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MARK E. NOVODVORSKY

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NEW UNIQUE MODELS OF REPRESENTATIONS OF UNITARY GROUPS

Mark E. Novodvorsky

Introduction

Let π be an irreducible admissible representation of a unitary group G over a p-adic field, with sufficiently small anisotropic kernel. This paper introduces a family of Z-eigen-functionals on the space of π, Z a subgroup of G, which are unique up to a scalar multiplier. This is equivalent to the uniqueness of corresponding Whittaker models of the representation π, and allows one to obtain Euler products for some meromorphic functions attached to automorphic forms on orthogonal groups – the details of the latter will appear in a forthcoming paper.

When G is a split orthogonal group of type B_2 realized as the group of 4×4 symplectic matrices, the functionals of this paper are essentially the same as those considered by I. I. Pyatetsky-Shapiro and the author in [5]. The proof follows the modification of arguments of I. M. Gelfand and D. A. Kajdan [3] introduced by the author in [6].

1. The result and its reduction

Let k be a locally compact non-discrete disconnected field, ℓ an extension of it,

$$\lambda \to \bar{\lambda}, \quad \lambda \in \ell, \tag{1}$$

an involution of ℓ trivial on k only (particularly, $\ell : k \leq 2$). Suppose the characteristic of k different from 2.

Algebraic groups defined over k will be identified throughout this paper with the groups of their k-points in the natural locally compact topology. The linear span of a set of vectors $\{e_i : 1 \leq i \leq n\}$ in a linear space over ℓ will be denoted $\{e_i | 1 \leq i \leq n\}$.

Let G denote the group of all automorphisms of a Hermitian form $(\ldots, \ldots)$ on a linear space over ℓ with base $\{e_i : 1 \leq i \leq n\}$. Let e be a linear combination of vectors e_i, $r \leq i \leq n-r$, with coefficients from k.

Suppose

$$(2)\qquad \begin{cases} (e_i, e_j) = (e_j, e_i) \forall_{i,j} \leq n; \\ (e_i, e_j) = \begin{cases} 0 & \text{if} \quad i+j \neq n+1, \\ 1 & \text{if} \quad i+j = n+1, \end{cases} \quad 1 \leq i \leq r, 1 \leq j \leq n; \\ (e, e) \neq 0. \end{cases}$$

Define

$$(3)\qquad \begin{aligned} &T = \{g \in G : ge = e; ge_i = e_i \quad \text{if} \quad i \leq r \quad \text{or if} \quad i > n-r\}, \\ &U = \{g = (a_{ij}) \in G : a_{ii} = 1 \ \forall_i; a_{ij} = 0 \quad \text{if} \quad j < i \quad \text{or} \\ &Z = T \times U. \qquad\qquad\qquad\qquad \text{if} \quad r < i < j \leq n-r\}, \end{aligned}$$

T and U are subgroups of G; T normalizes U; hence, Z is a subgroup too.

Take a non-trivial character χ of the additive group of the field k and a character τ of the group T trivial on the subgroup of unimodular matrices from T if $\ell \neq k$ and of order 2 if $\ell = k$.

Define

$$(4)\qquad \begin{cases} \theta(u) = \chi\left(\operatorname{tr}\left(\sum_{i=1}^{r-1} a_{i,i+1} + (e_{n-r+1}, ue)\right)\right), & u = (a_{ij}) \in U; \\ \alpha(z) = \tau(t) \cdot \theta(u), \quad z = t \cdot u \in Z. \end{cases}$$

Then θ and α are characters of the subgroups U and Z, respectively.

Assume that *the dimension r' of a maximal isotropic subspace of the form $(\ldots, \ldots)$ does not exceed $r+1$.*

THEOREM: *For any irreducible admissible representation π of the group G the dimension* $\dim(\pi, \alpha)$ *of all linear Z-eigenfunctionals with character α is, at most,* 1.

Define an involution of the group G:

$$(5)\qquad \begin{aligned} &\tilde{g} = I^{-1}(\bar{a}_{ij})^{-1} I, \quad g = (a_{ij}) \in G, \\ &Ie_i = \begin{cases} (-1)^i e_i & \text{if} \quad i \leq r, \\ (-1)^{r+1} e_i & \text{if} \quad r < i \leq n-r, \\ (-1)^{n+1-i} e_i & \text{if} \quad n-r < i. \end{cases} \end{aligned}$$

(Note that this involution preserves the subgroups U and T and the characters θ and τ.)

The main technical step of the proof is the following:

LEMMA: *For every $g \in G$ there exist $z_1, z_2 \in Z$ such that either both z_1, z_2 are unipotent and*

$$(6)\qquad z_1 g \tilde{z}_2 = g, \quad \alpha(z_1 z_2) \neq 1,$$

or

$$z_1 g \tilde{z}_2 = \tilde{g}, \quad \alpha(z_1 z_2) = 1. \tag{7}$$

This lemma, in view of Theorem 1 from [6], provides the inequality:

$$\dim(\pi, \alpha) \times \dim(\pi^*, \hat{\alpha}) \leq 1, \quad \hat{\alpha}(z) = \alpha(\tilde{z}^{-1}), \tag{8}$$

where π^* denotes the contragradient representation to π.

Now the theorem follows directly from the inequality (8) and the isomorphism $\hat{\pi} \simeq \pi^*$, $\hat{\pi}(g) = \pi(\tilde{g}^{-1})$, proved by I. M. Gelfand and A. D. Kajdan [2] providing that the elements g and $\tilde{g}$ are conjugate in G for all $g \in G$; the last fact is an immediate consequence of Milnor's classification of conjugate classes of G in [4].

The theorem is true for finite field k too; in that case it follows directly from the lemma, which, together with its proof, is true for any field k of characteristic different from 2.

2. The proof of the lemma

1. Note that if (1.6) is true for $g \in G$, $z_1(g)$, $z_2(g) \in U$, then for $g' = z'gz''$, z', $z'' \in Z$ the same formulas are true when $z_1(g') = z'z_1(g)(z')^{-1}$ and $z_2(g') = \tilde{z}''z_2(g)(\tilde{z}'')^{-1}$; similarly, if (1.7) is true for $g \in G$, $z_1(g)$, $z_2(g) \in Z$, then it is true for $g' = z'gz''$, z', $z'' \in Z$, when $z_1(g') = \tilde{z}''z_1(g)(z')^{-1}$, $z_2(g') = z'z_2(g)(\tilde{z}'')^{-1}$. Therefore, *it is enough to consider any complete set of representatives of the double cosets* $Z \backslash G / Z$.

The definition of the groups U, T, and Z and the characters θ and α does not depend on the base $\{e_i : r < i \leq n - r\}$ of the subspace $\{e_i | r < i \leq n - r\}$. Changing it, if necessary, we may assume that

$$(e_i, e_j) = \begin{cases} 0 & \text{if} \quad i + j \neq n + 1, \\ 1 & \text{if} \quad i + j = n + 1, \end{cases} \quad 1 \leq i \leq r', 1 \leq j \leq n; \\ \quad\quad\quad\quad 0 \quad \text{if} \quad i \neq j, r' < i \leq n - r', 1 \leq j \leq n; \tag{1}$$

and that the restriction of the form $(\ldots, \ldots)$ to the subspace $\{e_i | r' < i \leq n - r'\}$ is anisotropic; if $r' = r + 1$ we may assume that vectors e and e_{r+1} are not orthogonal. Then the representatives can be chosen in the form

$$\begin{gathered} g = m_1 w m_2, \quad we_i = \omega_i e_j, \quad \omega_i \in \ell, \quad we_i = e_i \quad \text{if both} \quad e_i, \\ we_i \in \{e_j | r < j \leq n - r\} \\ m_1, m_2 \in M = \{g \in G : ge_i = e_i, \quad \text{if} \quad i \leq r \quad \text{or} \quad i > n - r\}. \end{gathered} \tag{2}$$

Take an element g of the form (2) for which (1.6) is wrong for all unipotent $z_1, z_2 \in Z$. Define a function $\varphi(i)$, $1 \leq i \leq n$, by the equality

$$we_i = \omega_i e_{\varphi(i)}. \tag{3}$$

2. *If* $i < r$ *then* $\varphi(i) \neq r$.

PROOF: The vectors $\{m_1 e_j : r < j \leq n - r\}$ form a base in the space $\{e_j | r < j \leq n - r\}$. Let

$$e = \sum_j \lambda_j m_1 e_j,\ \lambda_j \in \ell,\ r < j \leq n - r. \tag{4}$$

Since $(e, e) \neq 0$ there exists an index $j = s$ such that either $\lambda_s \neq 0$ and $(e_s, e_s) \neq 0$ or $\lambda_s \neq 0$ and $\lambda_{n+1-s} \neq 0$. Therefore, putting either $\bar{e} = m_1 e_s$ or $\bar{e} = m_1 e_{n+1-s}$, we obtain

$$g^{-1}\bar{e} \in \{e_j | 1 \leq j \leq n - r\},\ (\bar{e}, e) \neq 0. \tag{5}$$

Define

$$z_1 e_j = \begin{cases} e_j & \text{if} \quad 1 \leq j \leq r \quad \text{or} \quad n + 2 - r \leq j \leq n; \\ e_j - \lambda\bar{e} - \mu e_r,\ \mu = \lambda \cdot \bar{\lambda} \cdot (\bar{e}, \bar{e})/2, & \text{if} \quad j = n + 1 - r; \\ e_j + (e_j, \lambda\bar{e})e_r & \text{if} \quad r < j \leq n - r; \end{cases} \tag{6}$$

$$\tilde{z}_2 = g^{-1} z_1^{-1} g,\ \lambda \in \ell.$$

Explicit formulas for $\bar{e}$, z_1, and z_2 allow one to check that if $we_i = e_r$, $i < r$, then $z_2 \in U$ and $\theta(z_2) = 1$; since $\theta(z_1) = \chi(\operatorname{tr}(\lambda\bar{e}, e))$, the elements g, z_1, z_2 satisfy (1.6) for a suitable λ, which contradicts our choice of g.

3. *If* $\varphi(i) > r$, $i \leq r$, *and* $i' < i$, *then* $\varphi(i') > r$.

PROOF: Otherwise there exist $i \leq r$, $i \geq 2$, such that $\varphi(i) > r$ and $\varphi(i - 1) \leq r$; hence, $\varphi(i - 1) < r$. Then g would have satisfied (1.6) when

$$z_2 e_j = \begin{cases} e_j & \text{if} \quad j \neq i, j \neq n + 2 - i, \\ e_j + \lambda e_{j-1}, & \text{if} \quad j = i, \\ e_j - \bar{\lambda} e_{j-1} & \text{if} \quad j = n + 2 - i; \end{cases} \tag{7}$$

$$z_1 = g(\tilde{z}_2)^{-1} g^{-1},\ \chi(\operatorname{tr}\lambda) \neq 1,\ \lambda \in \ell.$$

4. *If the set* $\{i : i \leq r,\ r < \varphi(i) \leq n - r\}$ *is empty*, then the set $\{i : i > n - r,\ r < \varphi(i) \leq n - r\}$ is empty too, and, consequently, the set $\{i : r < i \leq n - r\}$ is invariant, hence, stable under φ (cf. no. 1). Therefore,

$$g = mw = wm,\ m = m_1 \cdot m_2;\ \tilde{g} = \tilde{w}\tilde{m} = \tilde{m}\tilde{w}. \tag{8}$$

Arguments of R. Steinberg [2], proof of Theorem 49, show that $w = \tilde{w}$. Since

$$(me, e) = (e, m^{-1}e) = \overline{(m^{-1}e, e)} = (\tilde{m}e, e), \tag{9}$$

Witt's theorem (cf. [1], chap. 9, no. 3) provides an element

$$t \in T,\ tme = \tilde{m}e = \tilde{m}\tilde{t}e. \tag{10}$$

Changing g into tg (cf. no. 1) we obtain

$$me = \tilde{m}e. \tag{11}$$

Now g *satisfies* (1.7) *with*

$$z_1 = \tilde{m}m^{-1},\ z_2 = id. \tag{12}$$

(When $k = \ell$, $\tilde{m} = m^{-1}$, so z preserves both vectors e and me and coincides with m^{-2} on their orthogonal complement; if $k \neq \ell$, $\det z_1 = 1$; in both cases our restrictions on the character τ guarantee the equality $\alpha(z_1 z_2) = 1$.)

Now consider the case when *the set* $\{i : i \leq r,\ r < \varphi(i) \leq n - r\}$ *is non-empty*; then *it consists of only one element.* Denote it i_0. Necessarily $r' = r + 1$, and either $\varphi(i_0) = r + 1$ or $\varphi(i_0) = n - r$.

5. *If* $\varphi(i) > r$ *and* $i \geq i_0$, *then* $i = i_0$. Otherwise, in view of no. 3, $\varphi(i_0 + 1) > n - r$. Then g satisfies (1.6) with

$$z_2 e_j = \begin{cases} e_j & \text{if } \ j \neq i_0 + 1, j \neq n + 1 - i_0; \\ e_j + \lambda e_{j-1} & \text{if } \ j = i_0 + 1; \\ e_j - \bar{\lambda} e_{j-1} & \text{if } \ j = n + 1 - i_0; \end{cases} \tag{13}$$

$$z_1 = g(\tilde{z}_2)^{-1} g^{-1},\ \lambda \in \ell,\ \chi(\operatorname{tr} \lambda) \neq 1,$$

which contradicts our choice of g.

6. *If one of the formulas* (1.6), (1.7) *is valid for an element* $g \in G$, *it is valid for* $\tilde{g}$ *too.*

PROOF: Applying the antiautomorphism (1.5) to the equations (1.6) and (1.7) we obtain the statement immediately with

$$z_1(\tilde{g}) = z_2(g),\ \widetilde{z_2(\tilde{g})} = \widetilde{z_1(g)}. \tag{14}$$

Applying the statements of nos. 2–5 to $\tilde{g}$ we obtain:

$\varphi(r) \geq r$ (cf. no. 2);

$\varphi^{-1}(i) > n - r$ *if and only if* $i < i_0$ (cf. no. 3 and no. 5);
$\varphi^{-1}(i_0)$ *equals either* $r' = r + 1$ *or* $n - r$ (cf. no. 4).

7. *Matrices* m_1 *and* m_2 *can be chosen so that* $w = \tilde{w}$.

PROOF: Changing, if necessary, m_1 and m_2 obtain:

$$w(e_{r+1}) = -w^{-1}(e_{r+1}) = e_{i_0}. \tag{15}$$

Now the subspaces $\{e_i | i < i_0 \text{ or } i > n+1-i_0\}$ and $\{e_i | i_0 < i \leq r \text{ or } n-r < i \leq n-i_0\}$ are invariant under w. Applying Steinberg's arguments (loc. cit.) to the restrictions of w on these subspaces, we complete the proof.

8. Suppose w, m_1, and m_2 are chosen as in no. 7. If $i_0 < r$, then in view of no. 2 $\varphi(r) = r$, and $\varphi(n-r+1) = \varphi(n-r+1)$. Consequently, g satisfies (1.6) with

$$z_1 e_i = \begin{cases} e_i & \text{if} \quad i \neq n-r, i \neq n+1-r; \\ e_i + \lambda e_{r+1} & \text{if} \quad i = n+1-r; \\ e - \bar{\lambda} e_r & \text{if} \quad i = n-r; \end{cases} \tag{16}$$

$$z_2 = g^{-1} z_1 g$$

for a suitable $\lambda \in \ell$. Hence, $i_0 = r$.

If

$$(m_1 e_{r+1}, e) \neq 0. \tag{17}$$

Witt's theorem (Bourbaki, loc. cit.) provides an element $t \in T$ such that

$$t m_1 e_{r+1} = \lambda e_{r+1}, \ \lambda \in \ell. \tag{18}$$

Replacing g by tg we may suppose m_1 to be identity. If, in addition,

$$(m_2^{-1} e_{r+1}, e) \neq 0, \tag{19}$$

similar arguments lead us to equalities

$$g = wm, \ m e_{r+1} = \mu e_{r+1}, \ m e_{n-r} = \mu^{-1} e_{n-r}, \ \mu \in \ell. \tag{20}$$

Moreover, $\mu = 1$. Otherwise, g would have satisfied (1.6) with z_1, z_2 given by (16) for a suitable $\lambda \in \ell$ (remember that we assumed $(e_{r+1}, e) \neq 0$, cf. no. 1)).

Denote e' the orthogonal projection of the vector e on the subspace $\{e_i | r+1 < i < n-r\}$. According to our assumption (cf. no. 1) this subspace is anisotropic. Therefore, $(e', e') \neq 0$, and the arguments similar to those of no. 4 prove (1.7) for the element g.

If

$$(m_2^{-1} e_{r+1}, e) = 0 \tag{21}$$

(while (17) is valid, i.e. $m_1 = id$), g satisfies (1.6) with z_1, z_2 given by (16) again.

Similarly, if

$$(m_1 e_{r+1}, e) = 0, \; (m_2^{-1} e_{r+1}, e) \neq 0. \tag{22}$$

At last, let both

$$(m_j e_{r+1}, e) = 0, \; j = 1, 2. \tag{23}$$

Then the subspace $\{e_i | r+1 < i < n-r\}$ contains a vector $\bar{e}$ for which

$$(m_1 \bar{e}, e) \neq 0 \tag{24}$$

(otherwise e would have been proportional to $m_1 e_{r+1}$, which is impossible, for $(e, e) \neq 0$). Therefore, g satisfies (1.6) with

$$m_1 z_1 (m_1)^{-1} e_i = \begin{array}{l} e_i \quad \text{if} \quad i \leq r \quad \text{or} \quad i > n+1-r \\ e_i + \lambda \bar{e} - \mu e_r, \; \mu = \lambda \cdot \bar{\lambda} \cdot (\bar{e}, \bar{e})/2, \quad \text{if} \quad i = n+1-r \\ e_i - (e_i, \lambda \bar{e}) e_r \quad \text{if} \quad r < i \leq n-r, \end{array} \tag{25}$$

$$\tilde{z}_2 = g^{-1} z_1^{-1} g$$

for a suitable $\lambda \in \ell$.

This completes the proof of the lemma and, therefore, of the theorem.

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Purdue University
Division of Mathematical Sciences
West Lafayette, Indiana 47907