now if the dimension of V is countably infinite, V can be expressed as the countable union of finite dimensional subspaces L_n . If H is a hemispace in V , then $H \cap L_n$ is a hemispace in L_n with $H \cap L_n = \text{conv}(C_n)$. Letting $C = \bigcup_{n=1}^{\infty} C_n$, one can easily see that $H = \text{conv}(C)$. Moreover as the countable union of countable sets, C is countable. \square

An example

We shall give here an example of a totally ordered field which is completely sequential but does not contain a countable order dense subset. This example is based on the hyperreal function fields investigated by L. Gilman and M. Jerison [4, pp. 171–189]. Their terminology is employed below. Let R denote the field of real numbers and set $X = \{r \in R : r \geq 1\}$. Choose a free maximal ideal M [4, p. 54] in the ring $C(X)$ of continuous real-valued functions on X . The quotient ring $C(X)/M$ is a nonarchimedean ordered field K [4, p. 70]. In K , let F be the subfield generated by the quotient map images of the functions $x \rightarrow x^r$ where r is any real number. The ring generated in F by the functions x^r can be identified with the “pseudopolynomials”—functions of the form $p(x) = \sum_{i=1}^n c(i)x^{r(i)}$ where n is a positive integer, $c(i) \in R$, and the $r(i)$ are distinct real numbers. The largest $r(i)$ such that the corresponding $c(i)$ is nonzero, will be called the *degree* of $p(x)$ and the corresponding $c(i)$ the *leading coefficient*. Any element λ of F can be represented in the form $\lambda = p(x)/q(x)$ where p and q are pseudopolynomials and the leading coefficient of q is 1. Thus λ is positive only when the leading coefficient of p —which will also be called the leading coefficient of λ —is positive. We take the degree of λ to equal the degree of p minus the degree of q .

We shall now outline an argument that F is completely sequential but possesses no countable dense subset. The latter is easy to see. Note that for any real numbers $r < s$, if $\rho \in F$ has degree r and $\sigma \in F$ is positive of degree s , then $\rho < \sigma$. Hence the open order intervals of the form

$$\{\lambda \in F : x^r - 1 < \lambda < x^r + 1\}$$

are mutually disjoint and uncountable in number.

To show that F is completely sequential, we shall assume the opposite and derive an absurdity. Suppose that S is a bounded set of nonnegative elements of F with no increasing cofinal sequence. For each countable ordinal η , we shall construct an element λ_η of S and a real number d_η such that whenever γ precedes η , we have (i) $\lambda_\gamma < \lambda_\eta$ and (ii) $d_\gamma \geq d_\eta$ with strict inequality if γ is a nonlimit ordinal. (For a discussion of ordinals, see [8, p. 266], [4, p. 74].)

Let $\lambda_0 = 0$ and take d_0 to be any real number such that $\deg \sigma < d_0$ for all σ in S . (Such exists since $(x^r) r \in R$ is cofinal in F and S is bounded.)

Now suppose λ_γ and d_γ have been chosen for all ordinals γ less than a particular countable ordinal η . If η has an immediate predecessor β , let

$$T = \{\sigma - \lambda_\beta : \sigma \text{ is in } S \text{ and } \sigma \geq \lambda_\beta\}$$

and consider the set D of degrees of elements in T . Define $d_\eta = \sup D$. If d_η is not in D , select a sequence (τ_n) from T such that

$$\deg \tau_n > d_\eta - (1/n).$$

Such a sequence is cofinal in T , whence $(\tau + \lambda_\beta)$ is a cofinal sequence in S . As this contradicts the assumption on S , there must be an element of T with maximum degree. Let C be the set of all leading coefficients of elements of T which have degree d_η . Set $c = \sup C$. If c is not in C (or is infinite), select a *sequence* of elements in T of maximal degree whose leading coefficients have c as their supremum. This sequence is cofinal in T , again a contradiction. Hence c is finite and in T so there is some element μ of degree d_η with c as leading coefficient.

Taking $\lambda_\eta = \mu + \lambda_\beta$, we clearly get $\lambda_\eta > \lambda_\beta$ as required in condition (i). Now note that if

$$T' = \{\sigma - \lambda_\eta : \sigma \text{ is in } S \text{ and } \sigma > \lambda_\eta\},$$

then also

$$T' = \{\tau - \eta : \tau \text{ is in } T \text{ and } \tau > \mu\}.$$

From the choice of μ , it follows that every element of T' has degree strictly less than d_η . Inductive condition (ii) can be inferred from this.

If η is a (countable) limit ordinal, the construction is easier. Let

$$d_\eta = \inf \{d_\gamma : \gamma < \eta\}.$$

The countable set $\{\lambda_\gamma : \gamma < \eta\}$ cannot be cofinal in S by hypothesis on

S , so there is a λ_η in S such that $\lambda_\eta > \lambda_\gamma$ for all γ preceding η . Both inductive conditions (i) and (ii) are easily verified in this case.

Now for each countable ordinal η , we have an open interval of real numbers:

$$\{r \in R : d_{\eta+1} < r < d_\eta\}$$

(Here $\eta + 1$ denotes the successor of η .) By inductive property (ii), these are nonempty and disjoint. But this is impossible since there are uncountably many countable ordinals and every collection of disjoint open sets in R is countable. Hence F must be completely sequential.

Two Helly-type theorems

In this section we shall establish the second implication in the Main Theorem. This will be done in the general context of alignments, as developed in [6]. Only a brief review of the essentials required for the development will be given here. For more details and examples, see [6].

If X is any set, an *alignment* on X is a family \mathcal{L} of subsets of X (which will be called *convex subsets* of X) such that

- (A1) \emptyset and X are convex,
- (A2) the arbitrary intersection of convex sets is convex, and
- (A3) the union of a chain of convex sets (that is, a family of convex sets totally ordered by inclusion) is convex.

The pair (X, \mathcal{L}) is an aligned space. An alignment \mathcal{L} on X determines a convex hull operator on subsets of X

$$\mathcal{L}(S) = \bigcap \{L \in \mathcal{L} : S \subseteq L\}.$$

This operator can be shown to satisfy the useful *finitary property*:

$$(*) \quad \mathcal{L}(S) = \bigcup \{\mathcal{L}(E) : E \text{ is a finite subset of } S\}.$$

A direct proof of this may be given using transfinite induction [6, p. 5].

An alignment on X is, of course, a subset of the power set $\text{Pow}(X)$ of X . By identifying each subset of X with its characteristic function, $\text{Pow}(X)$ may be identified with the compact Hausdorff product of discrete two point spaces $\{0, 1\}$, indexed by X . The resulting topology on $\text{Pow}(X)$ will be called the *inclusion-exclusion topology*. Families of the form $\{A \subseteq X : I \subseteq A \text{ and } E \cap A = \emptyset\}$ where I and E are arbitrary finite subsets of X , form a base of open (and closed) sets for

this topology. The finitary property implies that any alignment \mathcal{L} on X is actually a *closed* subset of $\text{Pow}(X)$ under this topology. For if S is a subset of X that is not convex, there is a point p in $\mathcal{L}(S)$ that is not in S . By (*), there is a finite subset I of S with p in $\mathcal{L}(I)$. The family $\{A \subseteq X : I \subseteq A \text{ and } p \notin A\}$ is then an open family in $\text{Pow}(X)$ containing S but disjoint from \mathcal{L} .

WARNING: The inclusion-exclusion topology on $\text{Pow}(X)$ in no way reflects any topological or other structure X may possess. It is purely combinatorial in nature, depending only on the cardinality of X .

A set H in an aligned space (X, \mathcal{L}) is a *hemispace* provided H and its complement $X \sim H$ are both convex. Since the alignment \mathcal{L} is closed in $\text{Pow}(X)$ and the complementation map $A \rightarrow X \sim A$ is a homeomorphism of $\text{Pow}(X)$ onto itself, the family \mathcal{L}' of sets whose complements are convex is also closed. Thus the family of hemispaces in X , as the intersection of the closed families \mathcal{L} and \mathcal{L}' , is closed.

Hemispaces provide a natural notion to be used in describing separation properties of an aligned space. We require only one such property here (see [6, p. 24]).

SEPARATION AXIOM S_3 : Given any convex set K and point $p \notin K$, there is a hemispace H such that $K \subseteq H$ and $p \notin H$.

The ordinary alignment of convex sets in a vector space over a totally ordered field satisfies this axiom. To see this, take H to be any maximal convex set containing K but missing p . It is not hard to check that such an H must have a convex complement.

An alignment \mathcal{L} is said to have *Helly number* h provided

(†) for any finite family \mathcal{F} of \mathcal{L} -convex sets with empty intersection, some subfamily of at most h sets from \mathcal{F} has empty intersection.

This is a compactness type property. In fact, we shall shortly prove that the “finite” above can be replaced by “compact” in the sense of the inclusion-exclusion topology on $\text{Pow}(X)$. Helly’s theorem [5] asserts that $n + 1$ is the Helly number for the ordinary convex sets in an n -dimensional real vector space. A standard proof of this, such as given in [3, p. 33], is easily adapted for vector spaces over arbitrary ordered fields. An aligned space (X, \mathcal{L}) has *σ -finite Helly type* if X is the union of a countable family of subsets Y_n such that, for each n ,

the relative alignment

$$\mathcal{L}|Y_n = \{L \cap Y_n : L \in \mathcal{L}\}$$

has a finite Helly number. Thus any vector space of countable dimension over an ordered field has σ -finite Helly type.

The next Proposition will be used in the proof that (ii) implies (iii) from the Main Theorem. It also is of independent interest.

(B) PROPOSITION: *Let (X, \mathcal{L}) be an aligned space with Helly number h . Suppose \mathcal{K} is a family of \mathcal{L} -convex sets that is compact in the inclusion-exclusion topology on $\text{Pow}(X)$. If the sets in \mathcal{K} have empty intersection, then some subfamily of at most h sets from \mathcal{K} has empty intersection.*

PROOF: First we show by a Zorn's Lemma argument that among the closed subfamilies of \mathcal{K} which have empty intersection, there is a minimal one. For any decreasing chain (\mathcal{K}_γ) of closed subfamilies of \mathcal{K} , each with empty intersection, let

$$\mathcal{M} = \{K \subseteq X : K \in \mathcal{K}_\gamma \text{ for all } \gamma\}.$$

As an intersection of closed families in $\text{Pow}(X)$, \mathcal{M} is closed. Now consider any point p in X . For each γ , the family

$$\mathcal{K}_\gamma(p) = \{K \in \mathcal{K}_\gamma : p \notin K\}$$

is nonempty since the sets in \mathcal{K}_γ have void intersection and hence some set in \mathcal{K}_γ excludes p . Furthermore, as $\mathcal{K}_\gamma(p)$ is just the intersection of \mathcal{K}_γ with the basic closed family $\{S \subseteq X : p \notin S\}$, each $\mathcal{K}_\gamma(p)$ is closed and the collection $(\mathcal{K}_\gamma(p))$ is a descending chain. Since $\text{Pow}(X)$ is compact, this chain must have a nonvoid intersection. Any set in this intersection belongs to \mathcal{M} but misses p . Thus the intersection of the sets in \mathcal{M} is empty. One can now conclude from Zorn's lemma that there is a minimal closed subfamily of \mathcal{K} with empty intersection.

To prove the Proposition it now suffices to show that a minimal closed family of \mathcal{L} -convex sets whose intersection is empty must be finite. Proceeding by reductio ad absurdum, assume that \mathcal{M} is such a family but that \mathcal{M} contains infinitely many sets. Being both compact and infinite, \mathcal{M} must contain a set A which is an accumulation point of the other sets in \mathcal{M} (with respect to the inclusion-exclusion topology). Working toward a contradiction, we inductively choose $h + 1$ points from A as follows:

Let x_0 be any point in A . Having chosen x_0, x_1, \dots, x_n from A ,

consider the closed families

$$\mathcal{M}_i = \{M \in \mathcal{M} : x_i \notin M\}$$

defined for each i from 0 to n . Now define

$$\mathcal{C} = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_n \cup \{A\}.$$

Clearly \mathcal{C} is a closed subfamily of \mathcal{M} . We argue that it is properly included in \mathcal{M} . In $\text{Pow}(X)$,

$$\mathcal{U} = \{S \subseteq X : x_i \in S \text{ for all } i \text{ from } 0 \text{ to } n\}$$

is an open family containing A but no set from any \mathcal{M}_i . Thus $\mathcal{C} \cap \mathcal{U} = \{A\}$. But since A is not isolated relative to \mathcal{M} , there must be infinitely many sets in $\mathcal{M} \cap \mathcal{U}$. Thus \mathcal{C} is not all of \mathcal{M} . Therefore, the sets in \mathcal{C} must have nonempty intersection, by the minimality of \mathcal{M} . Let x_{n+1} be any point in this intersection. Then $x_{n+1} \in A$ since $A \in \mathcal{C}$. Furthermore, this choice also forces x_{n+1} to belong to every set in \mathcal{M}_i for each $i < n + 1$.

With x_0, x_1, \dots, x_n selected as above, define for each i from 0 to h an \mathcal{L} -convex set:

$$L_i = \mathcal{L}(\{x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_h\}).$$

Clearly any h of these $h + 1$ \mathcal{L} -convex sets have nonempty intersection. However, we shall show that all $h + 1$ of these sets have empty intersection, contradicting the hypothesis that h is a Helly number for \mathcal{L} .

Let $I = L_0 \cap L_1 \cap \cdots \cap L_h$. If M is in \mathcal{M} , we aver that $L_i \subseteq M$ for some i and hence that $I \subseteq M$. In fact, if M contains all the points x_i , then M contains all the sets L_i and hence I . If M misses some x_i , let j be the least index such that $x_j \notin M$. Then M belongs to \mathcal{M}_j . As observed in the construction of the x_i 's, for each $i > j$, the point x_i must lie in each set in \mathcal{M}_j . Thus $x_i \in M$ when $i > j$. However, for $i < j$, we have $x_i \in M$ by the minimal choice of j . Hence M contains all of the points x_i except for x_j and hence includes the corresponding convex hull L_j . Consequently $I \subseteq M$ for each M in \mathcal{M} . Thus I is contained in the intersection of all the sets in \mathcal{M} , which was assumed to be empty. This leads to the contradiction stated above and thereby completes the proof. \square

One might wonder if, for any alignment, a closed family of convex sets with empty intersection must always contain a finite subfamily with empty intersection. Were this so, the above Proposition could be

deduced as an easy corollary. However, as shown in the examples below, this need not happen in the absence of a finite Helly number.

(C) EXAMPLE: Let X be an infinite set with the *free* alignment: all subsets of X are convex. Let \mathcal{F} be the family of all subsets of the form $X \sim p$ where p is any point in X . Any net (not eventually constant) of sets in \mathcal{F} converges to X . Thus the family $\mathcal{F} \cup \{X\}$ is closed and has empty intersection, but no finite subfamily has empty intersection.

(D) EXAMPLE: The preceding example can be embedded in a vector space. Let V be an infinite dimensional real vector space and let X be an infinite linearly independent subset of V . For each p in X , let $C_p = \text{conv}(X \sim p)$ and set $C = \text{conv}(X)$. The family \mathcal{K} of all these convex sets has empty intersection, but any finite number always have a point in common. Furthermore, the family \mathcal{K} is closed in $\text{Pow}(V)$. (For any convergent net from \mathcal{K} , look at the corresponding net of extreme points. Note that $\text{ext}(C_p) = X \sim p$, and that a limit of convex sets must be convex.)

In [9] V. L. Klee gave a result for open convex sets in Euclidean space which is similar to Proposition B. His notion of compactness, however, seems to depend on the Hausdorff metric. In an earlier paper [7] Karlin and Shapley proved a related result for hemispheres in spheres. (Their result is phrased in terms of coverings.) Both of these results can be deduced from Proposition B. We shall not do so here since the original proofs are quite easy and it would lead us too far astray. For a survey of other related results, see [11].

(E) THEOREM: *Suppose that (X, \mathcal{L}) is an aligned space of σ -finite Helly type, and (X, \mathcal{L}) satisfies separation axiom S_3 . If every hemisphere in X is the convex hull of a countable set of points, then any family of convex sets with empty intersection has a countable subfamily with empty intersection.*

PROOF: Let \mathcal{F} be a family of convex subsets of X with empty intersection. For each point x of X , let C_x be a set in \mathcal{F} with $x \notin C_x$. By axiom S_3 a hemisphere H_x may be chosen such that $C_x \subseteq H_x$ and $x \notin H_x$. Let \mathcal{G} denote the collection of hemispheres so chosen as x ranges over X . To prove the theorem it suffices to show that some countable subfamily of \mathcal{G} has void intersection. To this end, define \mathcal{K} to be the closure of \mathcal{G} in $\text{Pow}(X)$ with respect to the inclusion-

exclusion topology. Note that since \mathcal{G} is included in the family of all hemispaces and, as observed earlier, this family is closed, it follows that each set in \mathcal{K} is a hemisphere.

Now the collection \mathcal{K} of hemispaces, as a closed subset of $\text{Pow}(X)$, is a compact Hausdorff space under the inclusion-exclusion topology. It is also first countable. To see this, consider any hemisphere H . By hypothesis, there are countable sets $S = \{s_1, s_2, \dots\}$ and $T = \{t_1, t_2, \dots\}$ such that $H = \mathcal{L}(S)$ and $X \sim H = \mathcal{L}(T)$. For each j , the family

$$\mathcal{N}_j = \{A \in \mathcal{K} : s_i \in A \text{ and } t_i \notin A \text{ for } i = 1, \dots, j\}$$

is a compact neighborhood of H relative to \mathcal{K} . If H' is any set in the intersection of this sequence of compact families, then H' is a hemisphere including S but missing T . Thus $H' \supseteq \mathcal{L}(S) = H$ since H' is convex, and $X \sim H' \supseteq \mathcal{L}(T) = X \sim H$, since $X \sim H'$ is convex. Consequently $H' = H$, so H is the only point in the intersection of the decreasing sequence of compact neighborhoods \mathcal{N}_j . It follows that \mathcal{K} is first countable. The information of importance to be gleaned from this is that each set in \mathcal{K} is the limit of a *sequence* of sets in \mathcal{G} .

Now write X as a countable union of subsets Y_n , where for each n the relative alignment $\mathcal{L}|Y_n$ has a finite Helly number. For each n ,

$$\mathcal{K}|Y_n = \{K \cap Y_n : K \text{ is in } \mathcal{K}\}$$

is a family of relatively convex sets in Y_n with empty intersection. Since intersection is continuous on $\text{Pow}(X)$ under the inclusion-exclusion topology [6, p. 43], $\mathcal{K}|Y_n$ is also closed in $\text{Pow}(Y_n)$. Since the Helly number of $\mathcal{L}|Y_n$ is finite, Proposition B implies that $\mathcal{K}|Y_n$ contains a finite subfamily with void intersection. Let the sets in this finite subfamily of $\mathcal{K}|Y_n$ be the restrictions to Y_n of the sets in a finite subfamily \mathcal{K}_n of \mathcal{K} . If A belongs to \mathcal{K}_n for some n , we may select a *sequence* of sets (A_i) from \mathcal{G} which converge to A . Let \mathcal{C} be the collection of all sets in \mathcal{G} which occur in some such chosen sequence. As a countable union of sequences, \mathcal{C} is countable.

We aver that the intersection of the sets in \mathcal{C} is empty. For if x is any point in X , then x lies in some Y_n . Hence there is a set A in \mathcal{K}_n with $x \notin A$. Since the chosen sequence (A_i) converges to A , for some i , x must be excluded from A_i . Thus x cannot belong to the intersection of all the sets in \mathcal{C} . Hence \mathcal{C} is a countable subfamily of \mathcal{G} with empty intersection. \square

As is inherent in the Main Theorem, the converse of Theorem E is valid in the case the space X is a vector space over an ordered field

and \mathcal{L} is the ordinary alignment. In general, however, the converse is not true. For example, take the space to be the closed unit disc D in the Euclidean plane with the alignment of ordinary convex sets. Then D is certainly a hemispace in itself, but D is not the convex hull of a countable set since D has uncountably many extreme-points. However, since Klee's theorem applies to any family of convex sets in the plane, it certainly applies to the convex sets contained in D . Consequently, in the general setting, the generating property concluded in Theorem A is strictly stronger than the intersection property expressed in Klee's theorem.

Conclusion of the proof of the main theorem

From Theorem A it follows that (i) implies (ii) in the Main Theorem. That (ii) implies (iii) is a special case of Theorem E. Now in any vector space of uncountable dimension, one can select per definitionem an uncountable linearly independent set X . One sees at once that the family of convex sets associated with this X as in Example D has empty intersection but every countable subfamily has nonvoid intersection. Hence if (iii) obtains, the vector space V must be of countable dimension. Furthermore, if (iii) obtains in V , then (iii) obtains in any linear subspace of V . Since a one-dimensional subspace can be identified with the underlying field F , it follows that (iii) obtains in F . From this it follows easily that F is completely sequential. This completes the proof of the Main Theorem.

Note: The referee has pointed out that an updated Russian translation of [2] is available: Danzer, Grünbaum, Klee, and Zalgaller, Helly's theorem and its applications (Russian). "Mir" Publishers, Moscow, 1968.

REFERENCES

- [1] C. CARATHÉODORY: Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen. *Rend. Circ. Mat. Palermo* 32 (1911) 193–217.
- [2] L. W. DANZER, B. GRÜNBAUM and V. L. KLEE: Helly's theorem and its relatives. *AMS Proc. Symp. CONVEXITY VII* (1963) 101–180.
- [3] H. G. EGGLESTON: *Convexity* Cambridge, Univ. Press, 1966.
- [4] L. GILLMAN and M. JERISON: *Rings of Continuous Functions*. Princeton, Van Nostrand, 1960.
- [5] E. HELLY: Über Mengen konvexen Körper mit gemeinschaftlichen Punkten. *Jber. Deutsch. Math. Verein.* 32 (1923) 175–176.

- [6] R. E. JAMISON: *A General Theory of Convexity*. Dissertation, Univ. of Washington, 1974.
- [7] S. KARLIN and L. S. SHAPLEY: Some applications of a theorem on convex functions. *Annals of Math.* 52 (1950) 148–153.
- [8] J. L. KELLEY: *Topology*. Princeton, Van Nostrand, 1955.
- [9] V. L. KLEE: Critical set of a convex body. *Amer. J. Math.* 75 (1953) 178–188.
- [10] V. L. KLEE: Structure of Semispaces. *Math. Scand.* (1956) 54–64.
- [11] V. L. KLEE: Infinite-dimensional intersection theorems. *AMS Proc. Symp. CONVEXITY VII* (1963) 349–360.
- [12] V. L. KLEE: *Mimeographed lecture notes*, Univ. of Washington, 1973.

(Oblatum 31–V–1976 & 5–XI–1976)

Mathematisches Institut
der Universität Erlangen
Bismarckstrasse 1½
D-8520 Erlangen
and
Mathematics Department
Louisiana State University