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ON THE SIMULTANEOUS APPROXIMATION OF a, b AND a^b

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1. Introduction

Let a and b be complex numbers with $a \neq 0$, $a \neq 1$ and $b \notin \mathbb{Q}$. Let a^b denote $\exp(b \log a)$ for some fixed branch of the logarithm. The problem is to determine whether it is possible that all three numbers a, b and a^b can be well approximated by algebraic numbers of bounded degree. We take d fixed and consider triples (α, β, γ) of algebraic numbers of degree at most d; H will denote the maximum of the heights of α, β and γ.

After initial results of Ricci [8] and Franklin [5], Schneider [9] stated that for any $\epsilon > 0$, there exist only finitely many triples (α, β, γ) with

$$\max(|a-\alpha|, |b-\beta|, |a^b-\gamma|) < \exp(-\log^{5+\epsilon} H).$$

Bundschuh [2] remarked that in Schneider's proof a condition like $\beta \notin \mathbb{Q}$ is needed and tried to prove a theorem without such a restriction. His assertion is that for any $\epsilon > 0$, there are only finitely many triples (α, β, γ) with

$$\max(|a-\alpha|, |b-\beta|, |a^b-\gamma|) < \exp(-\log^4 H \log_2^{-2+\epsilon} H),$$

where $\log_2$ means loglog. However, there is an error at the beginning of the proof of his Satz 2a, so that his result, too, is only valid if one makes some extra assumption.

Earlier Šmelev [10] had proved that only finitely many triples (α, β, γ) with $\beta \notin \mathbb{Q}$ have the property

$$\max(|a-\alpha|, |b-\beta|, |a^b-\gamma|) < \exp(-\log^4 H \log_2^3 H).$$

Cijsouw and Waldschmidt [4] recently improved upon the above results by showing that for any $\epsilon > 0$, there are only finitely many triples (α, β, γ) with $\beta \notin \mathbb{Q}$ and

$$\max(|a-\alpha|, |b-\beta|, |a^b-\gamma|) < \exp(-\log^3 H \log_2^{1+\epsilon} H).$$

The main purpose of this paper is to show that from all these theorems, the condition $\beta \notin \mathbb{Q}$ cannot be omitted. More precisely, the following will be proved:

THEOREM 1: *For any fixed natural number κ, there exist irrational numbers $a, b \in (0, 1)$ such that for infinitely many triples (α, β, γ) of rational numbers*

$$\max(|a-\alpha|, |b-\beta|, |a^b-\gamma|) < \exp(-\log^\kappa H),$$

where H denotes the maximum of the heights of α, β and γ.

In the theorems cited above, it would seem more natural to place a restriction upon the given number b than upon β. For instance, it is quite easy to see that the estimate of Cijsouw and Waldschmidt holds for arbitrary triples (α, β, γ) if one assumes that, for real b, the convergents p_n/q_n of the continued fraction expansion of b satisfy

$$q_{n+1} \ll \exp(\log^3 q_n), \quad n \to \infty.$$

(Note that the real numbers b for which this condition is not fulfilled, are U^*-numbers (see [9], III §3) and thus form a set of Lebesgue measure zero.) A sharper result in the same direction is given by the next theorem.

THEOREM 2: *Suppose $\epsilon > 0$, $d \in \mathbb{N}$, a, $b \in \mathbb{C}$, $a \neq 0$, $b \notin \mathbb{Q}$, l a branch of the logarithm with $l(a) \neq 0$. If $b \notin \mathbb{R}$, or if $b \in \mathbb{R}$ such that the convergents p_n/q_n of the continued fraction expansion of b satisfy*

$$q_{n+1} \ll \exp(q_n^3), \quad n \to \infty,$$

there are only finitely many triples (α, β, γ) of algebraic numbers of degree at most d with

$$\max(|a-\alpha|, |b-\beta|, |a^b-\gamma|) < \exp(-\log^3 H \log_2^{1+\epsilon} H),$$

where H denotes the maximum of the heights of α, β and γ and $a^b = \exp(bl(a))$.

In the proofs of Theorems 1 and 2, the following notations will be employed: if α is an algebraic number, $\overline{|\alpha|}$ denotes the maximum of the absolute values of the conjugates of α, $h(\alpha)$ the height of α, $\mathrm{dg}(\alpha)$ the degree of α and $\mathrm{den}(\alpha)$ the denominator of α. We shall make use of the relations $\mathrm{den}(\alpha) \leq h(\alpha)$ and $\overline{|\alpha|} \leq h(\alpha) + 1$.

2. Two sequences of rational numbers

LEMMA 1: *Let κ, $\lambda \in \mathbb{N}$ be given. There is a sequence $(\beta_n)_{n=1}^{\infty}$ of rational numbers in* $(0, 1)$ *such that for all $n \in \mathbb{N}$ the following inequalities hold*:

$$|\beta_n - \beta_{n+1}| < \exp(-h^{\kappa+2}(\beta_n)), \tag{1}$$

$$h(\beta_{n+1}) > \exp(h^{\kappa+2}(\beta_n)), \tag{2}$$

$$h(\beta_n) > \lambda. \tag{3}$$

PROOF: Let β_1 be any rational number in $(0, 1)$ with $h(\beta_1) > \lambda$. The sequence $(\beta_n)_{n=1}^{\infty}$ will be defined inductively; suppose β_n already chosen. Clearly there are infinitely many rational numbers $\beta^* \in (0, 1)$ with the property

$$|\beta_n - \beta^*| \leq \frac{1}{h(\beta^*)}.$$

Only finitely many rational numbers have heights bounded by $\exp(h^{\kappa+2}(\beta_n))$, so there exists a β_{n+1} with both $h(\beta_{n+1}) > \exp(h^{\kappa+2}(\beta_n))$ and

$$|\beta_n - \beta_{n+1}| \leq \frac{1}{h(\beta_{n+1})} < \exp(-h^{\kappa+2}(\beta_n)). \quad \square$$

LEMMA 2: *Let κ, $\lambda \in \mathbb{N}$ be given and let $(\beta_n)_{n=1}^{\infty}$ be a sequence of rational numbers in* $(0, 1)$ *such that* (1), (2) *and* (3) *are satisfied for all $n \in \mathbb{N}$. Put $\beta_n = v_n/w_n$, where v_n, $w_n \in \mathbb{N}$, $(v_n, w_n) = 1$. If λ is sufficiently large, there is a sequence $(\alpha_n)_{n=1}^{\infty}$ of rational numbers in* $(0, 1)$, *such that for all $n \in \mathbb{N}$ the following assertions hold*:

$$|\alpha_n - \alpha_{n+1}| < \exp(-\log^{\kappa+1} h(\alpha_n)), \tag{4}$$

$$h(\alpha_{n+1}) > h^2(\alpha_n), \tag{5}$$

$$h(\alpha_n) < (2w_n)^{2w_n}, \tag{6}$$

$$\alpha_n^{1/w_n} \in \mathbb{Q}. \tag{7}$$

PROOF: Choose $\alpha_1 := 2^{-w_1}$. The sequence $(\alpha_n)_{n=1}^{\infty}$ will be defined inductively; suppose $\alpha_1, \ldots, \alpha_n$ have already been chosen and possess the desired properties. By Bertrand's Postulate ([6], Theorem 418), there is a prime number u_{n+1} with

$$w_{n+1}^2 \leq u_{n+1} \leq 2w_{n+1}^2.$$

Notice that, if λ is sufficiently large,

$$\begin{aligned} \frac{u_{n+1}}{w_{n+1}} \geq w_{n+1} \geq \exp(w_n^{\kappa+2}) > \exp((2w_n)^{\kappa+1} \log^{\kappa+1}(2w_n)) \\ > \exp(\log^{\kappa+1} h(\alpha_n)). \end{aligned} \tag{8}$$

Consider the partition

$$D = \left(0, \frac{1}{u_{n+1}^{w_{n+1}}}, \frac{2^{w_{n+1}}}{u_{n+1}^{w_{n+1}}}, \ldots, \frac{(u_{n+1}-1)^{w_{n+1}}}{u_{n+1}^{w_{n+1}}}, 1\right)$$

of the interval $(0, 1)$. Take $t \in \{0, \ldots, u_{n+1} - 1\}$. Then

$$\frac{(t+1)^{w_{n+1}}}{u_{n+1}^{w_{n+1}}} - \frac{t^{w_{n+1}}}{u_{n+1}^{w_{n+1}}} \leq \frac{w_{n+1}(t+1)^{w_{n+1}-1}}{u_{n+1}^{w_{n+1}}} \leq \frac{w_{n+1}}{u_{n+1}};$$

therefore the width of the partition D does not exceed w_{n+1}/u_{n+1}. By (8) the interval $\{x \in (0, 1): |\alpha_n - x| < \exp(-\log^{\kappa+1} h(\alpha_n))\}$ has a length greater than w_{n+1}/u_{n+1}, so that this interval contains at least one of the points of D. This proves the existence of a $t_{n+1} \in \{1, \ldots, u_{n+1} - 1\}$ with

$$\left|\alpha_n - \frac{t_{n+1}^{w_{n+1}}}{u_{n+1}^{w_{n+1}}}\right| < \exp(-\log^{\kappa+1} h(\alpha_n)).$$

If one defines $\alpha_{n+1} := (t_{n+1}/u_{n+1})^{w_{n+1}}$, (4) is satisfied, and furthermore $\alpha_{n+1}^{1/w_{n+1}} \in \mathbb{Q}$. Finally,

$$h(\alpha_{n+1}) = u_{n+1}^{w_{n+1}} \leq (2w_{n+1}^2)^{w_{n+1}} < (2w_{n+1})^{2w_{n+1}},$$

and

$$h(\alpha_{n+1}) = u_{n+1}^{w_{n+1}} \geq (w_{n+1}^2)^{w_{n+1}} = \exp(2w_{n+1}\log w_{n+1}) \geq \exp(w_{n+1})$$
$$> \exp(\exp(w_n^{\kappa+2})) > \exp(4w_n \log(2w_n)) > h^2(\alpha_n). \quad \Box$$

3. Proof of Theorem 1

I. Take $\lambda \in \mathbb{N}$ and let $(\beta_n)_{n=1}^{\infty}$ be a sequence of rational numbers in $(0, 1)$ such that (1)–(3) are satisfied. Put $\beta_n = v_n/w_n$, where $v_n, w_n \in \mathbb{N}$, $(v_n, w_n) = 1$. We may suppose that λ is sufficiently large in the sense of Lemma 2; let $(\alpha_n)_{n=1}^{\infty}$ be a sequence of rational numbers in $(0, 1)$ such that (4)–(7) are satisfied. Put $\alpha_n = (t_n/u_n)^{w_n}$, where $t_n, u_n \in \mathbb{N}$, $(t_n, u_n) = 1$. Define $\gamma_n := \alpha_n^{\beta_n}$; we have

$$\gamma_n = \alpha_n^{\beta_n} = \left(\frac{t_n^{w_n}}{u_n^{w_n}}\right)^{v_n/w_n} = \frac{t_n^{v_n}}{u_n^{w_n}},$$

so $\gamma_n \in \mathbb{Q}$ and $h(\gamma_n) = u_n^{v_n} < u_n^{w_n} = h(\alpha_n)$. Therefore

$$H_n = \max(h(\alpha_n), h(\beta_n), h(\gamma_n)) = h(\alpha_n).$$

II. The sequence $(\alpha_n)_{n=1}^{\infty}$ has the property

$$\forall\, m > n: |\alpha_m - \alpha_n| < \exp(-\tfrac{1}{2}\log^{\kappa+1} h(\alpha_n)). \tag{9}$$

For put $I_k := \{x \in \mathbb{R}: |\alpha_k - x| < \exp(-\tfrac{1}{2}\log^{\kappa+1} h(\alpha_k))\}$. Then (9) can be written as $\forall\, m > n: \alpha_m \in I_n$. By (4), $\alpha_m \in I_{m-1}$, so it is sufficient to prove $\forall\, k > n: I_k \subset I_{k-1}$. Take $x \in I_k$, which means $|\alpha_k - x| < \exp(-\tfrac{1}{2}\log^{\kappa+1} h(\alpha_k))$. Then, by (4) and (5),

$$\begin{aligned}|\alpha_{k-1} - x| &\leq |\alpha_k - x| + |\alpha_{k-1} - \alpha_k| \\ &< \exp(-\tfrac{1}{2}\log^{\kappa+1} h(\alpha_k)) + \exp(-\log^{\kappa+1} h(\alpha_{k-1})) \\ &< \exp(-2^{\kappa}\log^{\kappa+1} h(\alpha_{k-1})) + \exp(-\log^{\kappa+1} h(\alpha_{k-1})) \\ &< 2\exp(-\log^{\kappa+1} h(\alpha_{k-1})) < \exp(-\tfrac{1}{2}\log^{\kappa+1} h(\alpha_{k-1}))\end{aligned}$$

if w_1 is sufficiently large, so $x \in I_{k-1}$.

III. From (9) we see that $(\alpha_n)_{n=1}^{\infty}$ is a Cauchy sequence; it converges to a limit, which we shall call a. Then

$$\forall\, n: |a - \alpha_n| \leq \exp(-\tfrac{1}{2}\log^{\kappa+1} h(\alpha_n)) = \exp(-\tfrac{1}{2}\log^{\kappa+1} H_n),$$

which, by Theorem 186 of [6], implies that a is irrational.

In the same way one can prove that $(\beta_n)_{n=1}^{\infty}$ is a Cauchy sequence and that its limit b satisfies

$$\begin{aligned}\forall\, n\colon |b-\beta_n| &\le \exp\left(-\tfrac{1}{2}h^{\kappa+2}(\beta_n)\right) = \exp\left(-\tfrac{1}{2}w_n^{\kappa+2}\right)\\ &\le \exp\left(-\tfrac{1}{2}(2w_n)^{\kappa+1}\log^{\kappa+1}(2w_n)\right) \le \exp\left(-\tfrac{1}{2}\log^{\kappa+1} h(\alpha_n)\right)\\ &= \exp\left(-\tfrac{1}{2}\log^{\kappa+1} H_n\right)\end{aligned}$$

so that b, too, is irrational.

IV. The function x^y is continuously differentiable on every compact subset K of $(0,1)\times(0,1)$, so that a constant C_K, only depending on K, can be found with

$$|x^y-\xi^\eta| < C_K \max\left(|x-\xi|, |y-\eta|\right) \text{ for } (x,y), (\xi,\eta)\in K.$$

From this follows the existence of a constant $C_{a,b}$, depending only on a and b, such that

$$\begin{aligned}|a^b-\alpha_n^{\beta_n}| &< C_{a,b}\max\left(|a-\alpha_n|, |b-\beta_n|\right)\\ &\le C_{a,b}\exp\left(-\tfrac{1}{2}\log^{\kappa+1}H_n\right) < \exp\left(-\log^{\kappa} H_n\right),\end{aligned}$$

for sufficiently large n. □

It may be of interest to note that a p-adic analogue of Theorem 1 can be proved with considerably less difficulty. Indeed, it suffices to construct a sequence $(\beta_n)_{n=1}^{\infty}$ of natural numbers with the properties $|\beta_n-\beta_{n+1}|_p < \exp(-\beta_n^{\kappa+1})$ and $|\beta_n|_p = p^{-2}$. If b is the p-adic limit of this sequence and $a=b+1$, infinitely many triples (α,β,γ) of natural numbers satisfy

$$\max\left(|a-\alpha|_p, |b-\beta|_p, |a^b-\gamma|_p\right) < \exp\left(-\log^{\kappa} H\right),$$

where $H=\max(\alpha,\beta,\gamma)$ and a^b is defined by means of the p-adic logarithm and exponential function.

4. A result on vanishing linear forms

LEMMA 3: *Suppose $d\in\mathbb{N}$, K a compact subset of the complex plane not containing* 0, *l_1 and l_2 branches of the logarithm, defined on K, such that l_1 does not take the value* 0. *Then only finitely many pairs $(\alpha,\gamma)\in K\times K$ of algebraic numbers of degree at most d have the* *property that a* $\beta \in \mathbb{Q}$ *exists with*

$$\beta l_1(\alpha) - l_2(\gamma) = 0$$

and

$$h(\beta) \geq \log H,$$

where $H = \max(h(\alpha), h(\gamma))$.

PROOF: I. Suppose $\alpha, \gamma \in K$, $\beta \in \mathbb{Q}$, such that the conditions of the lemma are fulfilled. By $c_1, c_2, \ldots$ we shall denote natural numbers depending only on d, K, l_1 and l_2; we suppose that H is greater than such a number, which will lead to a contradiction.

Put $B := h(\beta)$; then

$$\log H \leq B. \tag{10}$$

Define $L := [2dB \log^{-1/3} B] - 1$. We introduce the auxiliary function

$$\Phi(z) = \sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} p(\lambda_1, \lambda_2)\alpha^{\lambda_1 z}\gamma^{\lambda_2 z},\ z \in \mathbb{C},$$

where $\alpha^{\lambda_1 z} = \exp(\lambda_1 z l_1(\alpha))$, $\gamma^{\lambda_2 z} = \exp(\lambda_2 z l_2(\gamma))$ and where $p(\lambda_1, \lambda_2)$ are rational integers to be determined later. We have

$$\Phi^{(t)}(z) = l_1^t(\alpha) \sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} p(\lambda_1, \lambda_2)(\lambda_1 + \lambda_2\beta)^t \alpha^{\lambda_1 z}\gamma^{\lambda_2 z},$$
$$z \in \mathbb{C},\ t \in \mathbb{N} \cup \{0\}.$$

Now put $a := \operatorname{den}(\alpha)$, $b := \operatorname{den}(\beta)$, $c := \operatorname{den}(\gamma)$, $S := [\log^{1/3} B]$, $T := [B^2 \log^{-1} B]$ and consider the system of linear equations

$$(ac)^{sL} b^t l_1^{-t}(\alpha)\Phi^{(t)}(s) = 0,\ s = 0, \ldots, S-1,\ t = 0, \ldots, T-1.$$

These are ST equations in the $(L+1)^2$ unknowns $p(\lambda_1, \lambda_2)$; the coefficients are algebraic integers in the number field $\mathbb{Q}(\alpha, \gamma)$ of degree at most d^2. The absolute values of the conjugates of the coefficients are less than or equal to

$$(ac)^{LS} b^T \max(1, |\lambda_1 + \lambda_2\beta|^T) \max(1, \overline{|\alpha|}^{LS}) \max(1, \overline{|\gamma|}^{LS})$$
$$\leq H^{4LS} B^T c_1^{LS+T} L^T \leq \exp(c_2 B^2)$$

(here (10) is used).

As $(L+1)^2 \geq d^2B^2 \log^{-2/3} B \geq d^2ST$, Lemme 1.3.1 of [11] states that there is a non-trivial choice for the $p(\lambda_1, \lambda_2)$, such that

$$\Phi^{(t)}(s) = 0,\ s = 0, \ldots, S-1,\ t = 0, \ldots, T-1, \tag{11}$$

while

$$P := \max_{\substack{\lambda_1=0,\ldots,L \\ \lambda_2=0,\ldots,L}} |p(\lambda_1, \lambda_2)| \leq (c_3L^2 \exp(c_2B^2))^{d^2ST/((L+1)^2-d^2ST)} \leq \exp(c_4B^2).$$

II. For $k \in \mathbb{N} \cup \{0\}$ we put $T_k := 2^kT$; suppose $2^k \leq \log^{1/6} B$. Then, for our special choice of the $p(\lambda_1, \lambda_2)$, we have

$$\Phi^{(t)}(s) = 0,\ s = 0, \ldots, S-1,\ t = 0, \ldots, T_k - 1. \tag{12}$$

This is proved by induction; for $k = 0$ the assertion is precisely (11). Now suppose that (12) holds for some k, while $2^{k+1} \leq \log^{1/6} B$. By Lemma 7 of [3] we have

$$\max_{|z|\leq 2S} |\Phi(z)| \leq 2 \max_{|z|\leq 2BS} |\Phi(z)| \left(\frac{2}{B}\right)^{ST_k}. \tag{13}$$

Here

$$\max_{|z|\leq 2BS} |\Phi(z)| \leq (L+1)^2 P c_5^{2BLS} \leq \exp(c_6B^2) \tag{14}$$

and

$$\left(\frac{2}{B}\right)^{ST_k} \leq \exp\left(-\frac{1}{c_7} B^2 \log^{1/3} B\right).$$

Substitution in (13) gives

$$\max_{|z|\leq 2S} |\Phi(z)| \leq \exp\left(-\frac{1}{c_8} B^2 \log^{1/3} B\right).$$

For $s = 0, \ldots, S-1$, $t = 0, \ldots, T_{k+1} - 1$ we have

$$\Phi^{(t)}(s) = \frac{t!}{2\pi i} \int_{|z-s|=1} \frac{\Phi(z)}{(z-s)^{t+1}} dz,$$

so

$$|\Phi^{(t)}(s)| \leq \exp\left(t \log t - \frac{1}{c_8} B^2 \log^{1/3} B\right)$$
$$\leq \exp\left(c_9 B^2 \log^{1/6} B - \frac{1}{c_8} B^2 \log^{1/3} B\right),$$

from which we conclude

$$(15) |\Phi^{(t)}(s)| \leq \exp\left(-\frac{1}{c_{10}} B^2 \log^{1/3} B\right),\ s = 0, \ldots, S-1,\ t = 0, \ldots, T_{k+1} - 1.$$

However, $l_1^{-t}(\alpha)\Phi^{(t)}(s)$ is algebraic and formula (1.2.3) of [11] states that every non-zero algebraic number ξ has the property

$$(16) \qquad |\xi| \geq \exp(-2\,\mathrm{dg}(\xi) \max(\log \overline{|\xi|}, \log \mathrm{den}(\xi))).$$

Now for $s = 0, \ldots, S-1$, $t = 0, \ldots, T_{k+1} - 1$ we have

$$\mathrm{dg}(l_1^{-t}(\alpha)\Phi^{(t)}(s)) \leq d^2,$$
$$\mathrm{den}(l_1^{-t}(\alpha)\Phi^{(t)}(s)) \leq (ac)^{LS} b^{T_{k+1}} \leq H^{2LS} B^{T_{k+1}} \leq \exp(c_{11} B^2 \log^{1/6} B),$$
$$\overline{|l_1^{-t}(\alpha)\Phi^{(t)}(s)|} \leq P c_{12}^{LS+T_{k+1}} L^{T_{k+1}+2} H^{2LS} \leq \exp(c_{13} B^2 \log^{1/6} B),$$

so either $\Phi^{(t)}(s) = 0$ or

$$|l_1^{-t}(\alpha)\Phi^{(t)}(s)| \geq \exp(-c_{14} B^2 \log^{1/6} B);$$

in the latter case

$$(17) \qquad |\Phi^{(t)}(s)| \geq \exp(-c_{15} B^2 \log^{1/6} B).$$

Combining (15) and (17) gives $\Phi^{(t)}(s) = 0$ for $s = 0, \ldots, S-1$, $t = 0, \ldots, T_{k+1} - 1$. This completes the proof of (12).

III. Now let k be the *largest* natural number with $2^k \leq \log^{1/6} B$. From (12) it follows that

$$\Phi^{(t)}(s) = 0,\ s = 0, \ldots, S-1,\ t = 0, \ldots, T_k - 1.$$

Once more apply Lemma 7 of [3]; this gives (13) again and (14) also remains unchanged, but from the maximality of k we now get

$$\left(\frac{2}{B}\right)^{ST_k} \leq \exp\left(-\frac{1}{c_{16}} B^2 \log^{1/2} B\right),$$

so

$$\max_{|z|\le 2S} |\Phi(z)| \le \exp\left(-\frac{1}{c_{17}} B^2 \log^{1/2} B\right).$$

For $t = 0, 1, \ldots, (L+1)^2 - 1$ we have

$$\Phi^{(t)}(0) = \frac{t!}{2\pi i} \int_{|z|=1} \frac{\Phi(z)}{z^{t+1}}\, dz,$$

whence

$$\begin{aligned} |\Phi^{(t)}(0)| &\le \exp\left(t \log t - \frac{1}{c_{17}} B^2 \log^{1/2} B\right) \\ &\le \exp\left(c_{18} B^2 \log^{1/3} B - \frac{1}{c_{17}} B^2 \log^{1/2} B\right). \end{aligned}$$

Conclusion:

$$|\Phi^{(t)}(0)| \le \exp\left(-\frac{1}{c_{19}} B^2 \log^{1/2} B\right),\ t = 0, \ldots, (L+1)^2 - 1. \tag{18}$$

For these values of t we have

$$\begin{aligned} &\mathrm{dg}(l_1^{-t}(\alpha)\Phi^{(t)}(0)) \le d^2, \\ &\mathrm{den}(l_1^{-t}(\alpha)\Phi^{(t)}(0)) \le B^{(L+1)^2} \le \exp(c_{20} B^2 \log^{1/3} B), \\ &\overline{|l_1^{-t}(\alpha)\Phi^{(t)}(0)|} \le P(c_{21}L)^{c_{22}(L+1)^2} \le \exp(c_{23} B^2 \log^{1/3} B), \end{aligned}$$

so according to (16) either $\Phi^{(t)}(0) = 0$ or

$$|l_1^{-t}(\alpha)\Phi^{(t)}(0)| \ge \exp(-c_{24} B^2 \log^{1/3} B);$$

in the latter case

$$|\Phi^{(t)}(0)| \ge \exp(-c_{25} B^2 \log^{1/3} B). \tag{19}$$

Combining (18) and (19) gives

$$\Phi^{(t)}(0) = 0,\ t = 0, \ldots, (L+1)^2 - 1.$$

IV. For $t = 0, \ldots, (L+1)^2 - 1$ we now have

$$\sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} p(\lambda_1, \lambda_2)(\lambda_1 + \lambda_2\beta)^t = 0. \tag{20}$$

As the $p(\lambda_1, \lambda_2)$ are not all zero, it follows that the coefficient matrix of the system (20), which is of the Vandermonde type, must be singular. From this we deduce the existence of $\lambda_1, \lambda_2, \lambda_1', \lambda_2' \in \{0, \ldots, L\}$ with $\lambda_1 + \lambda_2\beta = \lambda_1' + \lambda_2'\beta$, or

$$\beta = \frac{\lambda_1' - \lambda_1}{\lambda_2 - \lambda_2'}.$$

This gives

$$B = h(\beta) \leq L = [2dB \log^{-1/3} B] - 1,$$

so we get a contradiction for sufficiently large H (and B). □

5. Proof of Theorem 2

I. The case $b \notin \mathbb{R}$ is trivial; we shall therefore suppose that b is real and that its continued fraction expansion has the property described in the theorem. Let (α, β, γ) be a triple fulfilling the conditions of the theorem; we suppose H to be greater than a certain bound depending only on ϵ, d, a, b and l. This will lead to a contradiction.

As $a \neq 0$ and $a^b \neq 0$, we may assume $\alpha \neq 0$ and $\gamma \neq 0$. For suitably chosen branches l_1 and l_2 of the logarithm we have

$$|l(a) - l_1(\alpha)| < \exp(-\log^3 H \log_2^{1+2\epsilon/3} H), \tag{21}$$

$$|bl(a) - l_2(\gamma)| < \exp(-\log^3 H \log_2^{1+2\epsilon/3} H); \tag{22}$$

from $l(a) \neq 0$ we thus get $l_1(\alpha) \neq 0$. As a consequence of (21), (22) and

$$|b - \beta| < \exp(-\log^3 H \log_2^{1+\epsilon} H)$$

we have

$$|\beta l_1(\alpha) - l_2(\gamma)| < \exp(-\log^3 H \log_2^{1+\epsilon/3} H).$$

If it were the case that $\beta l_1(\alpha) - l_2(\gamma) \neq 0$, Theorem 1 of [4] would imply

$$|\beta l_1(\alpha) - l_2(\gamma)| > \exp(-\log^3 H \log_2^{1+\epsilon/3} H),$$

which is a contradiction. Therefore $\beta l_1(\alpha) - l_2(\gamma) = 0$.

II. We have just proved that $l_1(\alpha)$ and $l_2(\gamma)$ are linearly dependent over the field of all algebraic numbers; using Theorem 1 of [1] we find that these numbers must also be linearly dependent over $\mathbb{Q}$. In other words, there are ξ, $\eta \in \mathbb{Q}$, not both zero, such that $\xi l_1(\alpha) + \eta l_2(\gamma) = 0$. Here $\eta \neq 0$ because $l_1(\alpha) \neq 0$, so

$$\beta = \frac{l_2(\gamma)}{l_1(\alpha)} = -\frac{\xi}{\eta} \in \mathbb{Q};$$

using Lemma 3 above we see that $h(\beta) < \log H$.

Put $q := \mathrm{den}(\beta)$; then $q < \log H$, so

$$|b - \beta| < \exp(-\log^3 H \log_2^{1+\epsilon} H) < \exp(-q^3 \log^{1+\epsilon} q). \tag{23}$$

As q must tend to infinity with H, we may assume

$$|b - \beta| < \frac{1}{2q^2},$$

and thus, by Satz 2.11 of [7], β is a convergent of b, say $\beta = p_n/q_n$. By (12) in §13 of [7], we have, for some constant c,

$$|b - \beta| > \frac{1}{q_n(q_n + q_{n+1})} \geq \frac{1}{q_n(q_n + c\exp(q_n^3))} > \exp(-q^3 \log^{1+\epsilon} q),$$

which contradicts (23). □

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