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A RESULT ON THE INTEGRAL CHOW RING OF A GENERIC PRINCIPALLY POLARIZED COMPLEX ABELIAN VARIETY OF DIMENSION FOUR

Charles Barton and C. H. Clemens

0. Introduction

In this paper, we wish to show that a certain positive algebraic two-cycle on a generic abelian variety of dimension four is not, in general, represented by an *effective* algebraic subvariety. This problem was suggested by the fact that this cycle is effectively representable if the abelian variety is the Jacobian of a curve or the intermediate Jacobian of a cubic threefold.

The method of proof is via a degeneration argument – we construct (in some detail) the "generic" degeneration of a family of principally polarized abelian varieties of dimension four, then we see what the existence of the effective two-cycle would imply in the limit.

1. A "generic" degeneration

Our purpose in this section is to construct a "generic" proper mapping of a holomorphic manifold J onto the unit disc Δ

$$\pi : J \to \Delta \tag{1.1}$$

such that:

(i) if $z \neq 0$, $J_z = \pi^{-1}(z)$ is a principally polarized abelian variety [4; Chapter 1] of dimension four;

(ii) J_0 is non-singular except that it crosses itself transversely along M, a principally polarized abelian variety of dimension three;

(iii) $\tilde{J}_0$, the normalization of J_0, is a bundle over M with fibre $\mathbb{P}_1$ (complex projective one-space).

To accomplish this, we begin with the set

$$(1.2)\qquad H=\left\{\left(\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix};\sigma,\tau\right)\in\mathbb{C}^3\times\mathbb{C}^2:\sigma\tau<1\right\}.$$

Let

$$(1.3)\qquad A:\Delta\to GL(3;\mathbb{C})$$

be a holomorphic mapping of the unit disc into the group of invertible 3×3 matrices over the complex numbers such that:

(i) $A(z)$ is symmetric for each $z\in\Delta$;

(ii) (imaginary part of $A(z)$) is positive definite for each $z\in\Delta$.

Let

$$E_1, E_2, E_3$$

be the standard basis of $\mathbb{R}_3$ and

$$A_1, A_2, A_3$$

the columns of $A(z)$. Let

$$L(z)=\{\Sigma m_jE_j+n_jA_j:m_j,n_j\in\mathbb{Z}\}.$$

Also let

$$(1.4)\qquad B:\Delta\to\mathbb{C}^3$$

be any holomorphic mapping,

$$B(z)=\begin{bmatrix}b_1(z)\\b_2(z)\\b_3(z)\end{bmatrix}.$$

Using $L(z)$ and $B(z)$ we define an equivalence relation on H as follows. We put

$$(u;\sigma,\tau)\sim(u';\sigma',\tau')$$

if

(i) $\sigma\cdot\tau=\sigma'\cdot\tau'=z\in\Delta$;

(ii) $(u-u')=\Sigma\,(m_jE_j+n_jA_j)\in L(z)$;

(iii) $\sigma=e^{2\pi i(\Sigma n_jb_j(z))}\cdot\sigma'$ and $\tau=e^{2\pi i(-\Sigma n_jb_j(z))}\cdot\tau'$.

Let

$$K=H/\{\sim\}.$$

Then K is a complex manifold and we have a natural mapping

$$(1.5)\qquad \kappa:K\to\Delta.$$

$$\{(u;\sigma,\tau)\}\mapsto\sigma\cdot\tau.$$

If $z \neq 0$, $\kappa^{-1}(z)$ is a $\mathbb{C}^*$-bundle over a principally polarized abelian variety

$$M_z = \mathbb{C}^3/L(z) \tag{1.6}$$

of dimension three. $\kappa^{-1}(0)$ is the union of two (mutually dual) line bundles over $M = \mathbb{C}^3/L(0)$.

The idea now, of course, is to construct J as a quotient of K. On K then, we define

$$\{(u; \sigma, \tau)\} \sim \{(u'; \sigma', \tau')\}$$

whenever

(i) $\sigma \cdot \tau = \sigma' \cdot \tau' = z$;

(ii) $\sigma' \tau = 1$;

(iii) $(u - u') = B(z)$.

Then "$\sim$" *generates* an equivalence relation and we can define

$$J = K/\{\sim\}$$

Clearly, J is smooth and the mapping κ in (1.5) induces a proper mapping

$$\pi : J \to \Delta.$$

Of the assertions (i)–(iii) following (1.1), (ii) and (iii) are clear for the mapping π we have just constructed. Assertion (i) is, in fact, only correct for sufficiently small values of z.

We will check this last fact by computing the period matrix for $J_z = \pi^{-1}(z)$. Let

$$\ell(\sigma) = \frac{1}{2\pi i} \log \sigma.$$

Then we can make a mapping

$$\kappa^{-1}(z) \to \mathbb{C} \times \mathbb{C}^3 \tag{1.7}$$

$$\{(u; \sigma, \tau)\} \mapsto (\ell(\sigma); u)$$

which is well-defined modulo integral combinations of the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ E_j \\ \\ \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} b_j \\ A_j \\ \\ \end{bmatrix}. \tag{1.8}$$

(See conditions (ii) and (iii) for the equivalence relation defining K.) To

pass from $\kappa^{-1}(z)$ to J_z we have introduced a second equivalence relation. Since $z \neq 0$, our equivalence is generated by the conditions
(i) $\sigma/\sigma' = z$,
(ii) $(u - u') = B(z)$;
in other words, the mapping (1.7) induces a mapping

$$J_z \to \mathbf{C} \times \mathbf{C}^3$$

which is well-defined modulo integral combinations of the vectors (1.8) and the vector

$$\begin{bmatrix} \ell(z) \\ B(z) \end{bmatrix}. \tag{1.9}$$

So J_z is simply the quotient of $\mathbf{C}^4$ by the subgroup generated by the vectors (1.8) and (1.9). If z is sufficiently small, the vector (1.9) is clearly linearly independent (over $\mathbf{R}$) from the others, so

$$J_z = \text{complex torus with period matrix } \Omega(z) \tag{1.10}$$

where

$$\Omega(z) = \begin{bmatrix} \ell(z) & {}^tB(z) \\ B(z) & A(z) \end{bmatrix}.$$

Also, if $z \neq 0$ is sufficiently small, the matrix

$$(\text{imaginary part of } \Omega(z))$$

is positive definite. This means that J_z does indeed have the structure of a principally polarized abelian variety. From here on, we assume that we have adjusted the parameter z so that this is the case for *all* $z \in (\Delta - \{0\})$. We call the family (1.1) a *generic degeneration* since the varieties J_0 constructed as above make up the "largest component" of a natural compactification of the moduli space of principally polarized abelian varieties of dimension four [5].

Finally we will need a family of theta-functions on the varieties J_z. We define these as functions on H (see (1.2)), but for $z \neq 0$ they will just give the usual theta functions of characteristic

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

on J_z. Let N be a positive integer and let

$$n = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

be a triple of integers such that $0 \leq n_j < N$ for $j = 1, 2, 3$. For $u \in \mathbb{C}^3$ define

$$\theta^{n,N}(u; z) = \sum_{m \in \mathbb{Z}^3} e^{\pi i\,{}^t(Nm+n)[A(z)(m+\frac{n}{N})+2u]} \tag{1.11}$$

where $A(z)$ is as in (1.3). Then for $0 \leq n_0 < N$, and $(u; \sigma, \tau) \in H$ (see (1.2)), define

$$\theta^{n_0,n,N}(u; \sigma, \tau) = \sum_{m_0 \in \mathbb{Z}} \Big[\sigma^{(N(m_0+1)+n_0)} (\sigma\tau)^{(N[m_0(m_0+1)/2]+n_0(m_0+1))} \cdot \theta^{n,N}\Big(u + \Big[m_0 + \frac{1}{2} + \frac{n_0}{N}\Big] B(\sigma\tau); \sigma\tau\Big)\Big] \tag{1.12}$$

where $B(z)$ is as in (1.4). From the definition itself, nothing is clear, not even the convergence of the series. Assume absolute convergence uniform on compact subsets of H. Then on the subset of H given by $\sigma\tau = 0$, the series in (1.12) reduces to

(1.13) (i)

$$\theta^{n,N}\Big[u - \frac{B(0)}{2}; 0\Big] + \sigma^N \theta^{n,N}\Big[u + \frac{B(0)}{2}; 0\Big] + \tau^N \theta^{n,N}\Big[u - \frac{3B(0)}{2}; 0\Big]$$

if $n_0 = 0$;

(ii)

$$\sigma^{n_0} \theta^{n,N}\Big[u + \Big[\frac{n_0}{N} - \frac{1}{2}\Big] B(0); 0\Big] + \tau^{(N-n_0)} \theta^{n,N}\Big[u + \Big[\frac{n_0}{N} - \frac{3}{2}\Big] B(0); 0\Big]$$

if $n_0 \neq 0$.

Now, to check convergence, we use the relations $\sigma\tau = z$ and $\sigma = e^{2\pi i u_0}$, which allow us to rewrite (1.12) as follows:

(1.14)

$$\theta^{n_0,n,N}(\tilde{u}; \sigma, \tau) = \sigma^{N/2} z^{n_0} e^{-\pi i (N(1/2+n_0/N)^2 \ell(z))} \sum_{\tilde{m}} e^{\pi i\,{}^t(N\tilde{m}+\tilde{n})[\Omega(z)(\tilde{m}+N^{-1}\tilde{n})+2\tilde{u}]}$$

where

$$\tilde{m} = \begin{bmatrix} m_0 + 1/2 \\ (m) \end{bmatrix}, \quad m_0 \in \mathbb{Z},\ m \in \mathbb{Z}^3,$$

$$\tilde{n} = \begin{bmatrix} n_0 \\ (n) \end{bmatrix} \in \{0, \ldots, (N-1)\}^4,$$

and

$$\tilde{u} = \begin{bmatrix} u_0 \\ (u) \end{bmatrix} \in \mathbb{C}^4.$$

Now on a set

$$|\sigma| = \delta > 0, \quad |\tau| = \epsilon > 0, \tag{1.15}$$
$$\tilde{u} \in (\text{compact subset of } \mathbb{C}^4),$$

the series (1.14) is absolutely and uniformly convergent – this is an immediate corollary of the proof of the uniform and absolute convergence of the Fourier series of N-th order theta-functions [2; page 96]. So the series (1.12) converges absolutely and uniformly on sets (1.15) and so on any compact subset of H.

Indeed, in the formation (1.14), the functions

$$\theta^{n_0,n,N}(\tilde{u}\,; z) = \sigma^{-N/2}\theta^{n_0,n,N}(u\,; \sigma, \tau)$$

for $\sigma\tau = z \neq 0$ give a basis for the N-th order theta-functions on J_z with characteristic

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Also these functions are invariant under the substitution

$$\ell(z) \mapsto (\ell(z) + 1);$$

thus the zero set of the function (1.12) on

$$(H - \{(u\,; \sigma, \tau) : \sigma\tau = 0\}) \tag{1.16}$$

is invariant with respect to the identifications used to define

$$(J - J_0)$$

as a quotient space of (1.16). Also from the formulas (1.13) it is clear that the zero set of a function (1.12) in H is simply the closure of its zero set in (1.16). These two facts imply that the zero set of (1.12) in H is invariant with respect to the identifications used to define J as a quotient space of H and so defines a divisor

$$\Theta^{n_0,n,N} \tag{1.17}$$

on J. The linear system spanned by the divisors (1.17) has projective dimension

$$N^4 - 1.$$

The rest of this section will be devoted to the study of this linear system, which we denote by

$$\mathscr{D}_N. \tag{1.18}$$

First of all, the formulas (1.13) immediately imply that

$$J_z \not\subseteq D$$

for any $D \in \mathcal{D}_N$ and any $z \in \Delta$. Thus the algebraic cycle (with multiplicity)

$$(J_z \cdot D)$$

always makes sense and for any $z \in \Delta$

$$(J_z \cdot D) \equiv (J_0 \cdot D) \tag{1.19}$$

in $H_6(J; \mathbb{Z})$. Also, by [1; §5–6], the semi-group $[0, 1] \times \mathbb{R}$ acts on J in such a way that

(i) $\pi((r, \theta) \cdot x) = re^{2\pi i\theta}\pi(x)$ for all $x \in J$ and $(r, \theta) \in [0, 1] \times R$;

(ii) $(r, \theta) \cdot x = x$ whenever $x \in J_0$.

So in $H_6(J; \mathbb{Z}) = H_6(J_0; \mathbb{Z})$:

$$(D \cdot J_z) \equiv (0, 0) \cdot (D \cdot J_z)$$

and therefore by (1.19)

$$(D \cdot J_0) \equiv (0, 0) \cdot (D \cdot J_z). \tag{1.20}$$

But we can explicitly compute the right-hand-side of (1.20). To do this, notice that the real coordinates

$$(\xi_0, \ldots, \xi_3, \eta_0, \ldots, \eta_3)$$

give a set of coordinates for J_z via the mapping

$$(\xi, \eta) \mapsto \sum_{j=0}^{3} \xi_j E_j + \sum_{j=0}^{3} \eta_j \Omega_j$$

where

$$E_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{etc.,}$$

and $\Omega_j = (j + 1)$-st column of the period matrix $\Omega(z)$. Let γ_j be the element of $H_1(J_z; \mathbb{Z})$ defined by fixing ξ_k for $k \neq j$ and all the η_k and letting ξ_j run from 0 to 1. Similarly define $\delta_j \in H_1(J_z; \mathbb{Z})$ by letting η_j run from 0 to 1. Then

$$\{\gamma_0, \ldots, \gamma_3, \delta_0, \ldots, \delta_3\} \tag{1.21}$$

is a basis for $H_1(J_z; \mathbb{Z})$. From the classical theory of theta-functions we have that if $D \in \mathcal{D}_N$ and $z \neq 0$:

$$(D \cdot J_z) \equiv N \cdot \sum_{j=0}^{3} \gamma_0 \times \delta_0 \times \cdots \times \widehat{\gamma_j \times \delta_j} \times \cdots \times \gamma_3 \times \delta_3 \tag{1.22}$$

where "×" denotes Pontriagin product in the topological group J_z and "^" means "delete."

Next let

$$\tilde{J}_0 = (\text{normalization of } J_0).$$

We then have a $\mathbb{P}_1$-bundle

(1.23) $$\mu : \tilde{J}_0 \to M = M_0$$

with fibre coordinate σ (see (1.5)–(1.6)). The bundle μ has distinguished sections

(1.24) $$\begin{gathered} M^0 \text{ given by } \sigma = 0 \\ M^\infty \text{ given by } \sigma = \infty \end{gathered}$$

which are identified (via translation by $B(0)$) under the normalization mapping

(1.25) $$\nu : \tilde{J}_0 \to J_0.$$

Their common image, which we will denote simply by M, is the double variety of J_0.

Topologically, for $z \neq 0$

$$J_z \cong \gamma_0 \times \delta_0 \times M$$

and the "collapsing" map

$$\begin{gathered} J_z \to J_0 \\ x \mapsto (0,0) \cdot x \end{gathered}$$

is given by fixing $0 \in \delta_0$ and collapsing

$$\gamma_0 \times \{0\} \times M$$

to $\{u\} \subseteq M$ for each point $u \in M$. So using (1.20) and (1.22), we can explicitly describe

(1.26) $$(D \cdot J_0) \in H_6(J_0; Z)$$

as follows. Abusing notation, let

(1.27) $$\{\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, \delta_3\}$$

denote the standard basis of $H_1(M; \mathbb{Z})$ with respect to the period matrix $A(0)$. Then by (1.22) and our description of the collapsing map, we have that

$$(0,0) \cdot (D \cdot J_z)$$

is given in $H_6(J_0; \mathbb{Z})$ by

$$N \cdot \left[\nu(M^0) + \nu\left(\mu^{-1}\left(\sum_{1 \leq j < k \leq 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k\right)\right)\right]. \tag{1.28}$$

So by (1.20) the class of $(D \cdot J_0)$ for $D \in \mathscr{D}_N$ must be given by the same formula. If P denotes a fibre of μ, then by (1.28) we have

$$\nu(P) \cdot D = N \tag{1.29}$$

which agrees with the formulas (1.13). (In (1.13) we can, for example, set $\tau = 0$ and use σ as the fibre coordinate of $\mu : \tilde{J}_0 \to M$.)

Now if $N = 1$, then $\mathscr{D}_N$ contains a unique divisor, which we will call

$$\Theta. \tag{1.30}$$

For $z \neq 0$, $(\Theta \cdot J_z)$ is called the *theta-divisor* of J_z.

THEOREM 1.31: *If the mappings A and B in* (1.3) *and* (1.4) *are chosen generically,* Θ *is smooth in a neighborhood of its intersection with* J_0. *Also for z near* 0, Θ *meets* J_z *transversely.*

PROOF: Let

$$\tilde{\Theta}_0 \subseteq \tilde{J}_0$$

be such that $\nu(\tilde{\Theta}_0) = (\Theta \cdot J_0)$. By elementary properties of analytic varieties, the theorem will be proved if we can show that $\tilde{\Theta}_0$ is a smooth subvariety (of multiplicity one) in $\tilde{J}_0$ which intersects M^0 and M^∞ transversely. By (1.13) (i), $(\Theta \cdot J_0)$ is given by the zero set of

$$\theta\left(u - \frac{B(0)}{2}\right) + \sigma\theta\left(u + \frac{B(0)}{2}\right) + \tau\theta\left(u - \frac{3B(0)}{2}\right) \tag{1.32}$$

where $\theta(u) = \theta^{0,1}(u; 0)$. So

$$\tilde{\Theta}_0 \cap (\tilde{J}_0 - M^\infty)$$

is given by setting $\tau = 0$ in (1.32) and looking at the zero set of the resulting function. If (u', σ') is a singular point of this zero set, then

(i) $\theta\left(u' + \frac{B(0)}{2}\right) = \theta\left(u' - \frac{B(0)}{2}\right) = 0,$

and for $j = 1, 2, 3$:

(ii) $\sigma' \frac{\partial \theta}{\partial u_j}\left(u' + \frac{B(0)}{2}\right) = -\frac{\partial \theta}{\partial u_j}\left(u' - \frac{B(0)}{2}\right).$

If $A(0)$ is chosen generically, there is no common zero of $\theta(u)$ and $(\partial \theta / \partial u_j)(u)$, $j = 1, 2, 3$. Otherwise, for example, the Riemann singularities theorem would imply that every curve of genus three is hyperelliptic. So, for general $A(0)$, the *Gauss map*

$$g: (\text{zero set of } \theta \text{ in } M) \to \mathbb{P}_2$$
$$u \mapsto \left[\frac{\partial \theta}{\partial u_j}(u)\right]_{j=1,2,3}$$

is a morphism and is surjective (recall that M is a Jacobian). But then one computes immediately that

$$\{u' - u'': g(u') = g(u'')\} \subseteq M$$

is a subvariety of dimension ≤ 2. If we choose $B(0)$ outside this subvariety (and $A(0)$ as above) then (i) and (ii) have no common solutions $(u', \sigma') \in (\tilde{J}_0 - M^\infty)$. Also by (1.32) (with $\tau \equiv 0$), $\tilde{\Theta}_0$ meets M^0 tranversely whenever $\theta(u)$ and the $(\partial\theta/\partial u_j)(u)$ have no common zeros. Putting $\sigma \equiv 0$ in (1.32), the analogous argument works for $(\tilde{\Theta}_0 \cap (\tilde{J}_0 - M^0))$. This proves the first statement of Theorem 1.31. The second statement then follows from the fact that $(\Theta \cap J_z)$ is given locally by the equation $\sigma = z$ or by the equation $\sigma\tau = z$.

THEOREM 1.33: *Suppose $N \geq 3$ and $B(0) \neq 0$ in M. Let*

$$F_N: J \to \mathbb{P}_{(N^4-1)}$$

be the mapping defined by the linear system $\mathcal{D}_N$ in (1.18). *The system $\mathcal{D}_N$ has no basepoints so that F_N is a regular mapping. In fact, the mapping*

$$G_N: J \to \mathbb{P}_{(N^4-1)} \times \Delta$$
$$x \to (F_N(x), \pi(x))$$

is an embedding.

PROOF: Except along J_0 this is a standard classical theorem. The same classical theorem says that the linear system spanned by the divisors of the functions (1.12) in M_z gives an embedding of M_z. Applying this for $z = 0$ and the formulas (1.13) (i), it is clear that F_N embeds $M \subseteq J_0$ in $\mathbb{P}_{(N^4-1)}$. To show that G_N is also an immersion at points of $M \subseteq J$, it suffices to note that, given $u' \in M$, there exists by (1.13) (ii) a divisor in $\mathcal{D}_N$ which is smooth and tangent to $\{(u; \sigma, \tau): \sigma = 0\}$ at u' and which contains M, as well as a divisor which is smooth and tangent to $\{(u; \sigma, \tau): \tau = 0\}$ at u' and which contains M. (We use again that $N \geq 3$.) Next, recall that to study the linear system cut out by $\mathcal{D}_N$ on $(J_0 - M)$ we can set $\tau = 0$ in (1.13) and use σ as the fibre coordinate of the $\mathbb{C}^*$-bundle

$$\mu: (J_0 - M) \to M. \tag{1.34}$$

So, by (1.13) (ii), $\mathscr{D}_N$ has no fix-points on $(J_0 - M)$ and

$$F_N((J_0 - M)) \cap F_N(M) = \phi.$$

Given $(u'; \sigma')$ and $(u''; \sigma'') \in (J_0 - M)$, (1.13) (ii) also shows that if $F_N((u'; \sigma')) = F_N((u''; \sigma''))$, then

$$u' = u'',$$

and, considering the cases $n_0 = 2$ and $n_p = 1$,

$$(\sigma'/\sigma'')^2 = (\sigma'/\sigma'')$$

so that

$$\sigma' = \sigma''.$$

Finally, to show that G_N is an immersion at a point of $(J_0 - M)$, it suffices to show that

$$F_N|_{(J_0 - M)}$$

is an immersion. But this follows immediately from (1.13) and the facts:

(i) the linear system spanned by the divisors of the functions (1.12) (in the case $z = 0$) embeds M;

(ii) given $(u'; \sigma') \in (J_0 - M)$, there exists a vector

$$(a_n) \in \mathbb{C}^{N^3}$$

such that

$$\sum_n a_n \left(\theta^{n,N} \left(u' - \frac{B(0)}{2}; 0 \right) + (\sigma')^N \theta^{n,N} \left(u' + \frac{B(0)}{2}; 0 \right) \right) = 0$$

but

$$\sum_n a_n \theta^{n,N} \left(u' + \frac{B(0)}{2}; 0 \right) \neq 0$$

(see (1.13) (i)). Notice that (ii) follows from the fact that $B(0) \neq 0$ in M which implies that the vectors

$$\left(\theta^{n,N} \left(u' - \frac{B(0)}{2}; 0 \right) + (\sigma')^N \theta^{n,N} \left(u' + \frac{B(0)}{2}; 0 \right) \right)_n \quad \text{and}$$

$$\left(\theta^{n,N} \left(u' + \frac{B(0)}{2}; 0 \right) \right)_n \quad \text{are not proportional.}$$

Notice that the argument in Theorem 1.31 can be applied inductively to show that a generic principally polarized abelian variety of dimension k has non-singular theta-divisor. The proof of Theorem 1.33 also applies, of course, in higher dimensions.

2. The "generic" Chow ring

On a complex torus J_1 of dimension four, a principal polarization is given by an element

(2.1) $$\Omega_1 \in H^2(J_1; \mathbb{Z}) \cong (\Lambda^2 H_1(J_1; \mathbb{Z}))^*$$

such that

(i) Ω_1 is a positive form of type (1, 1) in the Hodge decomposition of $H^2(J_1; \mathbb{C})$;

(ii) Ω_1 is unimodular as a bilinear form on $H_1(J_1; \mathbb{Z})$.

Given (J_1, Ω_1), we can choose a basis (1.21) for $H_1(J_1; \mathbb{Z})$ which is *sympletic*, that is,

$$\Omega_1(\gamma_j, \gamma_k) = \Omega_1(\delta_j, \delta_k) = 0,$$

$$\Omega_1(\gamma_j, \delta_k) = \text{Kronecker } \delta_{jk}.$$

If

(2.2) $$\omega = \begin{bmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix},$$

where $\{\omega_i\}_{i=0,\dots,3}$ is a basis for $H^{1,0}(J_1)$ such that

$$\int_{\gamma_j} \omega = E_j, \quad \int_{\delta_j} \omega = \Omega_j$$

where the E_j are the standard basis for $\mathbb{C}^4$, then the imaginary part of

(2.3) $$\Omega = (\Omega_0 \Omega_1 \Omega_2 \Omega_3)$$

is positive definite and the associated N-th order theta-functions

$$\theta^{n_o, n, N}(\vec{u}) = \sum_{\vec{m}} e^{\pi i^t(N\vec{m} + \vec{n})[\Omega(\vec{m} + N^{-1}\vec{n}) + 2\vec{u}]}$$

(see (1.14)) have zero sets on J_1 whose associated homology class is the Poincare dual of $N\Omega_1$ (see (1.22)). The question we wish to treat is the following:

(2.4) Which elements of $H_*(J_1; \mathbb{Z})$ are *always* representable by effective algebraic cycles (i.e. subvarieties) in J_1?

From what we have said so far, the duals of

$$\Omega_1, \Omega_1 \wedge \Omega_1, \Omega_1 \wedge \Omega_1 \wedge \Omega_1$$

are all representable by subvarieties. In terms of a sympletic basis

(1.21) for $H_1(J_1; \mathbb{Z})$, we can write these homology classes in the form

$$\sum_{0\le i<j<k\le 3} \gamma_i \times \delta_i \times \gamma_j \times \delta_j \times \gamma_k \times \delta_k$$

$$2! \cdot \sum_{0\le j<k\le 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k \tag{2.5}$$

$$3! \sum_{j=0}^{3} \gamma_j \times \delta_j$$

It is a theorem of Matsusaka [3] and Hoyt that $\Sigma\, \gamma_i \times \delta_i$ is representable by an algebraic curve if and only if (J_1, Ω_1) is the Jacobian variety of that (possibly reducible) curve. So since not all principally polarized abelian varieties of dimension four are (products of) Jacobians, the cycle $\Sigma_{j=0}^{3}\, \gamma_j \times \delta_j$ is not in general representable by a subvariety.

LEMMA 2.6 (Mattuck): *There exist principally polarized abelian varieties* (J_1, Ω_1) *of dimension four such that any element of* $H_*(J_1; \mathbb{Z})$ *which is representable by an algebraic subvariety is a positive rational multiple of one of the cycles* (2.5).

PROOF: For elements of $H_6(J_1; \mathbb{Z})$, the lemma is simply the classical fact that the Picard number of a generic principally polarized abelian variety is one. By duality, therefore, the lemma is also true for elements of $H_2(J_1; \mathbb{Z})$. We must only examine $H_4(J_1; \mathbb{Z})$. Suppose the lemma is false. Then for each family (1.1) there will exist an element

$$\alpha \in H_4(J_z; \mathbb{Z})$$

for each $z \neq 0$ and a three-dimensional closed analytic subvariety

$$S \subseteq J$$

such that:

(i) $$S_z = (J_z \cdot S)$$

represents the homology class α;

(ii) α is not an integral multiple of

$$\sum_{j<k} \gamma_j \times \delta_j \times \gamma_k \times \delta_k.$$

(This is because of Theorem 1.33 and the fact that the set of algebraic cycles in $\mathbb{P}_{(N^4-1)}$ of fixed degree forms a finite union of irreducible algebraic families of cycles.) Then just as in (1.19)–(1.29), we can conclude that there exists a finite set of cycles

$$\alpha_1, \ldots, \alpha_r \in H_4(J_0; \mathbb{Z})$$

and positive integers $p_1, \ldots, p_r$ such that:

(i)
$$(0,0)\cdot\alpha = \sum_{i=1}^{r} p_i\alpha_i\,;$$

(ii) each α_i is represented by an irreducible algebraic subvariety

$$S_i \subseteq J_0.$$

Let $\tilde{S}_i \subseteq \tilde{J}_0$ be such that

$$\nu(\tilde{S}_i) = S_i$$

(see (1.25)). If M has Picard number 1, then under

$$\mu : \tilde{J}_0 \to M$$

$\tilde{S}_i$ must go to an algebraic cycle whose homology class is a positive multiple of

$$\sum_{1\le j<k\le 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k \in H_4(M\,;\mathbb{Z}).$$

We can therefore conclude that

$$\alpha_i = \gamma_0 \times \beta_i + r_i \sum_{1\le j<k\le 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k$$

for some $r_i \ge 0$, and so by (i) above and the topological description of the degeneration $\alpha \mapsto (0,0)\cdot\alpha$ given in (1.19)–(1.29), we have that

$$\alpha = \gamma_0 \times \beta + r \sum_{1\le j<k\le 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k$$

for some $\beta \in H_3(J_z\,;\mathbb{Z})$ and some $r \ge 0$. Now we can arrange so that for some $z_0 \ne 0$, the period matrix for J_{z_0} is given by

$$\Omega_{z_0} = i\cdot(\text{identity matrix}) + \Omega'$$

where each entry in Ω' has small absolute value and each entry in

$$(\Omega_{z_0})^{-1} + i\cdot(\text{identity matrix})$$

has small absolute value. Therefore for each $j = 1, 2, 3$, J_{z_0} fits into a family (1.1) in which M has Picard number one and γ_j plays the role of γ_0. Therefore by elementary algebra

$$\alpha = r \sum_{1\le j<k\le 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k.$$

This completes the proof of Lemma 2.6.

The above lemma reduces the search for the answer to the question posed in (2.4) to the homology classes

$$\text{(2.7)}\quad \text{(i)}\qquad r\sum_{j=0}^{3}\gamma_j\times\delta_j,$$

$$\text{(ii)}\qquad s\sum_{0\le j<k\le 3}\gamma_j\times\delta_j\times\gamma_k\times\delta_k.$$

We have seen that, if $r=1$, the cycle (2.7) (i) is, in general, not representable by a subvariety. By an as yet unpublished result of A. Beauville, *every* principally polarized abelian variety of dimension four is the *Prym variety* associated to a two-sheeted covering of a (possibly singular) algebraic curve. The image of this two-sheeted cover in its Prym variety has homology class (2.7) (i) where $r=2$. Thus the only cycles (2.7) (i) which remain in doubt are those for which r is odd and greater than 1. Similarly, since $\Omega_1\wedge\Omega_1$ has as its dual the cycle (2.7) (ii) with $s=2$, the only cycles (2.7) (ii) which remain in doubt are those for which s is odd. Our next project is to eliminate the possibility $s=1$.
Suppose

$$\text{(2.8)}\qquad \Gamma=\sum_{0\le j<k\le 3}\gamma_j\times\delta_j\times\gamma_k\times\delta_k$$

is representable by a subvariety for all principally polarized abelian varieties of dimension four. Then in general the representing subvariety must be irreducible since no element in fourth homology which is not a positive integral multiple of Γ is generically representable. Therefore, by the general theory of the Chow ring of $\mathbb{P}_{(N^4-1)}$, [6], there must exist for each sufficiently general family (1.1) a closed, irreducible, three-dimensional analytic subvariety

$$\text{(2.9)}\qquad S\subseteq J$$

such that:

(i) if $z\neq 0$

$$(J_z\cap S)=S_z^{(1)}\cup\cdots\cup S_z^{(s)}$$

where each $S_z^{(j)}$ represents the homology class Γ,

(ii) for almost all z, the varieties $S_z^{(j)}$ are all distinct and irreducible.

For such a general family (1.1), consider the set

$$S'=\cup\{S_z^{(1)}: z \text{ real}, >0\}.$$

The topological closure $\overline{S'}$ of S' intersects J_0 in a union

$$\text{(2.10)}\qquad S_0=S_{(1)}\cup\cdots\cup S_{(r)}\subseteq J_0$$

of irreducible analytic subvarieties of dimension two. Just as in

(1.19)–(1.29), if $\tilde{S}_{(i)}$ is a subvariety of $\tilde{J}_0$ such that (counting multiplicities)

$$\nu(\tilde{S}_{(i)}) = S_{(i)}$$

and if $\tilde{S}_{(i)}$ has homology class α_i, then

$$\sum_{i=1}^{r} m_i\alpha_i = \sum_{1\leq j<k\leq 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k + \mu^{-1}\left(\sum_{j=1}^{3} \gamma_j \times \delta_j\right) \tag{2.11}$$

for some $m_i > 0$. If the double locus M of J_0 is chosen suitably generally, then for each i, the homology class of $(M^0 \cdot \tilde{S}_{(i)})$ is a non-negative multiple of $\Sigma_{j=1}^{3} \gamma_j \times \delta_j$ and the homology class of $\mu(\tilde{S}_{(i)})$ is a non-negative multiple of $\Sigma_{1\leq j<k\leq 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k$. Then the only possibilities in (2.11) are:

(i) $r = 1$ and $m_1 = 1$;

(ii) $r = 2$, $m_1 = m_2 = 1$ and

$$\alpha_1 = \sum_{1\leq j<k\leq 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k$$

$$\alpha_2 = \mu^{-1}\left(\sum_{j=1}^{3} \gamma_j \times \delta_j\right).$$

Assume that possibility (ii) holds for a general family (1.1). It is impossible that $S_{(1)} \subseteq M$, the double locus, because the multiplicity of any component of $(S \cap J_0)$ which lies in M must be greater than one. Thus

$$S_{(1)} \subseteq (J_0 - M)$$

and so

$$\tilde{S}_{(1)} \subseteq \tilde{J}_0 - (M^0 \cup M^\infty).$$

This implies that the bundle

$$\mu : \tilde{J}_0 \to M$$

is trivial when restricted to the theta-divisor Σ of M, since (up to translation)

$$\mu(\tilde{S}_{(1)}) = \Sigma$$

and

$$((\mu^{-1}(\text{point})) \cdot \tilde{S}_{(1)}) = 1$$

in $\mu^{-1}(\Sigma)$. Since the mapping

$$\text{Pic}^0(M) \to \text{Pic}^0(\Sigma)$$

is an isomorphism for non-singular Σ, possibility (ii) is ruled out unless $\tilde{J}_0$ is the trivial bundle over M which is in general not the case. Thus we can conclude that for our general family:

(2.12) S_0 is irreducible and lifts to a cycle $\tilde{S}_0$ in $\tilde{J}_0$ with homology class

$$\mu^{-1}\left(\sum_{j=1}^{3} \gamma_j \times \delta_j\right) + \sum_{1 \leq j < k \leq 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k.$$

Using (2.12), up to translation

$$\mu(\tilde{S}_0) = \Sigma \subseteq M.$$

Also the homology class of

$$(\tilde{S}_0 \cdot M^{\infty}) \text{ or } (\tilde{S}_0 \cdot M^{0})$$

in $H_2(M; \mathbb{Z})$ is

$$\sum_{j=1}^{3} \gamma_j \times \delta_j.$$

Also we can suppose that

$$(M, \Sigma) = (J(C), C^{(2)}),$$

the Jacobian of a non-singular, non-hyperelliptic curve C of genus three and that M has endomorphism ring $\mathbb{Z}$. Also $\mu(\tilde{S}_0 \cap M^{\infty})$ and $\mu(\tilde{S}_0 \cap M^{\infty})$ are homologous in the second symmetric product

$$C^{(2)} = \Sigma = \mu(\tilde{S}_0) \subseteq M.$$

With the help of the theorem of Matsusaka mentioned previously, we can therefore conclude that there are only two possibilities:

(i) there exist $P_0, P_\infty \in C$ such that

$$\mu(\tilde{S}_0 \cap M^{0}) = \{(P_0, P) \in C^{(2)} : P \in C\}$$

$$\mu(\tilde{S}_0 \cap M^{\infty}) = \{(P_\infty, P) \in C^{(2)} : P \in C\},$$

(ii) there exist $P_0, P_\infty \in C^{(2)}$ such that

$$\mu(\tilde{S}_0 \cap M^{0}) = \{(P_1, P_2) \in C^{(2)} : P + P_0 + P_1 + P_2$$

is a canonical divisor of C for some $P \in C\}$

$$\mu(\tilde{S}_0 \cap M^{\infty}) = \{(P_1, P_2) \in C^{(2)} : P + P_\infty + P_1 + P_2 \text{ is} \ldots\}.$$

Furthermore, since

$$\mu^{-1}(\text{point}) \cdot \tilde{S}_0 = 1$$

in $\mu^{-1}(\Sigma)$, $\tilde{S}_0$ gives a meromorphic section of the line bundle

$$\mu : (\mu^{-1}(\Sigma) - (M^{\infty} \cap \mu^{-1}(\Sigma))) \to \Sigma$$

whose associated divisor is

$$\mu(\tilde{S}_0 \cap M^0) - \mu(\tilde{S}_0 \cap M^{\infty}). \tag{2.13}$$

However since the natural mapping

$$\mathrm{Pic}^0(M) \to \mathrm{Pic}^0(\Sigma)$$

is bijective, our assumption that the cycle Γ in (2.8) is always representable by a subvariety forces a contradiction. For if we choose $B(0)$ in (1.4) sufficiently generally, the line bundle

$$\mu : (\tilde{J}_0 - M^{\infty}) \to M$$

will *not* restrict over Σ to a line bundle belonging to the two-parameter family of line bundles whose associated divisor has the form (2.13). Thus we have proved the following theorem.

THEOREM 2.14: *There exist principally polarized abelian varieties* (J_1, Ω_1) *of dimension four such that the cycle* $\Gamma = \Sigma_{0 \leq j < k \leq 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k$ *is not representable by a subvariety of* J_1.

Notice that the two possibilities for the families of divisors (2.13) correspond to the degenerations of $D_z^{(2)}$ and $-D_z^{(2)}$ respectively where D_z is a curve of genus four which acquires a double point as $z \mapsto 0$ and J_z is the Jacobian of D_z.

Left open is the very intriguing question as to the odd values of r and $s > 1$ in (2.7) for which the corresponding homology classes are always carried by subvarieties. Of course, if we find a value of r such that the cycle (2.7) (i) is carried by an algebraic curve D, the cycle (2.7) (ii) with $s = r^2$ will be carried by the image of $D^{(2)}$ in J_1 so the representability of the cycles (2.7) (i) and the cycles (2.7) (ii) are related. If it turns out, for example, that there exists an abelian variety J_1 on which no odd multiple of $\Sigma_{j=0}^{3} \gamma_j \times \delta_j$ is representable by a subvariety, one would have a new type of counter-example to the (false) Hodge conjecture over $\mathbb{Z}$, one that did not involve torsion cycles.

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