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A TAUBERIAN THEOREM FOR CESÀRO SUMMABILITY

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1

The "standard" Tauberian theorems applicable to Cesàro summability (C, α) continue to hold when (C, α) is replaced by Abel summability, though the proof for Abel summability is often harder. It is therefore of some interest to obtain a Tauberian theorem for summability (C, α) which becomes false when (C, α) is replaced by Abel summability. Such theorems have been given in [1], [2]; in the present paper we give such a result with a Tauberian condition of a somewhat different nature.

We consider a series (in general, of complex numbers),

$$\sum_{n=0}^{\infty} a_n, \tag{1}$$

and will write throughout

$$s_n = a_0 + a_1 + \cdots + a_n.$$

THEOREM 1: *Let $\alpha > 0$. Suppose that*

$$\sum_{k=0}^{n} |a_k| = O(|s_n|), \tag{2}$$

and that (1) *is bounded* (C, α). *Then* (1) *converges.*

We note that we need assume only the (C, α) boundedness of (1), and not its (C, α) summability.

I obtained this result in the course of certain investigations into Nörlund summability. It is familiar that the Nörlund transformation (N, a_n) is regular if and only if (2) holds and

$$a_n = o(|s_n|). \tag{3}$$

Thus (2) is relevant. However, the result appears to be of some interest for its own sake, and thus we shall not be concerned with Nörlund summability in the present paper.

This theorem becomes false when (C, α) is replaced by Abel summability, even if we assume Abel summability (instead of only boundedness), and even if we impose the additional restriction (3). In other words, we have the following result.

THEOREM 2: *There is a series satisfying* (2) *and* (3) *which is Abel summable but not convergent.*

The results for Abel summability become somewhat different if we restrict ourselves to real series.

THEOREM 3: *Suppose that a_n is real. Suppose that* (2) *and* (3) *hold, and that* (1) *is Abel bounded. Then* (1) *converges. This result becomes false when* (3) *is omitted, even if we replace Abel boundedness by Abel summability.*

2

We let S_n^α denote the (C, α) sum of (1); that is to say,

$$S_n^\alpha = \sum_{k=0}^{n} A_{n-k}^\alpha a_k,$$

where, as is usual, A_ν^α denotes the binomial coefficient

$$\frac{(\alpha+\nu)(\alpha+\nu-1)\dots(\alpha+1)}{\nu!}.$$

For any positive integer m, let δ_m denote the operator defined by

$$\delta_m t_n = t_{n+m} - t_n;$$

if α is a positive integer, let δ_m^α denote the result of applying the operator δ_m α times. With this notation, we have the following lemma.

LEMMA: *Let α be a positive integer. Then*

$$\delta_m^\alpha S_n^\alpha = \sum_{k=n+\alpha}^{n+\alpha m} K_{k-n,m}^\alpha s_k, \tag{4}$$

where

$$K_{\nu,m}^\alpha > 0 \quad (\alpha \le \nu \le \alpha m). \tag{5}$$

and where

(6)
$$\sum_{\nu=\alpha}^{\alpha m} K^{\alpha}_{\nu,m} = m^{\alpha}.$$

PROOF: The proof is by induction on α. The result is evident when $\alpha = 1$, since

$$\delta_m S^1_n = S^1_{n+m} - S^1_n = \sum_{k=n+1}^{n+m} s_k.$$

Now assume the result true for α (where $\alpha \geq 1$), and prove it true for $\alpha + 1$. We have

$$\begin{aligned}
\delta^{\alpha+1}_m S^{\alpha+1}_n &= \delta^{\alpha}_m(\delta S^{\alpha+1}_n) = \delta^{\alpha}_m(S^{\alpha+1}_{n+m} - S^{\alpha+1}_n)\\
&= \delta^{\alpha}_m \sum_{\mu=n+1}^{n+m} S^{\alpha}_{\mu} = \sum_{\mu=n+1}^{n+m} \delta^{\alpha}_m S^{\alpha}_{\mu}\\
&= \sum_{\mu=n+1}^{n+m} \sum_{k=\mu+\alpha}^{\mu+\alpha m} K^{\alpha}_{k-\mu,m} s_k\\
&= \sum_{n+\alpha+1}^{n+(\alpha+1)m} s_k \sum_{\mu=n+1}^{n+m} K^{\alpha}_{k-\mu,m}.
\end{aligned}$$

Here we adopt the convention that $K^{\alpha}_{\nu,m}$ is to be taken to mean 0 when $\nu < \alpha$ or $\nu > \alpha m$. Thus (4) holds with α replaced by $\alpha + 1$, and with

$$K^{\alpha+1}_{\nu,m} = \sum_{\rho=1}^{m} K^{\alpha}_{\nu-\rho,m}.$$

This gives us (4) and (5). In order to prove (6), we need only observe that, by (4) the sum on the left of (6) is equal to the value of $\delta^{\alpha}_m S^{\alpha}_n$ in the special case in which $s_k = 1$ for all k. But, in this special case, $S^{\alpha}_n = A^{\alpha}_n$, and (6) follows easily.

3

We can now prove Theorem 1. In view of the well known result that, if $\alpha' > \alpha > -1$, any series bounded (C, α) is also bounded (C, α'), there is no loss of generality in supposing that α is an integer. We will make this assumption throughout in what follows.

We note that it follows at once from (2) that there is a constant $c > 0$, which will be kept fixed throughout, such that, for all sufficiently large n,

(7)
$$|s_n| \geq c \sum_{k=0}^{n} |a_k|.$$

It follows that for sufficiently large n and all $m > n$,

$$|s_m| \geq c \sum_{k=0}^{m} |a_k| \geq c \sum_{k=0}^{n} |a_k| \geq c|s_n|. \tag{8}$$

It is enough to prove that $|s_n|$ is bounded, for the convergence (indeed, absolute) of (1) will then follow from (2). So we suppose that $|s_n|$ is unbounded, and show that this leads to a contradiction. Since $|s_n|$ is unbounded, it follows from (8) that

$$|s_n| \to \infty \tag{9}$$

as $n \to \infty$.

Now, for each n define $k(n)$ as the least value of $k > n$ such that

$$|\arg s_k - \arg s_n| > \tfrac{1}{4}\pi. \tag{10}$$

We will show that, for all sufficiently large n, $k(n)$ exists (that is to say, there is some $k > n$ satisfying (10)) and, further, that

$$\frac{k(n)}{n} \to 1 \tag{11}$$

as $n \to \infty$. This is equivalent to showing that, if $\delta > 0$ is given, then, for all sufficiently large n, there is some k with $n < k \leq (1+\delta)n$ satisfying (10). To prove this, suppose the assertion false; thus

$$|\arg s_k - \arg s_n| \leq \tfrac{1}{4}\pi \tag{12}$$

for all k with $n < k \leq (1+\delta)n$. Now let

$$\theta = \theta_n = \arg s_n,$$

and define $m = m(n)$ by $m = [n\delta/\alpha]$. Then, by the lemma together with (12) and (8),

$$\begin{aligned} \operatorname{Re}\{e^{-i\theta}\delta_m^\alpha S_n^\alpha\} &= \operatorname{Re}\left\{e^{-i\theta} \sum_{k=n+\alpha}^{n+\alpha m} K_{k-n,m}^\alpha s_k\right\} \\ &\geq \frac{1}{\sqrt{2}} \sum_{k=n+\alpha}^{n+\alpha m} K_{k-n,m}^\alpha |s_k| \\ &\geq \frac{c}{\sqrt{2}} |s_n| \sum_{k=n+\alpha}^{n+\alpha m} K_{k-n,m}^\alpha \\ &= \frac{c}{\sqrt{2}} m^\alpha |s_n| \\ &\sim \frac{c}{\sqrt{2}} \left(\frac{\delta}{\alpha}\right)^\alpha n^\alpha |s_n| \qquad (13) \end{aligned}$$

as $n \to \infty$ (δ being fixed). But, since (1) is bounded (C, α), there is a constant A such that

$$|\delta_m^\alpha S_n^\alpha| \le A(n^\alpha + m^\alpha)$$

for all $n \ge 1$, $m \ge 1$. In view of (9), this contradicts (13) whenever n is sufficiently large.

Now choose any sufficiently large fixed integer n_0, and define $\{n_r\}$ inductively by

$$n_r = k(n_{r-1}) \quad (r \ge 1).$$

We note that, by (8) we have, for $r \ge 1$,

$$|s(n_r)| \ge c|s(n_0)|; \quad s(n_{r-1}) \ge c|s(n_0)|.$$

[Here and elsewhere, in order to avoid complicated suffixes, we write $s(n)$ in place of s_n whenever n is replaced by a more complicated expression]. In view of the definition of n_r, it follows at once that

$$|s(n_r) - s(n_{r-1})| \ge 2c \sin\frac{\pi}{8}|s(n_0)|,$$

and hence that

$$\sum_{k=n_{r-1}+1}^{n_r} |a_k| \ge 2c \sin\frac{\pi}{8}|s(n_0)|, \tag{14}$$

Now let ρ be a fixed positive integer so chosen that

$$\mu = 2c^2\rho \sin\frac{\pi}{8} > 1.$$

Then, by (7) and (14)

$$|s(n_\rho)| \ge \mu|s(n_0)|.$$

Applying a similar argument with n_0 replaced by n_ρ, we find that

$$|s(n_{2\rho})| > \mu|s(n_\rho)|,$$

and so on. Thus, for all positive integers ν,

$$|s(n_{\nu\rho})| \ge \mu^\nu|s(n_0)|. \tag{15}$$

Now it follows from (11) that

$$\frac{n_{(\nu+1)\rho}}{n_{\nu\rho}} \to 1$$

as $\nu \to \infty$, so that

$$\log(n_{\nu\rho}) = o(\nu).$$

Thus, given any $\eta > 0$ we have

$$n_{\nu\rho} \le e^{\eta\nu}$$

for all sufficiently large ν. But the (C, α) boundedness of (1) implies that there is a constant B such that, for all sufficiently large n

$$|s_n| \le Bn^{\alpha}.$$

Thus, for all sufficiently large ν,

$$|s(n_{\nu\rho})| \le Be^{\eta\nu\alpha}.$$

If we choose η so that

$$e^{\eta\alpha} < \mu,$$

this contradicts (15) whenever ν is sufficiently large. The proof of Theorem 1 is thus completed.

4

In order to prove Theorem 2, let (1) be chosen so that, for $|z| < 1$,

$$\Phi(z) = \sum_{n=0}^{\infty} a_n z^n = (1-z)\sum_{n=0}^{\infty} s_n z^n = (1-z)^{1/2} \exp\left(\frac{-2iz}{1-z}\right).$$

It is immediately evident that (1) is Abel summable to 0. By (5.1.9) of [3] we have, with the notation for the Laguerre polynomials used in [3],

$$s_n = L_n^{(-1/2)}(2i).$$

It now follows from (5.6.1) of [3] that

$$s_n = (-1)^n \frac{H_{2n}(1+i)}{2^{2n} n!}. \tag{16}$$

We deduce from (16) with the aid of Theorem 8.22.7 of [3] and Stirling's formula that, for large n,

$$\begin{aligned} s_n &= \frac{(2n)!}{2^{2n}(n!)^2}[\cos(4n+1)^{1/2}(1+i)](\lambda + O(n^{-1/2})) \\ &= n^{-1/2}\exp[(4n+1)^{1/2}(1-i)](\lambda + O(n^{-1/2})), \end{aligned} \tag{17}$$

where λ denotes a non-zero constant (which is different at each occurrence). We deduce from (17) that

$$a_n = O\{n^{-1}\exp(4n+1)^{1/2}\}, \tag{18}$$

and it follows easily from (17) and (18) that (2) and (3) hold. This completes the proof of Theorem 2.

The first part of Theorem 3 is trivial, but it is stated for the sake of completeness. Suppose the hypothesis of the theorem satisfied. Then, as in the case of Theorem 1, it is enough to prove that s_n is bounded. Supposing this false, it follows that (9) holds. But, since a_n is real, (3) implies that s_n is ultimately of constant sign. Hence either $s_n \to +\infty$ as $n \to \infty$ or $s_n \to -\infty$ as $n \to \infty$. This implies that

$$\Phi(x) = \sum_{n=0}^{\infty} a_n x^n$$

tends to either $+\infty$ or $-\infty$ as $x \to 1-$, in contradiction to the assumption that $\Phi(x)$ is bounded.

For the second part of Theorem 3, let δ be a fixed positive number with $e^{\delta} < 2$. Define

$$s_n = (-1)^r e^{\delta r} (r^2 \leq n < (r+1)^2,\ r = 0, 1, 2 \ldots).$$

Thus

$$a_n = \begin{cases} 1 \quad (n = 0); \\ (-1)^r (1 + e^{-\delta}) e^{\delta r} (n = r^2, r = 1, 2, \ldots); \\ 0 \quad (\text{otherwise}) \end{cases}$$

It is clear that (2) holds. Also

$$\Phi(x) = \sum_{n=0}^{\infty} a_n x^n = -e^{-\delta} + (1 + e^{-\delta}) \sum_{r=0}^{\infty} (-1)^r e^{\delta r} x^{r^2}$$

Now if

$$b_n = \begin{cases} (-1)^r e^{-\delta r} & (n = r^2, r = 1, 2, \ldots) \\ 0 & (\text{otherwise}), \end{cases}$$

then $\Sigma_{n=0}^{\infty} b_n$ converges and is thus certainly Abel summable. Hence, as $x \to 1-$,

$$\Phi(x) = (1 + e^{-\delta}) \psi(x) + c + o(1),$$

where c is a constant, and where

$$\psi(x) = \sum_{r=-\infty}^{\infty} (-1)^r e^{\delta r} x^{r^2}. \tag{19}$$

With the notation for the theta functions used, for example, in [4], we can write (19) as

$$\Psi(x) = \vartheta_4(\tfrac{1}{2} i\delta, x).$$

By Jacobi's product for the theta function (see, e.g., [4], §§21.3 and 21.42), we have

$$\Psi(x) = G\prod_{n=1}^{\infty}(1-2x^{2n-1}\cosh\delta + x^{4n-2}),$$

$$G = \prod_{n=1}^{\infty}(1-x^{2n}).$$

Now for $0 < x < 1$,

$$1-2x^{2n-1}\cosh\delta + x^{4n-2} = (1-e^{\delta}x^{2n-1})(1-e^{-\delta}x^{2n-1}).$$

Since $e^{\delta} < 2$, the first factor lies between ± 1; the second clearly lies between 0 and 1. Thus

$$\left|\prod_{n=1}^{\infty}(1-2x^{2n-1}\cosh\delta + x^{4n-2})\right| < 1.$$

Since $G \to 0$ as $x \to 1-$, it is clear that $\psi(x) \to 0$ as $x \to 1-$. Thus (1) is Abel summable. Since (1) is clearly not convergent, the theorem is proved.

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