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A DECOMPOSITION THEOREM FOR COMODULES

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Injective comodules over coalgebras can be decomposed as a direct sum of indecomposable injective comodules, in a fashion similar to the dual decomposition of projective modules over algebras, [1]. This paper gives an elementary proof of this theorem, avoiding the use of idempotents.

1. Preliminaries and definitions

Let $k$ be a field of unspecified characteristic. A coalgebra $(C, \Delta, e)$ is a $k$-space $C$ together with a comultiplication or diagonal map $\Delta : C \rightarrow C \otimes C$, and a counit (or augmentation) $e : C \rightarrow k$ such that the following properties are satisfied.

\begin{align*}
CA 1. \quad & (\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta \quad \text{Coassociativity} \\
CA 2. \quad & (e \otimes I)\Delta = (I \otimes e)\Delta = I
\end{align*}

A comodule $(W, T)$ for a coalgebra $C$ is a $k$-space $W$ together with a map $T : W \rightarrow W \otimes C$ such that the following properties are satisfied.

\begin{align*}
CM 1. \quad & (T \otimes I)T = (I \otimes \Delta)T \\
CM 2. \quad & (I \otimes e)T = I
\end{align*}

A subcomodule (subcoalgebra) is a subspace which has a comodule (coalgebra) structure under the restricted structure maps. If $S$ is a subset of a comodule (coalgebra) the subcomodule (subcoalgebra) generated by $S$, denoted by $\langle S \rangle$, is defined to be the smallest subcomodule (subcoalgebra) containing $S$. If $S$ is a finite set

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or spans a finite dimensional subspace, \( \langle S \rangle \) is in fact a finite dimensional subcomodule (subcoalgebra).

If \( W \) is a comodule and \( V \) is a subcomodule, then \( W/V \) has a comodule structure. If \( (W, T) \) and \( (W', T') \) are comodules and \( f: W \to W' \) is a \( k \)-map, then \( f \) is a comodule map if \( (f \otimes I)T = T'f \). The usual isomorphism theorems hold.

A comodule (coalgebra) will be called simple if it contains no proper non-zero subcomodules (subcoalgebras). Every comodule contains a simple subcomodule, and every coalgebra contains a simple subcoalgebra. If \( W \) is a comodule for \( C \), define the socle of \( W, s(W) \) to be the sum of all simple subcomodules of \( W \). Define the coradical \( R \) of the coalgebra \( C \) to be the sum of all simple subcoalgebras of \( C \). If \( C \) is considered as the \( C \)-comodule \( (C, \Delta) \), then \( s(C) = R \). If \( V \) is a subcomodule of \( W \) such that \( T(V) \subseteq V \otimes R \), then \( V \subseteq s(W) \). \( s(W) \) has the property that it decomposes as a direct sum of simple subcomodules. \( R \) decomposes as a direct sum of simple subcoalgebras.

The notion of the socle can be extended. Define \( s_n(W) \) inductively by setting \( s_0(W) = 0 \), and \( s_n(W)/s_{n-1}(W) = s(W/s_{n-1}(W)) \). Since every non-zero subcomodule contains a simple subcomodule, the chain \( s_0(W) \subseteq s_1(W) \subseteq s_2(W) \subseteq \ldots \) is strictly ascending unless \( s_k(W) \) is the whole of \( W \) for some \( k \). Since every element \( w \) of \( W \) is contained in the finite dimensional subcomodule \( \langle \langle w \rangle \rangle \), \( W = \bigcup_{n=1}^{\infty} s_n(W) \).

The socle can be described in another way. For subspaces \( X \subseteq W \), and \( Y \subseteq C \), define the wedge of \( X \) and \( Y \), \( X \wedge Y \) to be the kernel of the map

\[
W \xrightarrow{T} W \otimes C \longrightarrow W/X \otimes C/Y
\]

Thus \( X \wedge Y = T^{-1}(W \otimes Y + X \otimes C) \). It can be shown that \( 0 \wedge R = s(W) \). If we define \( \wedge^0 R = 0 \) and \( \wedge^\alpha R = (\wedge^{\alpha-1} R) \wedge R \), then it follows that \( \wedge R = s_n(W) \).

A comodule \( (I, T) \) is injective if for every comodule \( (W, T') \) and every subcomodule \( U \subseteq W \), every comodule map \( f: U \to I \) extends uniquely to a map \( f: W \to I \). \( C \) itself is an injective \( C \)-comodule. Direct summands of injective comodules are injective.

2. The theorem

**Theorem:** Let \( (W, T) \) be an injective comodule. Let \( s(W) = \sum_{\mu \in M} X_\mu \) be a direct decomposition of the socle of \( W \) as a sum of

1For elementary properties of comodules and coalgebras, see Sweedler, [2].
simple subcomodules. This decomposition of \( s(W) \) can be extended to a direct decomposition of \( W \) as a sum of indecomposable injective subcomodules, \( W = \sum_{\mu \in M} J_\mu \) such that \( s(J_\mu) = X_\mu \).

The theorem is proved by constructing inductively a decomposition of \( s_n(W) \) which extends the decomposition of \( s_{n-1}(W) \).

For every \( \mu \) in \( M \), let \( J_\mu^1 = X_\mu \). Suppose we have \( J_\mu^{n-1} \) defined for some \( n \geq 2 \) such that

(i) \( s(J_\mu^{n-1}) = X_\mu \)

(ii) \( \sum_{\mu \in M} J_\mu^{n-1} = s_{n-1}(W) \)

(iii) The sum \( \sum_{\mu \in M} J_\mu^{n-1} \) is direct.

We wish to define \( J_\mu^n \). Set \( Z_\mu = \Sigma_{\lambda \in M \setminus \mu} X_\lambda \). Define

\[ \mathcal{B}_\mu = \{ S \leq J_\mu^{n-1} \land R : S \geq J_\mu^{n-1}, S \cap Z_\mu = 0 \} \]

\( \mathcal{B}_\mu \) is nonempty, since \( J_\mu^{n-1} \) is in \( \mathcal{B}_\mu \), and by Zorn’s lemma \( \mathcal{B}_\mu \) has maximal elements. Choose \( J_\mu^n \) to be a maximal element of \( \mathcal{B}_\mu \). It remains to show that the set \( \{ J_\mu^n \}_{\mu \in M} \) satisfies the three conditions of the inductive hypothesis.

(i) \( s(J_\mu^n) \geq X_\mu \), since \( J_\mu^n \geq J_\mu^{n-1} \). If \( s(J_\mu^n) \not\geq X_\mu \), it follows that \( J_\mu^n \cap Z_\mu \neq 0 \), a contradiction. So \( s(J_\mu^n) = X_\mu \).

(ii) It is enough to show that the sum \( \Sigma_{\lambda \in \Lambda} J_\lambda^n \) is direct for all finite subsets \( \Lambda \leq M \). This can be done by induction on \( |\Lambda| \). Assume now that for any subset \( \Lambda \) of \( M \) with \( |\Lambda| < r \), the sum \( \Sigma_{\lambda \in \Lambda} J_\lambda^n \) is direct. If \( \Gamma \leq M, |\Gamma| = r \), and the sum \( \Sigma_{\lambda \in \Gamma} J_\lambda^n \) is not direct then there is some \( \lambda \) in \( \Gamma \) and some simple comodule \( U \leq J_\lambda^n \) such that \( U = X_\lambda \leq s(\Sigma_{\mu \in \Gamma \setminus \lambda} J_\mu^n) = \Sigma_{\mu \in \Gamma \setminus \lambda} s(J_\mu^n) \leq \Sigma_{\mu \in \Gamma \setminus \lambda} X_\mu \leq Z_\lambda \), which contradicts the directness of the decomposition of the socle, and completes the inductive step. (The second equality follows from the directness of the sum \( \Sigma_{\mu \in \Gamma \setminus \lambda} J_\mu^n \), by the inductive hypothesis.)

(iii) This condition is shown in three steps.

Step 1. \( J_\mu^{n-1} \land R = J_\mu^n \oplus Z_\mu \)

Step 2. \( \sum_{\mu \in M} J_\mu^n = \sum_{\mu \in M} (J_\mu^{n-1} \land R) \)

Step 3. \( \sum_{\mu \in M} (J_\mu^{n-1} \land R) = \left( \sum_{\mu \in M} J_\mu^{n-1} \right) \land R = s_{n-1}(W) \land R = s_n(W) \).
Step 1. Clearly \( J^n_\mu + Z_\mu \leq J^{n-1}_\mu \land R \). To see the converse, it is sufficient to show that if \( U \supseteq J^{n-1}_\mu \) is a subcomodule of \( W \) such that \( U/J^{n-1}_\mu \) is simple, then \( U \leq J^n_\mu + Z_\mu \). Suppose that \( U \supseteq J^n_\mu + Z_\mu \). Then \( U + J^n_\mu \nleq J^{n-1}_\mu \land R \), so by the maximality of \( J^n_\mu \) in \( \mathcal{B}_\mu \), it must be that \((U + J^n_\mu) \cap Z_\mu \neq 0 \). We may pick \( z \neq 0 \) in \( Z_\mu \) such that \( z = u + j \) with \( u \) in \( U \) and \( j \) in \( J^n_\mu \). Now \( u \) is not in \( J^{n-1}_\mu \) (otherwise \( z \) would be in \( J^n_\mu \cap Z_\mu \) contrary to the conditions in \( \mathcal{B}_\mu \)) and hence not in \( J^{n-1}_\mu \). Therefore \( u + J^{n-1}_\mu \) must generate \( U/J^{n-1}_\mu \). Thus

\[
U = \langle \langle u \rangle \rangle + J^{n-1}_\mu \leq \langle \langle j \rangle \rangle + J^{n-1}_\mu \leq J^n_\mu + Z_\mu
\]

which is a contradiction. Thus it must be that \( U \leq J^n_\mu + Z_\mu \), and therefore \( J^n_\mu + Z_\mu = J^{n-1}_\mu \land R \). Since \( J^n_\mu \) is in \( \mathcal{B}_\mu \), \( J^n_\mu \cap Z_\mu = 0 \) and the sum is direct.

Step 2. This is a direct consequence of step 1 and the definition of \( J^n_\mu \).

Step 3. The last equality is a property of the wedge, the second uses the inductive hypothesis, that \( \Sigma_{\mu \in M} J^{n-1}_\mu = S_{n-1}(W) \). Since \( J^{n-1}_\mu \leq \Sigma_{\lambda \in M} J^{n-1}_\lambda \), we have that \( J^{n-1}_\mu \land R \leq (\Sigma_{\lambda \in M} J^{n-1}_\lambda) \land R \) for all \( \mu \) in \( M \), and \( \Sigma_{\mu \in M} (J^{n-1}_\mu \land R) \leq (\Sigma_{\mu \in M} J^{n-1}_\mu) \land R \).

Now let \( U \leq (\Sigma_{\mu \in M} J^{n-1}_\mu) \land R \). We may assume that \( U \) is finite dimensional. Then

\[
U + \sum_{\mu \in M} J^{n-1}_\mu \bigg/ \sum_{\mu \in M} J^{n-1}_\mu \equiv U/\bigcup_{\mu \in M} J^{n-1}_\mu \equiv U + \sum_{\mu \in M'} J^{n-1}_\mu \bigg/ \sum_{\mu \in M'} J^{n-1}_\mu
\]

Where \( M' \) is a finite subset of \( M \) such that \( U \cap (\Sigma_{\mu \in M} J^{n-1}_\mu) \leq \Sigma_{\mu \in M'} J^{n-1}_\mu \). Since \( U \leq (\Sigma_{\mu \in M} J^{n-1}_\mu) \land R \), \( U + \sum_{\mu \in M'} J^{n-1}_\mu/\Sigma_{\mu \in M'} J^{n-1}_\mu \) is completely reducible. Let

\[
U + \sum_{\mu \in M'} J^{n-1}_\mu \bigg/ \sum_{\mu \in M'} J^{n-1}_\mu \equiv \sum_{i=1}^k \left( U_i \bigg/ \sum_{\mu \in M'} J^{n-1}_\mu \right)
\]

be a direct decomposition as simple comodules. It is sufficient to show each \( U_i \) is contained in \( \Sigma_{\mu \in M}(J^{n-1}_\mu \land R) \).

Take \( U = U_i \), and set \( Q = \Sigma_{\mu \in M} J^{n-1}_\mu \), and \( Q_\mu = \Sigma_{\lambda \in M', \lambda \neq \mu} J^{n-1}_\lambda \), for all \( \mu \) in \( M' \). We have projections (which are comodule maps)

\[
p_\mu: U \to U/Q_\mu \text{ for all } \mu \text{ in } M'.
\]
These can be used to get a comodule homomorphism

$$p: U \to \sum_{\mu \in M'} \frac{U}{Q_\mu} \text{ (external direct sum)}.$$  

If $a$ is in ker$(p)$, then $p_\mu(a) = 0$ for all $\mu$ in $M'$. That is, $a$ is in $Q_\mu$ for all $\mu$ in $M'$. But the sum $\Sigma_{\mu \in M'} J_\mu$ is direct, and so $\cap_{\mu \in M'} Q_\mu = 0$, whence $a = 0$ and $p$ is injective.

Let $U' = \text{im}(p)$ in $\Sigma_{\mu \in M'} \frac{U}{Q_\mu}$. $p$ is an isomorphism of $U$ onto $U'$. Let $\varphi: U' \to W$ be the inverse to $p$ on $U'$. Since $W$ is injective we can extend $\varphi_0$ to a map

$$\varphi: \sum_{\mu \in M'} \frac{U}{Q_\mu} \to W$$

$\text{Im}(\varphi) \supseteq U$ and $\text{Im}(\varphi) \subseteq \Sigma_{\mu \in M'} r(U/Q_\mu)$.

It remains to show that $r(U/Q_\mu)$ is contained in $J_\mu^{-1} \wedge R$. We have a series

$$U/Q_\mu \subseteq Q/Q_\mu \subseteq 0$$

The bottom factor is isomorphic to $J_\mu^{-1}$ and the top factor $(U/Q_\mu)/(Q/Q_\mu)$ is simple. Moreover,

$$r(Q/Q_\mu) = r_\mu(p(J_\mu^{-1})) = J_\mu^{-1}$$

(Notice that $p_\mu(J_\mu^{n-1}) = 0$ if $\lambda \neq \mu$, and thus $p(J_\mu^{n-1}) \subseteq Q/Q_\mu \subseteq U/Q_\mu$.)

We have an induced homomorphism

$$\tilde{r}: U/Q_\mu \to r(U/Q_\mu)/r(Q/Q_\mu) = r(U/Q_\mu)/J_\mu^{-1}$$

Thus $r(U/Q_\mu)$ is a homomorphic image of a simple comodule and must therefore be simple or 0. If $r(U/Q_\mu)$ is simple, then $r(U/Q_\mu) \leq J_\mu^{-1} \wedge R$, by a property of the wedge. If $r(U/Q_\mu)$ is 0, then $r(U/Q_\mu) \leq J_\mu^{-1} \wedge R$.

Thus $r(U/Q_\mu) \leq J_\mu^{-1} \wedge R$ for all $\mu$ in $M'$ and $U \leq \Sigma_{\mu \in M'} r(U/Q_\mu) \leq \Sigma_{\mu \in M}(J_\mu^{-1} \wedge R)$, which completes step 3.

Let $J_\mu = \bigcup_{n=1}^\infty J_\mu^n$. The sum $\Sigma_{\mu \in M} J_\mu$ is direct, since the sum $\Sigma_{\mu \in M} J_\mu^n$ is direct for all $n$, and it is the whole of $W$ since $\Sigma_{\mu \in M} J_\mu^n = s_n(W)$ and $\bigcup_{n=1}^\infty s_n(W) = W$. $s(J_\mu) = (\Sigma_{\lambda \in \mu} J_\mu) \cap J_\mu = J_\mu$, by directness of the sum $\Sigma_{\lambda \in \mu} J_\mu$. The $J_\mu$ are indecomposable since each $J_\mu$ contains a unique
simple subcomodule. Each $J_\mu$ is injective since direct summands of injective comodules are injective.

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