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H. HERING

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SUBCRITICAL BRANCHING DIFFUSIONS

H. Hering*

With a suitable concept of positive regularity it is possible to prove limit theorems for Markov branching processes with an infinite set of types which are as strong as the best results known for Bienaymé-Galton-Watson processes. For critical and supercritical processes this has been demonstrated – with differing degrees of generality – in [6] and [2], respectively. The present note concerns the subcritical case. We shall restrict ourselves to continuous parameter processes with local branching laws. Our standard example will be branching diffusions on simply connected bounded domains in $\mathbb{R}^n$. Absorption at the boundary will be admitted.

1. Statement of results

Let X be a locally compact Hausdorff space with countable open base, $X \cup \{\partial\}$ the one-point compactification if X is non-compact, $\mathfrak{A}$ the topological Borel algebra on X, and $(\hat{X}, \hat{\mathfrak{A}})$ the corresponding population space, i.e.

$$\hat{X} := \bigoplus_{n=0}^{\infty} X^{(n)},$$

where $X^{(n)}$ is the symmetrization of the n-fold direct product X^n, $n \geq 1$, $X^{(0)} := \{\theta\}$ with some extra point θ, and $\hat{\mathfrak{A}}$ the σ-algebra on $\hat{X}$ induced by $\mathfrak{A}$.

Let $\mathcal{B}$ be the Banach algebra of all bounded, complex-valued, $\mathfrak{A}$-measurable functions ξ with supremum-norm $\|\xi\|$, $\mathcal{B}_+$ the non-negative cone in $\mathcal{B}$, $\mathcal{C}^0$ the algebra of all continuous $\xi \in \mathcal{B}$, $\mathcal{C}_0 := \mathcal{C}^0$ if

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X is compact, $\mathscr{C}_0 := \{\xi \in \mathscr{C}^0 : \xi(x) \to 0 \text{ as } x \to \partial\}$ if X is non-compact, further $\bar{\mathscr{S}} := \{\xi \in \mathscr{B} : \|\xi\| \le 1\}$ and $\bar{\mathscr{S}}_+ := \bar{\mathscr{S}} \cap \mathscr{B}_+$. Define $\mathbf{1}(x) := 1$ and $\mathbf{0}(x) := 0$ for $x \in X$, and let 1_A, $1_{\hat{A}}$ be the indicator functions of $A \subset X$, $\hat{A} \subset \hat{X}$, respectively. Finally, denote

$$\begin{aligned} \hat{x}[\xi] &:= 0; \qquad \hat{x} = \theta, \\ &:= \sum_{\nu=1}^{n} \xi(x_\nu); \quad \hat{x} = \langle x_1, \ldots, x_n \rangle \in X^{(n)}, n > 0, \end{aligned}$$

for $\xi \in \mathscr{B}$.

Suppose to be given

(a) *a contraction semigroup of non-negative linear operators* $\{T_t\}_{t \ge 0}$ *on* $\mathscr{B}$, *strongly continuous on* $\mathscr{C}_0$ *with* $T_t\mathscr{C}_0 \subseteq \mathscr{C}_0$, $t \ge 0$,

(b) *a termination density* $k \in \mathscr{B}_+$ *and a branching law of the form*

$$\pi(x, \hat{A}) = p_0(x) 1_{\hat{A}}(\theta) + \sum_{n>1} p_n(x) 1_{\hat{A}}(\langle \overbrace{x, \ldots, x}^{n} \rangle); \quad x \in X, \hat{A} \in \hat{\mathfrak{A}},$$

$$p_n \in \bar{\mathscr{S}}_+, \quad \sum_{n \ne 1} p_n = \mathbf{1},$$

with

$$m := \sum_{n>1} n p_n \in \mathscr{B}_+.$$

Let $\{\hat{x}_t, \boldsymbol{P}^{\hat{x}}\}$ be the Markov branching process determined by $[T_t, k, \pi]$, see [8], [11], and denote by $\boldsymbol{E}^{\hat{x}}$ the expectation with respect to $\boldsymbol{P}^{\hat{x}}$. The assumptions guarantee that $\{\boldsymbol{E}^{(\cdot)}\hat{x}_t[\cdot]\}_{t \ge 0}$ exists as a semigroup of linear-bounded operators on $\mathscr{B}$. In fact, we assume throughout that $\{\hat{x}_t, \boldsymbol{P}^{\hat{x}}\}$ is positively regular in the sense that the following condition is satisfied:

(M) *The first moment semigroup can be represented in the form*

$$\boldsymbol{E}^{(x)}\hat{x}_t[\xi] = \rho^t \varphi^*[\xi]\varphi(x) + Q_t[\xi](x); \quad x \in X, t \ge 0, \xi \in \mathscr{B}$$

where $\rho \in (0, \infty)$, φ^* *is a non-negative linear-bounded functional on* $\mathscr{B}$, $\varphi \in \mathscr{B}_+$, *and* Q_t: $\mathscr{B} \to \mathscr{B}$ *such that*

$$\begin{gathered} \varphi^*[\varphi] = 1, \\ \varphi^*[Q_t[\cdot]] \equiv 0, \quad Q_t[\varphi] \equiv \mathbf{0}; \quad t > 0, \\ |Q_t[\xi](x)| \le \alpha_t \varphi^*[\xi]\varphi(x); \quad x \in X, \quad \xi \in \mathscr{B}_+, t > 0, \end{gathered}$$

with some $\alpha.$: $T \to [0, \infty)$ *satisfying*

$$\rho^{-t}\alpha_t \downarrow 0; \quad t \uparrow \infty.$$

EXAMPLE: Let Ω be a simply connected bounded open set in $\mathbb{R}^n$ with sufficiently smooth boundary – see below –, let $\bar{\Omega}$ be the closure of Ω, and $\mathscr{C}^q(\bar{\Omega})$, $\mathscr{C}^q(\bar{\Omega}\backslash\Omega)$ the set of all q times continuously differentiable functions on $\bar{\Omega}$, $\bar{\Omega}\backslash\Omega$, respectively. Suppose $\Omega \subset X \subset \bar{\Omega}$, and let $\{T_t\}$ be the transition semigroup of a diffusion process with formally selfadjoint differential generator A defined on

$$\mathscr{D} := \left\{\xi|_X : \xi \in \mathscr{C}^2(\bar{\Omega}) \wedge \left(a\xi + b\frac{\partial}{\partial n}\xi\right)\Big|_{\bar{\Omega}\backslash\Omega} = 0\right\},$$

where $a \geq 0$, $b \geq 0$, $a + b > 0$, $\bar{\Omega}\backslash X = \{b = 0\}$, and $\partial/\partial n$ means differentiation in the direction of the exterior normal. If Ω is of class $\mathfrak{C}^{2q}$, $q = 2[(n+2)/4] + 1$, if the coefficients of A as well as the functions k and m are in $\mathscr{C}^1(\bar{\Omega}) \cap \mathscr{C}^{2(q-1)}(\bar{\Omega})$, and if $a, b \in \mathscr{C}^{2q-1}(\bar{\Omega}\backslash\Omega)$, then (M) is satisfied with $\varphi(x) > 0$ for $x \in X$. If X is non-compact, i.e., if $\{b = 0\} \neq \phi$, then $\varphi(x) \to 0$ as $x \to \partial$. A proof for general n is contained in [7], for the one-dimensional case see [2], [6].

Let $F_t[\cdot|\hat{x}]$ be the generating functional of $\boldsymbol{P}^{\hat{x}}(\hat{x}_t \in \cdot)$, cf. [5], and define $F_t[\cdot]: \bar{\mathscr{S}} \to \bar{\mathscr{S}}$ by

$$F_t[\cdot](x) := F_t[\cdot|\langle x\rangle]; \quad x \in X.$$

THEOREM 1: *Suppose $\rho < 1$. Then there exists a non-negative non-increasing bounded functional γ on $\bar{\mathscr{S}}_+$ such that*

$$\lim_{t\to\infty} \|\rho^{-t}(\mathbf{1} - F_t[\xi]) - \gamma[\xi]\varphi\| = 0; \quad \xi \in \bar{\mathscr{S}}_+. \tag{1.1}$$

Given $\xi \in \bar{\mathscr{S}}_+ \cap \{\varphi^*[\mathbf{1} - \xi] > 0\}$, we have $\gamma[\xi] > 0$ if and only if

$$\varphi^*\left[k\varphi \sum_{n>1} p_n n \log n\right] < \infty. \tag{1.2}$$

A proof of this result is to be found in §3.

REMARK: Relation (1.2) is also a necessary and sufficient condition for the non-degeneracy of $W := \lim_{t\to\infty} \rho^{-t}\hat{x}_t[\varphi]$ a.s., in case $\rho > 1$. Furthermore, it has been shown in this context that (1.2) is equivalent to

$$\varphi^*[\boldsymbol{E}^{\langle\cdot\rangle}\hat{x}_t[\varphi] \log \hat{x}_t[\varphi]] < \infty$$

for any $t > 0$, regardless of the value of ρ. For details see [2].

By the branching property, i.e., by

$$F_t[\cdot|\langle x_1,\dots,x_n\rangle]=\prod_{\nu=1}^{n}F_t[\cdot|\langle x_\nu\rangle],$$

it follows from (1.1) that

$$\begin{aligned}\lim_{t\to\infty}&\rho^{-t}(1-F_t[\xi|\hat{x}])\\&=\lim_{t\to\infty}\rho^{-t}\left(1-\prod_{\nu=1}^{n}(1-(1-F_t[\xi|\langle x_\nu\rangle]))\right)\\&=\lim_{t\to\infty}\rho^{-t}\sum_{\nu=1}^{n}(1-F_t[\xi|\langle x_\nu\rangle])=\gamma[\xi]\hat{x}[\varphi]\end{aligned}$$

uniformly in $\hat{x}=\langle x_1,\dots,x_n\rangle\in X^{(n)}$ for every $n>0$.

Corollary 1: *If $\rho<1$, then*

$$\lim_{t\to\infty}\rho^{-t}\boldsymbol{P}^{\hat{x}}(\hat{x}_t\neq\theta)=\gamma[\mathbf{0}]\hat{x}[\varphi] \tag{1.3}$$

uniformly in $\hat{x}\in X^{(n)}$ for every $n>0$.

Proof: Notice that $\boldsymbol{P}^{\hat{x}}(\hat{x}_t\neq\theta)=1-F_t[\mathbf{0}|\hat{x}]$.

If $\gamma[\mathbf{0}]>0$, a limit result of Yaglom type is a corollary of (1.1). However, in order to cover the case $\gamma[\mathbf{0}]=0$, we have to assume some additional well-behaviour. One way of expressing it is the following condition:

(C) *There exists a compactification $\bar{X}$ of X such that for every $t>0$ and $\xi\in\bar{\mathcal{S}}_+$ the function $(\mathbf{1}-F_t[\xi])/\varphi$ possesses a continuous extension defined on $\bar{X}$.*

Notice that $1-F_t[\xi|\langle x\rangle]\equiv 0$ if $\varphi(x)=0$, see §2. Hence it can be assumed w.l.o.g. that $\varphi(x)>0\ \forall\, x\in X$.

Example: In our diffusion example (C) is satisfied, see section 2.3 below.

Theorem 2: *Suppose $\rho<1$ and $\varphi(x)>0\ \forall\, x\in X$. If $\gamma[\mathbf{0}]=0$, assume in addition that* (C) *is satisfied. Then there exists a proper generating functional G on $\bar{\mathcal{S}}$ such that*

$$\lim_{t\to\infty}\left\|\frac{F_t[\xi]-F_t[\mathbf{0}]}{\mathbf{1}-F_t[\mathbf{0}]}-G[\xi]\mathbf{1}\right\|=0;\quad \xi\in\bar{\mathcal{S}}_+. \tag{1.4}$$

If $\gamma[\mathbf{0}]>0$, *G has the first moment functional*

$$M[\xi]:=\varphi^*[\xi]/\gamma[\mathbf{0}];\quad \xi\in\mathscr{B}. \tag{1.5}$$

If $\gamma[\mathbf{0}]=0$, *G does not have a bounded first moment functional.*

This theorem will be proved in §4. Notice that we have admitted the case that $\inf\varphi=0$.

Again using the branching property, we deduce from (1.4) that

$$\lim_{t\to\infty}\frac{F_t[\xi|\hat{x}]-F_t[\mathbf{0}|\hat{x}]}{1-F_t[\mathbf{0}|\hat{x}]}=1-\lim_{t\to\infty}\frac{1-F_t[\xi|\hat{x}]}{1-F_t[\mathbf{0}|\hat{x}]}$$

$$=1-\lim_{t\to\infty}\frac{\sum_{\nu=1}^{n}(1-F_t[\mathbf{0}|\langle x_\nu\rangle])\dfrac{1-F_t[\xi|\langle x_\nu\rangle]}{1-F_t[\mathbf{0}|\langle x_\nu\rangle]}}{\sum_{\nu=1}^{n}(1-F_t[\mathbf{0}|\langle x_\nu\rangle])}=G[\xi]$$

uniformly in $\hat{x}=\langle x_1,\ldots,x_n\rangle\in X^{(n)}$ for every $n>0$.

COROLLARY 2: *Under the assumptions of Theorem* 2 *there exists a probability measure* $\boldsymbol{P}$ *on* $\mathfrak{A}$ *such that*

$$\lim_{t\to\infty}\boldsymbol{P}^{\hat{y}}(\hat{x}_t[1_{A_\nu}]=j_\nu;\quad \nu=1,\ldots,j|\hat{x}_t\neq\theta)$$
$$=\boldsymbol{P}(\hat{x}[1_{A_\nu}]=j_\nu;\quad \nu=1,\ldots,j) \tag{1.6}$$

for every decomposition $\{A_1,\ldots,A_j\}$ *of* X, $A_\nu\in\mathfrak{A}$, $\nu=1,\ldots,j$, $j>0$, *and uniformly in* $\hat{y}\in X^{(n)}$ *for every* $n>0$.

PROOF: Note that $(1-F_t[\mathbf{0}|\hat{y}])^{-1}(F_t[\Sigma_{\nu=1}^{j}1_{A_\nu}s_\nu|\hat{y}]-F_t[\mathbf{0}|\hat{y}])$ and $G[\Sigma_{\nu=1}^{j}1_{A_\nu}s_\nu]$; $|s_\nu|\leq 1$, $\nu=1,\ldots,j$, are ordinary j-dimensional generating functions, and use the standard argument for these. Of course, $\boldsymbol{P}$ is the probability measure generated by G.

2. Some auxiliary facts

2.1. Define

$$f(s):=\sum_{n\neq 1}p_ns^n;\quad |s|\leq 1.$$

Since $m\in\mathscr{B}$, there exists a mapping $r(\cdot)$: $\{|s|\leq 1\}\to\mathscr{B}$, $r(1)=0$, $r(s)(x)$

continuous in s for every x, such that

$$0 \leq r(s') \leq r(s) \leq m; \quad 0 \leq s \leq s' \leq 1, \tag{2.1}$$

$$1 - f(s) = (1-s)m - (1-s)r(s); \quad |s| \leq 1. \tag{2.2}$$

Similarly, given $\boldsymbol{E}^{\langle \cdot \rangle}\hat{x}_t[\mathbf{1}] \in \mathcal{B}$, there exists a mapping $R^t(\cdot)[\cdot]$: $\bar{\mathcal{S}} \otimes \mathcal{B} \to \mathcal{B}$, $R^t(\mathbf{1})[\cdot] \equiv 0$, which is sequentially continuous with respect to the product topology on bounded regions in $\bar{\mathcal{S}} \otimes \mathcal{B}$, linear-bounded with respect to the second variable, and satisfies

$$0 \leq R^t(\xi')[\zeta] \leq R^t(\xi)[\zeta'] \leq \boldsymbol{E}^{\langle \cdot \rangle}\hat{x}_t[\zeta'], \quad \mathbf{0} \leq \xi \leq \xi' \leq \mathbf{1}, \quad \mathbf{0} \leq \zeta \leq \zeta', \tag{2.3}$$

$$\mathbf{1} - F_t[\xi] = \boldsymbol{E}^{\langle \cdot \rangle}\hat{x}_t[\mathbf{1} - \xi] - R^t(\xi)[\mathbf{1} - \xi]; \quad \xi \in \bar{\mathcal{S}}. \tag{2.4}$$

See [10], [5].

2.2. Let $\{x_t, P^x\}$ be the Markov process determined up to equivalence by $\{T_t\}$. This process is defined as a conservative process either on X, or on $X \cup \{\partial^*\}$, where ∂^* is an extra point serving as trap. If X is non-compact, take $\partial^* = \partial$. Denote by E^x the expectation with respect to P^x. With $[T_t, k, \pi]$ as defined, $\xi \in \bar{\mathcal{S}}$, $\zeta(\partial^*) := k(\partial^*) := 0$, and

$$T_t^0\zeta(x) := E^x\left(\zeta(x_t) \exp\left\{-\int_0^t k(x_s)ds\right\}\right); \quad \zeta \in \mathcal{B}, x \in X,$$

the function $F_t[\xi](x)$; $t \geq 0$, $x \in X$, is the unique solution of

$$u_t(x) = T_t^0\xi(x) + H_t(x) + \int_0^t T_s^0\{kf(u_{t-s})\}(x)ds, \tag{2.5}$$

where

$$H_t(x) := 1 - T_t^0\mathbf{1}(x) - \int_0^t T_s^0 k(x)ds.$$

Recall that $m \in \mathcal{B}$; thus for any $\xi \in \mathcal{B}$ the function $\boldsymbol{E}^{\langle x \rangle}\hat{x}_t[\xi]$; $t \leq 0$, $x \in X$, is the unique solution of

$$v_t(x) = T_t^0\xi(x) + \int_0^t T_s^0\{kmv_{t-s}\}(x)ds. \tag{2.6}$$

See [8], [11].

As an immediate consequence of (2.6)

$$T_t^0\xi \le \boldsymbol{E}^{(\cdot)}\hat{x}_t[\xi]; \quad t \ge 0, \xi \in \mathscr{B}_+, \tag{2.7}$$

and, given (M),

$$\varphi^*[T_t^0\xi] \le \rho^t\varphi^*[\xi]; \quad t \ge 0, \xi \in \mathscr{B}_+. \tag{2.8}$$

Moreover, using (2.6), (2.8), and (M),

$$\varphi^*[T_t^0\xi] \ge \rho^t(1-\|km\|t)\varphi^*[\xi]; \quad t \ge 0, \xi \in \mathscr{B}_+. \tag{2.9}$$

2.3. To verify (C) in the diffusion example, let $\bar{X}$ be the closure of X. The assumptions guarantee that T_t^0 has the kernel

$$U(t,x,y) = \sum_{\nu=1}^{\infty} e^{-\lambda_\nu t}\varphi_\nu(x)\varphi_\nu(y); \quad t \ge 0, x, y \in X, \tag{2.10}$$

where φ_ν is the regular eigenfunction of $A - k$ with eigenvalue $-\lambda_\nu$; $\nu = 1, 2, \ldots$, and

$$\sum_{\nu:\, \lambda_\nu \le \lambda} 1 \sim c\lambda^{n/2}; \quad \lambda \to \infty, \tag{2.11}$$

for some $c > 0$, cf. [1; Theorem 14.6]. Moreover, it can be deduced from Sobolev's lemma that

$$\sup_X \left(|\varphi_\nu| + \sum_{j=1}^{n} \left|\frac{\partial \varphi_\nu}{\partial x_j}\right|\right) = O(1+|\lambda_\nu|^\kappa); \quad \nu = 1, 2, \ldots \tag{2.12}$$

with some $\kappa > 0$, cf. [7].

$$\mathbf{1} - F_t[\xi] = T_t^0(\mathbf{1}-\xi) + \int_0^t T_s^0\{k(\mathbf{1} - f(F_{t-s}[\xi]))\}ds,$$

and it follows by (2.10)–(2.12) that $\mathbf{1} - F_t[\xi]$ has continuous extension on $\bar{X}$ for $t > 0$, $\xi \in \bar{\mathscr{S}}$. If $X = \bar{X}$, we are done, since φ is continuous and positive on X, cf. [7]. If $X \neq \bar{X}$, we know that φ is positive on X and that the first derivatives of φ possess continuous extensions on $\bar{X}$ such that $\partial\varphi/\partial n < 0$ on $\bar{X}\backslash X$, cf. [7]. By (2.10)–(2.12), the first derivatives of $T_t^0(\mathbf{1}-\xi)$ and $\int_\delta^t T_s^0\{k(\mathbf{1}-f(F_{t-s}[\xi]))\}ds$ also have continuous extensions on $\bar{X}$ for $t > 0$, $\delta > 0$, $\xi \in \bar{\mathscr{S}}$. Hence, it suffices to secure

$$\sup_{x\in X}\int_X \sum_{j=1}^{n}\left|\frac{\partial}{\partial x_j}U(s,x,y)\right| dy = O(s^{-1/2}); \quad s>0.$$

But this is obtained in exactly the same way as the estimate (3.43) in [9; p. 65].

3. Proof of Theorem 1

3.1. By (2.4), (2.3), and (M) with $\rho<1$

$$\begin{aligned}(3.1)\qquad \|\mathbf{1}-F_t[\xi]\| &\le \|\mathbf{1}-F_t[\mathbf{0}]\| + \|F_t[|\xi|]-F_t[\mathbf{0}]\| \\ &\le 2\|\mathbf{1}-F_t[\mathbf{0}]\| \le \rho^t(1+\rho^{-t}\alpha_t)\varphi^*[\mathbf{1}]\|\varphi\|\to 0; \quad t\to\infty,\end{aligned}$$

uniformly in $\xi\in\bar{\mathcal{S}}$.

LEMMA 1: *For every $t>0$ there exists a mapping $g_t[\cdot]\colon \bar{\mathcal{S}}_+\to\mathcal{B}$ such that*

$$(3.2)\qquad R^t(\xi)[\mathbf{1}-\xi] = g_t[\xi]\rho^t\varphi^*[\mathbf{1}-\xi]\varphi,$$

$$\lim_{\|\mathbf{1}-\xi\|\to 0}\|g_t[\xi]\| = 0,$$

where the convergence is uniform with respect to t in every closed interval $[a,b]$; $a>0$.

PROOF: It follows from (2.2), (2.4), (2.5), and (2.6) that for every $\epsilon>0$ and $\xi\in\bar{\mathcal{S}}_+$ the function $R^t(\xi)[\mathbf{1}-\xi](x)$; $t\ge\epsilon$, $x\in X$, solves

$$(3.3)\qquad w_t(x) = A_t(x) + B_t^\epsilon(x) + \int_0^{t-\epsilon} T_s^0\{kmw_{t-s}\}(x)ds,$$

$$A_t(x) := \int_0^t T_s^0\{kr(F_{t-s}[\xi])(\mathbf{1}-F_{t-s}[\xi])\}(x)ds,^1$$

$$B_t^\epsilon(x) := \int_0^\epsilon T_{t-s}^0\{kmR^s(\xi)[\mathbf{1}-\xi]\}(x)ds.$$

As is readily verified, it is the only solution bounded on $[\epsilon,b]\otimes X$

[1]For $\xi\in\bar{\mathcal{S}}$ we define $r(\xi)(x) := r(\xi(x))(x)$; $x\in X$.

and therefore equals the limit of the (non-decreasing) iteration sequence $\{w_t^\nu(x)\}_{\nu\geq 0}$, $w_t^0 \equiv 0$, which we now estimate.

Take $0<\delta<\epsilon/2$, and write

$$A_t(x) = \int_0^\delta + \int_\delta^{t-\delta} + \int_{t-\delta}^t (\ldots)ds$$

with the same integrand as above. Using (2.1), (2.4), (2.3), (2.7), and (M) in this order, we get

$$\begin{aligned}\int_0^\delta + \int_{t-\delta}^t (\ldots)ds &\leq 2\delta \|km\| \boldsymbol{E}^{\langle x\rangle}\hat{x}_t[\mathbf{1}-\xi] \\ &\leq 2\delta\|km\|(1+\rho^{-\epsilon}\alpha_\epsilon)\rho^t\varphi^*[\mathbf{1}-\xi]\varphi; \quad t\geq\epsilon.\end{aligned}$$

By (2.4) and (2.6), or (M), there is a constant $c_1<\infty$ such that

$$F_{t-s}[\xi] \geq \mathbf{1} - c_1\|\mathbf{1}-\xi\|\mathbf{1}; \quad \delta\leq s\leq t-\delta, t\leq b.$$

Thus

$$\begin{aligned}\int_\delta^{t-\delta}(\ldots)ds &\leq \int_\delta^{t-\delta} \boldsymbol{E}^{\langle x\rangle}\hat{x}_s[kr(1-c_1\|\mathbf{1}-\xi\|)\boldsymbol{E}^{\langle\cdot\rangle}\hat{x}_{t-s}[\mathbf{1}-\xi]]ds \\ &\leq (t-2\delta)(1+\rho^{-\delta}\alpha_\delta)(1+\rho^{\delta-\epsilon}\alpha_{\epsilon-\delta})\|k\varphi\| \\ &\quad\times\varphi^*[r(1-c_1\|\mathbf{1}-\xi\|)]\rho^t\varphi^*[\mathbf{1}-\xi]\varphi(x).\end{aligned}$$

Since (M) implies that $\varphi^*[1_A]\|\mathfrak{A}$ is a measure, and since $r(s)(x)\to 0$; $s\to 1$, $r(\cdot)\leq m$, we have

$$\lim_{\|\mathbf{1}-\xi\|\to 0}\varphi^*[r(1-c_1\|\mathbf{1}-\xi\|)] = 0.$$

Hence, for every $\epsilon'>0$ we can first fix a $\delta>0$ such that $\int_0^\delta + \int_{t-\delta}^t(\ldots)ds \leq t\epsilon'\varphi^*[\mathbf{1}-\xi]\varphi/2$ for all $\xi\in\bar{\mathcal{S}}_+$ and then choose a $\delta'>0$ such that $\int_\delta^{t-\delta}(\ldots)ds\leq t\epsilon'\varphi^*[\mathbf{1}-\xi]\varphi/2$, whenever $\|\mathbf{1}-\xi\|<\delta'$. That is,

(3.4) $$A_t \leq t\Delta_\epsilon[\xi]\rho^t\varphi^*[\mathbf{1}-\xi]\varphi; \quad t\geq\epsilon,$$

(3.5) $$\lim_{\|\mathbf{1}-\xi\|\to 0}\Delta_\epsilon[\xi] = 0; \quad \epsilon>0.$$

Using (2.3) and (2.7),

(3.6) $$B_t^\epsilon \leq \epsilon\|km\|\boldsymbol{E}^{\langle\cdot\rangle}\hat{x}_t[\mathbf{1}-\xi]; \quad t\geq\epsilon.$$

From (3.3), (3.4), and (3.6) it follows by induction and use of (2.7), (M), that

$$\lim_{\nu\to\infty} w_t^\nu(x) \le e^{\|km\|t}\{t\Delta_\epsilon[\xi]\rho^t\varphi^*[\mathbf{1}-\xi]\varphi(x) + \epsilon\|km\|\boldsymbol{E}^{\langle x\rangle}\hat{x}_t[\mathbf{1}-\xi]\}; \quad t \ge \epsilon.$$

In view of (3.5), (M), and the fact that ϵ can be chosen arbitrarily small, this proves the lemma. □

LEMMA 2: *Given that* $\|\mathbf{1} - F_t[\mathbf{0}]\| \to 0$ *as* $t \to \infty$, *there exists for every* $t > 0$ *a mapping* $h_t\colon \bar{\mathcal{S}}_+ \to \mathcal{B}$ *such that*

$$(3.7)\qquad \begin{aligned} &\mathbf{1} - F_t[\xi] = (\mathbf{1} + h_t[\xi])\varphi^*[\mathbf{1} - F_t[\xi]]\varphi; \quad t > 0, \xi \in \bar{\mathcal{S}}_+,\\ &\lim_{t\to\infty} \|h_t[\xi]\| = 0 \text{ uniformly in } \xi \in \bar{\mathcal{S}}_+, \end{aligned}$$

where $\varphi^*[\mathbf{1} - F_t[\xi]] > 0$ *for all* $t > 0$ *and* $\xi \in \bar{\mathcal{S}}_+ \cap \{\varphi^*[\mathbf{1} - \xi] > 0\}$.

PROOF: If $\varphi^*[\mathbf{1} - \xi] = 0$, then by (2.4), (2.3), and (M) also $\varphi^*[\mathbf{1} - F_t[\xi]] = 0$; $t > 0$, and we may take $h_t[\xi] \equiv \mathbf{0}$. Now let $\varphi^*[\mathbf{1} - \xi] > 0$ and $0 < \epsilon < 1/\|km\|$. Then by (2.5) and (2.9) there is a constant $c_2 > 0$ such that

$$\begin{aligned}\varphi^*[\mathbf{1} - F_t[\xi]] &= \varphi^*[T_t^0(\mathbf{1} - \xi)] + \varphi^*\Big[\int_0^t T_s^0\{k(\mathbf{1} - f(F_{t-s}[\xi]))\}ds\Big]\\ &\ge \varphi^*[T_t^0(\mathbf{1} - \xi)] \ge c_2\varphi^*[\mathbf{1} - \xi] > 0; \quad 0 \le t \le \epsilon.\end{aligned}$$

Combining the branching relation with the Chapman-Kolmogorov equation, we get

$$(3.8)\qquad F_{t+s}[\xi] = F_t[F_s[\xi]]; \quad t, s \ge 0, \xi \in \bar{\mathcal{S}}.$$

Hence, by induction $\varphi^*[\mathbf{1} - F_t[\xi]] > 0$; $t \ge 0$.

The remaining argument is essentially the same as in [5] and [6][1]: From (3.8)

$$\mathbf{1} - F_t[\xi] = \boldsymbol{E}^{\langle\cdot\rangle}\hat{x}_s[\mathbf{1} - F_{t-s}[\xi]] - R^s(F_{t-s}[\xi])[\mathbf{1} - F_{t-s}[\xi]]; \quad t > s > 0, \xi \in \bar{\mathcal{S}},$$

[1] In the middle of the last formula of [6; Proof of Lemma 2] a "$-\varphi_1$" is missing.

and from this by (M) and Lemma 1

$$(1-\rho^{-s}\alpha_s - \|g_s[F_{t-s}[\xi]]\|)\rho^s\varphi^*[\mathbf{1}-F_{t-s}[\xi]]\varphi$$
$$\leq \mathbf{1}-F_t[\xi] \leq (1+\rho^{-s}\alpha_s)\rho^s\varphi^*[\mathbf{1}-F_{t-s}[\xi]]\varphi; \quad t>s>0, \xi\in\bar{\mathcal{S}}_+.$$

Combining these two inequalities with those obtained by applying φ^* to them, we get

$$-\frac{2\rho^{-s}\alpha_s + \|g_s[F_{t-s}[\xi]]\|}{1+\rho^{-s}\alpha_s}\varphi \leq \frac{\mathbf{1}-F_t[\xi]}{\varphi^*[\mathbf{1}-F_t[\xi]]} - \varphi$$
$$\leq \frac{2\rho^{-s}\alpha_s + \|g_s[F_{t-s}[\xi]]\|}{1-\rho^{-s}\alpha_s - \|g_s[F_{t-s}[\xi]]\|}\varphi;$$
$$t\geq t^*(s), s\geq s^*, \xi\in\bar{\mathcal{S}}_+\cap\{\varphi^*[\mathbf{1}-\xi]>0\},$$

for some $t^*(s)<\infty$, $s^*<\infty$. By (M), (3.1), (3.2) this proves the lemma. □

By (3.8), (M), and (2.4)

(3.9) $$\rho^{-t-s}\varphi^*[\mathbf{1}-F_{t+s}[\xi]]$$
$$=\rho^{-t}\varphi^*[\mathbf{1}-F_t[\xi]] - \rho^{-t-s}\varphi^*[R^s(F_t[\xi])[\mathbf{1}-F_t[\xi]]]\geq 0, \quad \xi\in\bar{\mathcal{S}}_+.$$

Hence, recalling (2.3), there exists a non-negative, non-increasing bounded functional γ on $\bar{\mathcal{S}}_+$ such that

$$\rho^{-t}\varphi^*[\mathbf{1}-F_t[\xi]]\downarrow\gamma[\xi]; \quad t\uparrow\infty, \xi\in\bar{\mathcal{S}}_+.$$

By Lemma 2 this implies (1.1) for $\xi\in\bar{\mathcal{S}}_+\cap\{\varphi^*[\mathbf{1}-\xi]>0\}$. The case $\varphi^*[\mathbf{1}-\xi]=0$ is trivial, since $\mathbf{1}-F_t[\xi]=\mathbf{0}$; $t>0$, if $\varphi^*[\mathbf{1}-\xi]=0$, by (2.4), (2.3), and (M).

3.2. Fix $\xi\in\bar{\mathcal{S}}_+\cap\{\varphi^*[\mathbf{1}-\xi]>0\}$. By Lemma 2 and (3.9)

$$\rho^{-n}\varphi^*[\mathbf{1}-F_n[\xi]]$$
$$=\rho^{-1}\varphi^*[\mathbf{1}-F_1[\xi]]\prod_{\nu=1}^{n-1}\{1-\rho^{-1}\varphi^*[R^1(F_\nu[\xi])[(\mathbf{1}+h_\nu[\xi])\varphi]]\};$$

where $\varphi^*[\mathbf{1}-F_1[\xi]]>0$. Thus $\gamma[\xi]>0$ if and only if

$$\sum_{\nu=1}^{\infty}\varphi^*[R^1(F_\nu[\xi])[(\mathbf{1}+h_\nu[\xi])\varphi]]<\infty.$$

If $\gamma[\xi] > 0$, there exists by (1.1) a positive real $\epsilon < 1/\|\varphi\|$ such that $\mathbf{1} - F_\nu[\xi] \geq \epsilon\varphi\rho^\nu$ for all sufficiently large ν and thus by (2.3)

$$\sum_{\nu=1}^{\infty} \varphi^*[R^1(\mathbf{1} - \epsilon\varphi\rho^\nu)[\varphi]] < \infty. \tag{3.10}$$

On the other hand, if $\gamma[\xi] = 0$, there is for every $\epsilon > 0$ a ν_0 such that $\mathbf{1} - F_\nu[\xi] \leq \epsilon\varphi\rho^\nu$ for all $\nu \geq \nu_0$, and (3.10) cannot hold. Hence $\gamma[\xi] > 0$ if and only if (3.10) is satisfied for some $\epsilon < 1/\|\varphi\|$. This is the same argument as for processes with a finite set of types, [10]. Two additional steps now lead to (1.2).

LEMMA 3: *Given $\rho < 1$, (3.10) is satisfied for some positive $\epsilon < 1/\|\varphi\|$ if and only if*

$$\varphi^*\left[k \sum_{\nu=1}^{\infty} r(\mathbf{1} - \epsilon'\varphi\rho^\nu)\varphi\right] < \infty \tag{3.11}$$

for some positive $\epsilon' < 1/\|\varphi\|$.

PROOF: For any integer n the function $\Sigma_{\nu=1}^{n} R^t[\mathbf{1} - \epsilon\varphi\rho^\nu)[\varphi](x)$; $t \geq 0$, $x \in X$, solves

$$\begin{aligned} w_{n,t}(x) &= w_{n,t}^1(x) + \int_0^t T_s^0\{kmw_{n,t-s}\}(x)ds, \\ w_{n,t}^1(x) &= \int_0^t T_s^0\left\{k \sum_{\nu=1}^{n} r(F_{t-s}[\mathbf{1} - \epsilon\varphi\rho^\nu]) \frac{\mathbf{1} - F_{t-s}[\mathbf{1} - \epsilon\varphi\rho^\nu]}{\epsilon\rho^\nu}\right\}(x)ds, \end{aligned} \tag{3.12}$$

cf. (3.3). Moreover, it is the only solution bounded on $[0, \Lambda] \otimes X$ for every $\Lambda > 0$. Thus

$$w_{n,t}(x) = \lim_{\nu\to\infty} w_{n,t}^\nu(x); \quad t > 0, x \in X, \tag{3.13}$$

where

$$\mathbf{0} \equiv w_{n,t}^0 \leq w_{n,t}^1 \leq w_{n,t}^2 \leq \cdots \tag{3.14}$$

is the iteration sequence of (3.12).

Using (3.12), (3.13), (3.14), (2.7), and (M) with $\rho < 1$,

$$\varphi^*[w_{n,t}^1] \leq \varphi^*[w_{n,t}] \leq e^{\|km\|t} \sup_{0\leq s\leq t} \varphi^*[w_{n,s}^1]; \quad t > 0. \tag{3.15}$$

Appealing to (2.4), Lemma 1, and (2.3),

$$\epsilon(\mathbf{1}-g_s[\mathbf{1}-\epsilon\varphi\rho^\nu])\rho^{\nu+s}\varphi \le \mathbf{1}-F_s[\mathbf{1}-\epsilon\varphi\rho^\nu] \le \epsilon\varphi\rho^{\nu+s}; \quad s>0, \nu=1,2,\dots \tag{3.16}$$

Now choose a real $\tau > 0$ such that $0 < 1-\tau < 1/\|km\|$, and define

$$\Gamma_\nu := \sup_{\tau\le s\le 1} \|g_s[\mathbf{1}-\epsilon\varphi\rho^\nu]\|.$$

By Lemma 1 there is for every positive $\delta < 1$ an integer μ such that $\Gamma_\nu \le \delta$; $\nu \le \mu$. By (3.16) and (2.9) there then exists a real $c_3 > 0$ for which

$$\begin{aligned} \varphi^*[w^1_{n,1}] &\ge \varphi^*\left[\int_0^{1-\tau} T^0_s\left\{k\sum_{\nu=\mu}^n r(F_{1-s}[\mathbf{1}-\epsilon\varphi\rho^\nu])\frac{\mathbf{1}-F_{1-s}[\mathbf{1}-\epsilon\varphi\rho^\nu]}{\epsilon\rho^\nu}\right\}ds\right] \\ &\le c_3(1-\delta)\varphi^*\left[k\sum_{\nu=\mu}^n r(\mathbf{1}-\epsilon(1-\delta)\varphi\rho^{\nu+1})\varphi\right]; \quad n\ge\mu. \end{aligned} \tag{3.17}$$

Conversely, by (2.7), (M), and (3.16)

$$\varphi^*[w^1_{n,s}] \le s\varphi^*\left[k\sum_{\nu=1}^n r(\mathbf{1}-\epsilon\varphi\rho^\nu)\varphi\right]; \quad s\ge 0. \tag{3.18}$$

Since $\varphi^*[1_A]|\mathfrak{A}$ is a measure and $r(\cdot)\le m$, (3.15), (3.17), and (3.18) prove the lemma. □

LEMMA 4: *Given $\rho < 1$, (3.11) holds for some $\epsilon' < 1/\|\varphi\|$ if and only if*

$$\varphi^*\left[k\varphi\sum_{n=1}^\infty p_n n \log n\right] < \infty. \tag{3.19}$$

PROOF: The argument involves a trick that has already been used in the different context of [3]. In view of (2.1),

$$\int_0^\infty r(\mathbf{1}-\epsilon'\varphi\rho)dt - m \le \sum_{\nu=1}^\infty r(\mathbf{1}-\epsilon'\varphi\rho^\nu) \le \int_0^\infty r(\mathbf{1}-\epsilon'\varphi\rho^t)dt$$

pointwise on X. Fix $x\in X$ and change variables according to

$1-\epsilon'\varphi(x)\rho^t =: e^{-s}$. Then by (2.2) and the definition of f

$$\int_0^\infty r(1-\epsilon'\varphi\rho^t)dt = \frac{1}{|\log \rho|}\int_0^{|\log(1-\epsilon'\varphi)|}\Big\{\sum_{n\neq 1} p_n s^{-2}(e^{-ns}-1+ns) + ms^{-2}(1-e^{-s}-s)\Big\} s^2(1-e^{-s})^{-2}e^{-s}ds$$

pointwise on X. Note that $s^{-2}(1-e^{-s}-s)$ and $s^2(1-e^{-s})^{-2}e^{-s}$ are bounded on $(0,\infty)$, rewrite

$$\int_0^{|\log(1-\epsilon'\varphi)|}\sum_{n\neq 1} p_n s^{-2}(e^{-ns}-1+ns)ds = \sum_{n\neq 1} p_n n\int_0^{n|\log(1-\epsilon'\varphi)|}\sigma^{-2}(e^{-\sigma}-1+\sigma)d\sigma,$$

and consider the function

$$K(\omega) := \frac{\int_0^\omega \sigma^{-2}(e^{-\sigma}-1+\sigma)d\sigma}{\log(1+\omega)};\quad \omega>0,$$
$$:= \frac{1}{2};\qquad \omega = 0.$$

Clearly, K is positive and continuous on $[0,\infty)$, also $K(\omega)\to 1$, as $\omega\to\infty$. Consequently

$$\inf_{\omega\geq 0} K(\omega) > 0,\quad \sup_{\omega\geq 0} K(\omega) < \infty.$$

Thus (3.11) holds if and only if

$$\varphi^*\Big[k\varphi\sum_{n>1} p_n n\log(1+n|\log(1-\epsilon'\varphi)|)\Big] < \infty. \tag{3.20}$$

Since for every integer $N\geq 2$

$$-\infty < -\varphi^*[km\varphi|\log|\log(1-\epsilon'\varphi)||]$$
$$\leq \varphi^*\Big[k\varphi\sum_{n=2}^N p_n n\log(1+n|\log(1-\epsilon'\varphi)|)\Big] - \varphi^*\Big[k\varphi\sum_{n=2}^N p_n n\log n\Big]$$
$$\leq \varphi^*[km\varphi]\log(1+|\log(1-\epsilon'\|\varphi\|)|) < \infty,$$

(3.20) is equivalent to (3.19). □

4. Proof of Theorem 2

4.1. Assume $\gamma[\mathbf{0}]>0$ and $\varphi(x)>0$; $x\in X$. Then it follows immediately from (1.1) that (1.4) is true with

$$G[\xi]=1-\gamma[\xi]/\gamma[\mathbf{0}];\quad \xi\in\bar{\mathcal{S}}_+,$$

Let $\{\xi_n\}$ be any sequence in $\bar{\mathcal{S}}_+$ such that $\xi_n(x)\to 1$; $x\in X$, as $n\to\infty$. Then $F_1[\xi_n](x)\to 1$; $x\in X$, as $n\to\infty$. Since

$$\rho^{-1}\varphi^*[\mathbf{1}-F_1[\xi]]\geq\gamma[\xi]\geq 0;\quad \xi\in\bar{\mathcal{S}}_+,$$

and $\varphi^*[1_A]|\mathfrak{A}$ is a measure, it follows that

$$\lim_{n\to\infty}G[\xi_n]=1.$$

Hence, G is a proper generating functional, [4], [12].

Using (3.8), Lemma 2 with $\varphi(x)>0$; $x\in X$, (2.4), and (3.1),

$$\begin{aligned}1-G[F_t[\xi]]&=\lim_{s\to\infty}\frac{\varphi^*[\mathbf{1}-F_t[F_s[\xi]]]}{\varphi^*[\mathbf{1}-F_s[\mathbf{0}]]}\\ &=\rho^t(1-G[\xi])-\lim_{s\to\infty}\varphi^*\left[R^t(F_s[\xi])\left[\frac{\mathbf{1}-F_s[\xi]}{\varphi^*[\mathbf{1}-F_s[\mathbf{0}]]}\right]\right]\\ &=\rho^t(1-G[\xi]);\quad \xi\in\bar{\mathcal{S}}_+,t>0.\end{aligned}\tag{4.1}$$

In particular

$$G[F_t[\mathbf{0}]]=1-\rho^t.\tag{4.2}$$

Since $(\mathbf{1}-F_t[\mathbf{0}])^{-1}\boldsymbol{E}^{(\cdot)}\hat{x}_t[\mathbf{1}]$ converges (strongly), as $t\to\infty$, to $\varphi^*[\mathbf{1}]/\gamma[\mathbf{0}]<\infty$, G must have a bounded first moment functional M. From (4.1)

$$M[\boldsymbol{E}^{(\cdot)}\hat{x}_t[\xi]]=\rho^tM[\xi];\quad t\geq 0,\xi\in\mathcal{B}.$$

By (M) therefore $M=\beta\varphi^*$, β a positive real constant. Using (4.2) and expanding G similarly as F_t in (2.4),

$$1=\rho^{-t}(1-G[F_t[\mathbf{0}]])=M[\rho^{-t}(\mathbf{1}-F_t[\mathbf{0}])]-R(F_t[\mathbf{0}])[\rho^{-t}(\mathbf{1}-F_t[\mathbf{0}])]$$

where $R(\zeta)[\xi]$ is linear-bounded in ξ and tends to 0, as $\|\mathbf{1}-\zeta\|\to 0$.

Hence, appealing to Theorem 1, $M[\gamma[\mathbf{0}]\varphi]=1$, i.e., $\beta=1/\gamma[\mathbf{0}]$.

4.2. Now assume (C) and $\varphi(x)>0$; $x\in X$. Fix $\xi\in\bar{\mathcal{S}}_+$, and define

$$G_t[\xi]:=\frac{F_t[\xi]-F_t[\mathbf{0}]}{\mathbf{1}-F_t[\mathbf{0}]}=1-\frac{\mathbf{1}-F_t[\xi]}{\mathbf{1}-F_t[\mathbf{0}]};\quad t>0.$$

By (C) the function $h_t[\xi](x)$; $x\in X$, of Lemma 2 has a continuous extension $\bar{h}_t[\xi](x)$ to $\bar{X}$ for every $t>0$. Hence there exists a t_0 such that $G_t[\xi](x)$; $x\in X$, has a continuous extension $\bar{G}_t[\xi](x)$ to $\bar{X}$ for all $t\geq t_0$. Since $\bar{X}$ is a compactification of X, there is for every $t\geq t_0$ an $\bar{x}_t\in\bar{X}$ such that

$$\bar{G}_t[\xi](\bar{x}_t)=\sup_{x\in X}G_t[\xi](x).$$

It follows by the same argument as in [10; p. 421] that $\bar{G}_t[\xi](\bar{x}_t)$ is decreasing, as $t\to\infty$. Thus

$$G[\xi]:=\lim_{t\to\infty}\bar{G}_t[\xi](\bar{x}_t) \tag{4.3}$$

exists. However, for all $t\geq t_0$

$$\mathbf{1}-G_t[\xi]=\frac{\mathbf{1}+h_t[\xi]}{\mathbf{1}+h_t[\mathbf{0}]}\cdot\frac{1+\bar{h}_t[\mathbf{0}](\bar{x}_t)}{1+\bar{h}_t[\xi](\bar{x}_t)}(1-\bar{G}_t[\xi](\bar{x}_t)),$$

so that by (3.7) and (4.3)

$$\lim_{t\to\infty}\|G_t[\xi]-G[\xi]\mathbf{1}\|=0;\quad \xi\in\bar{\mathcal{S}}_+.$$

Now let $\{\xi_n\}$ be any sequence in $\bar{\mathcal{S}}_+$ such that $\xi_n(x)\to 1$; $x\in X$, as $n\to\infty$. Fix $\delta>0$, $s>0$, and $n_0>0$ such that

$$\rho^{-\delta}\alpha_\delta<1,$$

$$C:=\sup_{\xi\in\bar{\mathcal{S}}_+:\,\|\mathbf{1}-\xi\|\leq\|\mathbf{1}-F_s[\mathbf{0}]\|}\|g_\delta[\xi]\|<1-\rho^{-\delta}\alpha_\delta,$$

$$(\rho+\alpha_1)\varphi^*[\mathbf{1}-\xi_n]\leq\rho^\delta(1-\rho^{-\delta}\alpha_\delta-C)\varphi^*[\mathbf{1}-F_s[\mathbf{0}]];\quad n\geq n_0.$$

This is clearly possible by (3.1), Lemma 1, the fact that $\varphi^*[1_A]|\mathfrak{A}$ is a measure, and $\varphi^*[\mathbf{1}-F_s[\mathbf{0}]]>0$, see Lemma 2. Appealing to (3.1), the monotony of $F_t[\mathbf{0}]$, (3.8), (2.3), (2.4), (M), and Lemma 1, there then exists a sequence of integers $\{l(n)\}_{n=1,2,\dots}$ such that $l(n) \geq s$ if $n \geq n_0$, further $l(n) \to \infty$, as $n \to \infty$, and finally

$$\mathbf{1} - F_1[\xi_n] \leq (\rho + \alpha_1)\varphi^*[\mathbf{1} - \xi_n]\varphi$$
$$\leq \rho^{\delta}(1 - \rho^{-\delta}\alpha_{\delta} - C)\varphi^*[\mathbf{1} - F_{l(n)}[\mathbf{0}]]\varphi \leq \mathbf{1} - F_{\delta + l(n)}[\mathbf{0}]; \quad n \geq n_0.$$

Hence by (4.1), (4.2),

$$1 \geq G[\xi_n] = 1 - \rho^{-1}(1 - G[F_1[\xi_n]])$$
$$\geq 1 - \rho^{-1}(1 - G[F_{\delta + l(n)}[\mathbf{0}]]) = 1 - \rho^{\delta + l(n) - 1}; \quad n \geq n_0,$$

so that $G[\xi_n] \to 1$, as $n \to \infty$, i.e., G is a proper generating functional.
Define

$$\epsilon_n := \varphi^*[\mathbf{1} - F_n[\mathbf{0}]]/\varphi^*[\mathbf{1}]; \quad n = 1, 2, \dots .$$

By (3.1) and the monotony of $F_n[\mathbf{0}]$, we have $0 < \epsilon_n \downarrow 0$; $n \uparrow \infty$. Fix $t > 0$, $n_1 > 0$, and $s > 0$ such that

$$\rho^{-t}\alpha_t < 1,$$
$$C^* := \sup_{n \geq n_1} \|g_t[\mathbf{1} - \epsilon_n \mathbf{1}]\|,$$
$$\rho^{-s}\alpha_s < 1, \quad (\rho^t - \alpha_t - \rho^t C^*)/(\rho^s + \alpha_s) \geq 1.$$

This is possible by Lemma 1 and (M), $\rho < 1$. Then, using (2.4), Lemma 1, and (M),

$$\mathbf{1} - F_t[\mathbf{1} - \epsilon_n \mathbf{1}] \geq (\rho^t - \alpha_t - \rho^t C^*)[\epsilon_n \mathbf{1}]\varphi$$
$$\geq \mathbf{1} - F_s[F_n[\mathbf{0}]]; \quad n \geq n_1,$$

and by (4.1) and (3.8)

$$(1 - G[\mathbf{1} - \epsilon_n])/\epsilon_n = \rho^{-t}(1 - G[F_t[\mathbf{1} - \epsilon_n \mathbf{1}]])/\epsilon_n$$
$$\geq \rho^{-t}(1 - G[F_s[F_n[\mathbf{0}]]])/\epsilon_n = \rho^{s-t+n}\varphi^*[\mathbf{1}](\varphi^*[\mathbf{1} - F_n[\mathbf{0}])^{-1}; \quad n \geq n_1.$$

If $\gamma[\mathbf{0}] = 0$, the expression on the far right tends to infinity, as $n \to \infty$, by Theorem 1. That is, in this case G cannot have a bounded first moment functional.

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Fachbereich Mathematik
Universität Regensburg
Universitätsstraße 31
D-8400 Regensburg 2