

COMPOSITIO MATHEMATICA

C. H. HOUGHTON

Restricted subgroups of wreath products of groups

Compositio Mathematica, tome 33, n° 2 (1976), p. 209-225

<http://www.numdam.org/item?id=CM_1976__33_2_209_0>

© Foundation Compositio Mathematica, 1976, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (<http://http://www.compositio.nl/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

RESTRICTED SUBGROUPS OF WREATH PRODUCTS OF GROUPS

C. H. Houghton

1. Introduction

Hartley [5] investigated the conjugacy classes of baseless subgroups of wreath products of groups, that is, subgroups which intersect the base group trivially. In [8], it was shown that his results are related to the theory of ends. Here we consider the conjugacy classes of those subgroups of a wreath product whose intersection with the base group consists of functions with support of size less than some fixed infinite cardinal.

The wreath product $W = A \operatorname{Wr} B$ of groups A and B may be taken as the split extension by B of the left B -group $F = A^B$ of functions from B to A , with $x(fg) = (xf)(xg)$ and $x({}^b f) = (xb)f$, for $f, g \in F$, $x, b \in B$. Thus W consists of all pairs fb , with $f \in F$, $b \in B$, and $(fb)(gc) = (f {}^b g)bc$, for $f, g \in F$ and $b, c \in B$; we shall assume throughout that A and B are non-trivial. Let $\sigma(f)$ denote the support of $f \in F$. For an infinite cardinal α , we define F_α to consist of those $f \in F$ such that $|\sigma(f)| < \alpha$ and we put $W_\alpha = BF_\alpha \leq W$. When $\alpha = \aleph_0$, F_α consists of the functions with finite support and W_α is the restricted wreath product $A \operatorname{wr} B$ of A and B .

A subgroup L of W will be called α -restricted if $L \cap F \leq F_\alpha$; in the case $\alpha = \aleph_0$, we simply say that L is restricted. Clearly all subgroups of W_α are α -restricted and the question we consider is when an α -restricted subgroup L of W is conjugate in W to a subgroup of W_α .

We define the B -image of a subgroup L of W to be the image of L under the natural map from W to B . Our first result shows that if the B -image C of an α -restricted subgroup L is sufficiently small, then L is conjugate to a subgroup of W_α . Let β be the least cardinal such that α is the sum of β cardinals each $< \alpha$.

THEOREM (A): *If $\alpha > \aleph_0$ and $|C| \leq \beta$ or if $\alpha = \aleph_0$ and C is countable and locally finite, then all α -restricted subgroups of $W = A \operatorname{Wr} B$ with B -image C are conjugate in W to subgroups of W_α .*

The remaining results are concerned with the case $\alpha = \aleph_0$ and are related to work of Farrell [3, 4] and Bieri [1]. We show that if C has a normal finitely presented infinite subgroup N of infinite index, then, in most cases, the problem reduces to finding the number of ends of N and C/N . We summarise these results, using $e(G)$ to denote the number of ends of the group G .

THEOREM (B): *Every restricted subgroup of $W = A \operatorname{Wr} B$ with B -image C is conjugate to a subgroup of $A \operatorname{wr} B$ if C has a finitely generated free subgroup of finite index or C has a finitely presented normal subgroup N of infinite index such that either $e(N) = 1$, or $e(N) = 2$ and $e(C/N) = 1$, or $e(N) = \infty$, $e(C/N) = 1$ and C is finitely generated. There exist restricted subgroups with B -image C which are not conjugate to subgroups of $A \operatorname{wr} B$ if C has a finitely presented normal subgroup N with $e(N) > 1$ and $e(C/N) > 1$. For a polycyclic by finite group C , every restricted subgroup with B -image C is conjugate to a subgroup of $A \operatorname{wr} B$ if and only if C has Hirsch number 2.*

If $A^{(B)}$ denotes the B -group of functions from B to A with finite support, then $A \operatorname{wr} B$ is the split extension of $A^{(B)}$ by B . The previous results imply the following theorem.

THEOREM (C): *Let A be any non-trivial group. All extensions of $A^{(B)}$ by B split if B is countable and locally finite or is finitely generated free by finite. If B is polycyclic by finite, all extensions of $A^{(B)}$ by B split if and only if B has Hirsch number different from 2.*

We make use of the theory of groupoids, details of which may be found in Higgins [6]. Our definition of the wreath product has been chosen to correspond to the natural multiplication in the covering groupoid associated with a permutation representation of a given group.

2. The general case

Suppose C acts as a group of permutations of a set X . The associated covering groupoid of C is the set $X \times C$ with vertex set X and multiplication $(x, c)(xc, d) = (x, cd)$ for $x \in X$, $c, d \in C$. A map θ

from $X \times C$ to a group A will be called an almost homomorphism if, for each pair $c, d \in C$, $(x, c)\theta(xc, d)\theta = (x, cd)\theta$, for almost all $x \in X$; that is, the exceptions form a set of cardinal less than α . All homomorphisms are almost homomorphisms; in particular, this applies to the trivial map. The almost homomorphisms θ and ϕ are defined to be equivalent if there is a map γ from X to A such that, for each $c \in C$, $(x, c)\phi = (x\gamma)^{-1}(x, c)\theta(xc\gamma)$, for almost all $x \in X$. We note that if θ is any almost homomorphism and γ is any map from X to A , then the corresponding ϕ will be an almost homomorphism. Also, for any almost homomorphism θ , we have $(x, 1)\theta = 1$, for almost all $x \in X$, and so θ is equivalent to an almost homomorphism ϕ such that $(x, 1)\phi = 1$, for all $x \in X$.

Let W be the wreath product of A and C relative to the action of C on X ; that is, W consists of all pairs fc , with f in the group F of functions from X to A and $c \in C$, and the multiplication is given by $(fc)(gd) = (f \circ g)cd$, where $x(\circ g) = (xc)g$. As before, F_α denotes the subgroup of F consisting of those f such that $|\sigma(f)| < \alpha$ and a subgroup L of W is α -restricted if $L \cap F \leq F_\alpha$. If $f, g \in F$ are congruent modulo F_α then they differ on a set of cardinal $< \alpha$. We say f is almost equal to g and write $f =^a g$. We shall consider the case where C acts semiregularly on X , that is, the stabiliser of each point is trivial and so the representation can be thought of as a sum of regular representations.

THEOREM (1): *Let C act semiregularly on the set X and let W be the wreath product of A and C relative to X . The conjugacy classes of α -restricted subgroups containing F_α and having C as image under the projection from W to C correspond bijectively to the equivalence classes of almost homomorphisms from $X \times C$ to A .*

Let $W_1 = A \text{ Wr } B$ be the standard wreath product of A and B and suppose C is a subgroup of B . Every α -restricted subgroup of W_1 with B -image C is conjugate to a subgroup of the α -restricted wreath product W_α of A and B if and only if all almost homomorphisms from $B \times C$ to A are equivalent.

PROOF: Given an almost homomorphism θ from $X \times C$ to A , we define $f_c \in F$, for each $c \in C$, by $xf_c = (x, c)\theta$. For $c, d \in C$, we have $x(f_c \circ f_d) = (x, c)\theta(xc, d)\theta$ and $xf_{cd} = (x, cd)\theta$ and so $f_c \circ f_d =^a f_{cd}$. Let $R = R(\theta)$ be the subgroup of W generated by all $f_c c$, with $c \in C$, and by F_α . Now $f_c \circ f_d F_\alpha = f_{cd} F_\alpha$ so $(f_c c)(f_d d) F_\alpha = f_{cd} cd F_\alpha$ and $R = \{f_c c : c \in C\} F_\alpha$. Hence $R \cap F = F_\alpha$ and $R = R(\theta)$ is an α -restricted subgroup of W . Suppose ϕ is an almost homomorphism equivalent to θ and so, for each $c \in C$, $(x, c)\phi = (x\gamma)^{-1}(x, c)\theta(xc\gamma)$, for almost all

$x \in X$. Putting $xh_c = (x, c)\phi$, we have $h_c = {}^a\gamma^{-1}f_c{}^c\gamma$ and $(h_c c)F_\alpha = (\gamma^{-1}(f_c c)\gamma)F_\alpha$ so

$$R(\phi) = \{h_c c : c \in C\}F_\alpha = \gamma^{-1}\{f_c c : c \in C\}F_\alpha \gamma = \gamma^{-1}R(\theta)\gamma.$$

Suppose R is an α -restricted subgroup of W containing F_α and having image C under the projection map from W to C . If T is a transversal of the cosets of F_α in R , then $T = \{f_c c : c \in C\}$ and $f_c {}^c f_d = {}^a f_{cd}$. Defining $(x, c)\theta = xf_c$ gives an almost homomorphism from $X \times C$ to A . We note that θ depends on the choice of transversal as well as on R . Suppose S is a subgroup of W conjugate to R . We shall show that if ϕ is an almost homomorphism associated with S then ϕ is equivalent to θ . For some $fb \in W$, we have $S = fbRb^{-1}f^{-1}$ and

$$bRb^{-1} = bTb^{-1}F_\alpha = \{{}^b f_c (bcb^{-1}) : c \in C\}F_\alpha = \{{}^b f_c d : d \in C\}F_\alpha,$$

where $d = bcb^{-1}$. Putting $e = b^{-1}$,

$${}^b f_c = {}^a {}^b f_{edb} = {}^a {}^b f_e {}^b e f_d {}^b e d f_b = {}^a {}^b f_e f_d {}^d f_b.$$

Also $f_b {}^b f_e = {}^a 1$, so ${}^b f_c = {}^a f_b^{-1} f_d {}^d f_b$ and $bRb^{-1} = f_b^{-1}\{f_d d : d \in C\}f_b F_\alpha$. Thus $S = f_b^{-1}Rf_b f^{-1}$. Putting $g = f_b f^{-1}$ and choosing a transversal U for F_α in S , we have $U = \{k_c c : c \in C\}$ with $k_c = {}^a g^{-1}f_c {}^c g$. Taking $(x, c)\phi = xk_c$, we have, for each $c \in C$, $(x, c)\phi = (xg)^{-1}(x, c)\theta(xc)g$, for almost all $x \in X$, and hence ϕ is equivalent to θ . Thus the conjugacy classes of restricted subgroups containing F_α correspond to the equivalence classes of almost homomorphisms.

Suppose every almost homomorphism from $B \times C$ to A is equivalent to the trivial one and let R be an α -restricted subgroup of W_1 with B -image C . Then RF_α is contained in the wreath product W of A and C with $X = B$ and therefore RF_α is conjugate to a subgroup of W_α and so also is R . Conversely, if θ is an almost homomorphism from $B \times C$ to A , then there is a corresponding α -restricted subgroup R of W containing F_α . Now B normalises W_α so if R is conjugate to a subgroup of W_α , we have $R^b \leq W_\alpha$ and hence $R^f \leq W_\alpha$, for some $f \in F, b \in B$. Since R^f is in the conjugacy class of R in W , the first part implies that θ is equivalent to the trivial homomorphism.

We note that if the subgroup R above is baseless, that is, $R \cap F = 1$, then $f_c {}^c f_d = f_{cd}$, for all $c, d \in C$, and then the corresponding θ is a homomorphism. The next result shows that Theorem C is a consequence of Theorems A and B.

THEOREM (2): *Let F_α be the B -group of functions f from B to A with $|\sigma(f)| < \alpha$ and $x({}^b f) = (xb)f$, for all $x, b \in B$. All extensions of F_α by B split if and only if all α -restricted subgroups of $W = A \text{ Wr } B$ which contain F_α and have B -image B are conjugate.*

PROOF: Let K be an extension of F_α by B and let ρ be the natural map from K to B . For each $b \in B$, we may choose $b\tau \in K$ such that $b\tau\rho = b$ and $(b\tau)k(b\tau)^{-1} = {}^bk$, for all $k \in K$. We define $\omega: K \rightarrow W$ by $k\omega = f_k k\rho$, where $f_k \in F$ and $bf_k = 1((b\tau)k((b \cdot k\rho)\tau)^{-1})$, for $b \in B$; we note that the last expression is the value at 1 of some element of F_α . If $m \in K$ then

$$k\omega m\omega = f_k {}^{k\rho}f_m(km)\rho$$

and

$$b(f_k {}^{k\rho}f_m) = 1((b\tau)k((b \cdot k\rho)\tau)^{-1})1((b \cdot k\rho)\tau)m((b \cdot k\rho \cdot m\rho)\tau)^{-1} = bf_{km}.$$

Thus ω is a homomorphism. If $k \in F_\alpha$ then $k\omega = f_k$ with $bf_k = 1((b\tau)k(b\tau)^{-1}) = 1({}^bk) = bk$, for all $b \in B$, so $k\omega = k$. Then ω is injective and K is isomorphic to a subgroup L of W with B -image B and $L \cap F = F_\alpha$. We call such a subgroup a full α -restricted subgroup of W and note that we have shown that any extension of F_α by B is isomorphic to one of these.

A full α -restricted subgroup L of W which is conjugate to $W_\alpha = BF_\alpha$ is a split extension of F_α . Suppose conversely that L splits as an extension of F_α by B . Then L has a subgroup $M = \{f_b b : b \in B\}$, with $f_b \in F$, and M is baseless, that is, $M \cap F = 1$. From Lemma 3.2(i) of [5], M is conjugate in W to B and so L is conjugate to W_α . (This also follows from Lemma 3 below, since the θ corresponding to M is a homomorphism.) Thus all extensions split if and only if all full α -restricted subgroups are conjugate.

We now explain the relation between our work and that of Farrell [3, 4] and Bieri [5]. For abelian A , all extensions of F_α by B split if and only if $H^2(B, F_\alpha) = 0$. Suppose R is a commutative ring with identity, A is a free R -module and $\alpha = \aleph_0$. Then the B -group F_α of functions from B to A with finite support is B -isomorphic to $A \otimes_R RB$. Thus $H^2(B, F_\alpha)$ is isomorphic to $H^2(B, A \otimes_R RB)$, which is the group studied by Farrell and Bieri. Farrell considers the case where R is a field and obtains results which are not included in the results proved here for general A .

We now consider the classification of the equivalence classes of almost homomorphisms from $X \times C$ to A , under the assumption that C acts semiregularly on X .

LEMMA (3): Suppose θ is an almost homomorphism from $X \times C$ to A and G is a subgroupoid of $X \times C$. If the restriction of θ to G is a homomorphism, there exists $\gamma: X \rightarrow A$ such that $(x, c)\theta = (x\gamma)^{-1}(xc)\gamma$, for $(x, c) \in G$, and θ is equivalent to an almost homomorphism trivial on G .

PROOF: Let U be a subset of X containing one vertex from each connected component of G . Since C acts semiregularly, all vertex groups of G are trivial and there is a unique element joining one vertex to another in the same component of G . If x is a vertex of G then $x = ud$ for a unique $(u, d) \in G$ with $u \in U$. For $c \in C$, with $(x, c) \in G$, we have $(x, c) = (x, d^{-1})(u, dc) = (u, d)^{-1}(u, dc)$, so $(u, dc) \in G$ and $(x, c)\theta = ((u, d)\theta)^{-1}(u, dc)\theta$. Taking $x\gamma = (u, d)\theta$ gives $(x, c)\theta = (x\gamma)^{-1}(xc)\gamma$, for $(x, c) \in G$. Let $x\gamma$ be defined in this way for all vertices of G and let $x\gamma = 1$ for all other $x \in X$. If $(x, c)\phi = (x\gamma)(x, c)\theta((xc)\gamma)^{-1}$, then ϕ is an almost homomorphism equivalent to θ and trivial on G .

If D is a subgroup of C , we shall refer to a connected component of $X \times D$ as a D -sheet of $X \times C$. Thus each D -sheet consists of all (xd, e) , with $d, e \in D$ and x a fixed vertex of X . We recall that β was defined as the least cardinal such that α is the sum of β cardinals each $< \alpha$.

LEMMA (4): *Suppose $|C| < \beta$ and θ is an almost homomorphism from $X \times C$ to A . Then θ is a homomorphism on almost all C -sheets of $X \times C$, that is, there exists a subset T of X with $|T| < \alpha$, such that the restriction of θ to $xC \times C$ is a homomorphism for all $x \in X \setminus TC$.*

PROOF: For $c, d \in C$, let $X(c, d)$ be the set of all $x \in X$ such that $(x, c)\theta(xc, d)\theta \neq (x, cd)\theta$. Then $|X(c, d)| < \alpha$ and if T is the union of all $X(c, d)$, with $c, d \in C$, then $|T| < \alpha$. Clearly the restriction of θ to $(X \setminus TC) \times C$ is a homomorphism.

Theorem 1 shows that the next result implies Theorem A.

THEOREM (5): *Suppose $\alpha > \aleph_0$ and $|C| \leq \beta$ or $\alpha = \aleph_0$ and C is countable and locally finite. If C acts semiregularly on X , all almost homomorphisms from $X \times C$ to A are equivalent.*

PROOF: Considering α as an ordinal which is not equivalent to any of its predecessors, our assumption implies that we can express C as $\bigcup_{i < \beta} C_i$, with $C_i \leq C_j$ for $i \leq j$, $C_\lambda = \bigcup_{i < \lambda} C_i$ for λ a limit ordinal, and $|C_i| < \beta$ for all i . Let θ be an almost homomorphism from $X \times C$ to A . For $x \in X$, let $J(x)$ be the set of ordinals $j < \beta$ such that the restriction of θ to the C_j -sheet containing x is not a homomorphism. If $J(x)$ is non-empty, it has a first element $j(x)$. For a limit ordinal λ , if θ is a homomorphism on all C_j -sheets containing x with $j < \lambda$, then θ is a homomorphism on the C_λ -sheet containing x . Thus $j(x)$ is not a limit

ordinal and hence has an immediate predecessor. So for each $x \in X$ there is a maximal ordinal i such that θ restricted to the C_i -sheet containing x is a homomorphism; if $J(x)$ is empty, $i = \beta$. Suppose y is another element of X with corresponding maximal ordinal j . If the maximal sheets containing x and y intersect, then $xC_i \cap yC_j \neq \emptyset$ so, assuming $i \leq j$, we have $xC_j = yC_j$ and hence $i = j$. Thus X is partitioned into maximal subsets xC_i such that θ is a homomorphism on the C_i -sheet with vertex set xC_i . If G denotes the subgroupoid which is the union of all these sheets, then θ is a homomorphism on G . Let X_i be the set of all vertices of X not contained in a C_i -sheet of G . From Lemma 4, $|X_i| < \alpha |C_i| = \alpha$. From Lemma 3, we can choose γ so that $(x, c)\theta = (x\gamma)^{-1}(xc)\gamma$, for $(x, c) \in G$. Then for $c \in C_i$, $(x, c)\theta = (x\gamma)^{-1}(xc)\gamma$ for $x \in X \setminus X_i$ and hence for almost all $x \in X$. So θ is equivalent to the trivial homomorphism.

3. The case $\alpha = \aleph_0$

From now on we restrict our attention to the case $\alpha = \aleph_0$. Although some of the lemmas hold for a general cardinal, we can only make significant deductions in this case. We begin by describing the results we need from the theory of ends. For further details, see Cohen [2].

A subset S of a group G is almost invariant if $Sg \cap (G \setminus S)$ is finite, for all $g \in G$. The number of ends of G , denoted by $e(G)$, is the supremum of the number of parts in a partition of G into infinite almost invariant subsets. Then $e(G) = 0, 1, 2$, or ∞ , with $e(G) = 0$ if and only if G is finite and $e(G) = 2$ if and only if G is infinite cyclic by finite. Finitely generated groups G with $e(G) = \infty$ have been characterised and any countable locally finite group G has $e(G) = \infty$. If G has an ascendant subgroup which is not locally finite and has no non-abelian free subgroups, then $e(G) = 1$, unless G is infinite cyclic by finite. Further results may be found in [2] and [9].

Let G act semiregularly on X and let θ be a homomorphism from $X \times G$ to A . We say θ is almost trivial if, for each $g \in G$, $(x, g)\theta = 1$, for almost all $x \in X$. Two such homomorphisms θ and ϕ will be called equivalent if, for some function f from X to A , almost equal to a function constant on all xG , we have $(x, g)\phi = (xf)^{-1}(x, g)\theta(xg)f$, for all $(x, g) \in X \times G$. We note that this definition of equivalence is much weaker than equivalence between almost homomorphisms. From Lemma 3, there is a function h from X to A such that $(x, g)\theta = (xh)^{-1}(xg)h$, for all $(x, g) \in X \times G$. Since θ is almost trivial, $h = {}^a g h$, for all $g \in G$. Such a function h is said to be almost G -invariant and we note that any such function defines an almost trivial homomorphism

from $X \times G$ to A by putting $(x, g)\theta = (xh)^{-1}(xg)h = x(h^{-1}{}^g h)$. Then each almost trivial homomorphism θ is given by $(x, g)\theta = x(h^{-1}{}^g h)$, for some almost invariant function h , and if $(x, g)\phi = x(k^{-1}{}^g k)$, with k almost invariant, then θ and ϕ are equivalent if and only if $k^{-1}{}^g k = f^{-1}h^{-1}{}^g h{}^g f$, for some f almost equal to a function constant on all xG and for all $g \in G$.

For any function f from X to A , let $\{S_i: i \in I\}$ be the decomposition of X into constancy sets of f , that is, $x, y \in S_i$ if and only if $xf = yf$. Then f is almost invariant if and only if $\cup_{i \in I} (S_i g \cap (X \setminus S_i))$ is finite, for all $g \in G$. In the case where $X = G$ and f is almost invariant, the sets S_i are almost invariant subsets of G . Detailed analysis gives the following results.

LEMMA (6): *Let G act semiregularly on X . If $e(G) = 0$ or 1 , every almost invariant function from G to A is almost equal to a constant function and every almost trivial homomorphism from $X \times G$ to A is equivalent to the trivial homomorphism. If $e(G) = 2$ and $\{S, T\}$ is a partition of G into infinite almost invariant subsets, then any almost invariant function from G to A is almost equal to a function constant on S and T . If $e(G) = \infty$ and G is finitely generated, for any almost invariant function f from G to A , there is a finite partition $\{S_1, \dots, S_r\}$ of G into infinite almost invariant subsets such that f is almost equal to a function constant on each S_i . If $e(G) > 1$, the set Z of equivalence classes of almost trivial homomorphisms from $X \times G$ to A contains a subset bijective with the set $A_0^{(X/G)}$, consisting of the functions with finite support from X/G to the set A_0 of conjugacy classes of A ; if $e(G) = 2$, this subset is the whole of Z .*

PROOF: If $e(G) = 0$, G is finite and the results are immediate. For infinite G , the remarks about almost invariant functions follow from Lemmas 4.3 and 4.4 of [7]. Translation of Theorem 3 of [8] from the language of wreath products to that of groupoids, shows that, if $e(G) = 1$, every almost trivial homomorphism from $X \times G$ to A is equivalent to the trivial homomorphism. Finally, we must consider the set Z when $e(G) > 1$.

Let S be an infinite almost invariant subset of G such that $T = G \setminus S$ is infinite and let U be a transversal of the orbits of X under G . For any function $k \in A^{(U)}$ with support V , we can define an almost invariant function $h = h(k)$ from X to A by taking $(vS)h = vk$, for $v \in V$, and $(X \setminus VS)h = 1$. If m is conjugate to k in $A^{(U)}$ and $t = h(m)$, then for some function f from X to A , constant on all xG , we have $t = fhf^{-1}$. For $g \in G$, $t^{-1}{}^g t = fh^{-1}f^{-1}{}^g f{}^g h{}^g f^{-1} = fh^{-1}{}^g h{}^g f^{-1}$, and thus the homo-

morphisms corresponding to h and t are equivalent. Conversely, if the homomorphisms corresponding to h and t are equivalent, there are functions e, f from X to A , with f constant on all xG and $e = {}^a f$, such that $t^{-1} {}^g t = eh^{-1} {}^g h {}^g e^{-1}$, for $g \in G$. Then

$$tfh^{-1}f^{-1} = {}^a teh^{-1}f^{-1} = {}^g (teh^{-1})f^{-1} = {}^a {}^g (tfh^{-1}f^{-1}),$$

for all $g \in G$, and so $tfh^{-1}f^{-1}$ is constant on all xG . For $u \in U$, $(uT)tfh^{-1}f^{-1} = 1$, so $tfh^{-1}f^{-1} = 1$ and $t = fhf^{-1}$. Then m is conjugate to k and so we have shown that Z has a subset bijective with $A_0^{(U)}$.

In the case where $e(G) = 2$, a standard argument (see Theorem 3 of [8] or Lemma 2.3 of [2]) shows that any almost invariant function f from X to A is constant on almost all uG , $u \in U$. Let V be the finite set consisting of the remaining $u \in U$. Since f is almost invariant on each vG , $v \in V$, it is almost equal to a function d with $(vS)d = a_v$, $(vT)d = b_v$, for $v \in V$, and $(uG)d = a_u$, for $u \in U \setminus V$. Then the almost trivial homomorphism associated with f is equivalent to the almost homomorphism ϕ associated with d and defined by $(x, g)\phi = (xd)^{-1}(xg)d$. Now $(x, g)\phi = 1$, for $x \in X \setminus VG$, and if $x \in vG$ with $v \in V$, $(x, g)\phi = 1$ for $x \in v(S \cap Sg^{-1}) \cup v(T \cap Tg^{-1})$, $(x, g)\phi = a_v^{-1}b_v$ for $x \in v(S \cap Tg^{-1})$, and $(x, g)\phi = b_v^{-1}a_v$ for $x \in v(T \cap Sg^{-1})$. Let $k \in A^{(U)}$ be given by $vk = a_v^{-1}b_v$ and let $h = h(k)$. Then $(x, g)\phi = (xh)^{-1}(xg)h$ and hence every almost trivial homomorphism is equivalent to one associated with $A_0^{(U)}$.

We now return to the analysis of almost homomorphisms.

LEMMA (7): *Suppose N is a finitely presented subgroup of C and let θ be an almost homomorphism from $X \times C$ to A . Then θ is equivalent to an almost homomorphism ψ trivial on almost all N -sheets of $X \times C$. If N is free, ψ may be taken as trivial on all N -sheets.*

PROOF: Let $\langle P : Q \rangle$ be a presentation of N with P and Q finite; if N is free, let Q be empty. A relator r in Q is expressed as a word $y_1 \cdots y_s$ with $y_i \in P \cup P^{-1}$. For almost all $x \in X$,

$$1 = (x, 1)\theta = (x, y_1 \cdots y_s)\theta = (x, y_1)\theta(xy_1, y_2)\theta \cdots (xy_1 \cdots y_{s-1}, y_s)\theta,$$

and also, for $p \in P$, $1 = (x, 1)\theta = (x, p)\theta((x, p)^{-1})\theta$. Now P and Q are finite, so there is a finite subset V of X such that, if $x \in Y = X \setminus VN$, an equation of the previous kind holds for all relators in Q and $(xp, p^{-1})\theta = ((x, p)^{-1})\theta = ((x, p)\theta)^{-1}$, for $p \in P$. If N is free then, replacing θ by an equivalent almost homomorphism, we may assume V is empty.

With $y \in Y$ and a word $w = p_1 \cdots p_t$, where $p_i \in P \cup P^{-1}$, we can associate the product $v(y, w) = (y, p_1)\theta(y p_1, p_2)\theta \cdots (y p_1 \cdots p_{t-1}, p_t)\theta$. Now $(x, p)\theta(xp, p^{-1})\theta = 1 = (xp, p^{-1})\theta(x, p)\theta$, for $x \in Y$, so deletion from w or insertion in w of products pp^{-1} or $p^{-1}p$, with $p \in P$, gives a word w' with $v(y, w') = v(y, w)$. Similarly, if w' is obtained from w by deleting a relator, then $v(y, w') = v(y, w)$. Suppose $n \in N$ has two expressions in terms of $P \cup P^{-1}$, $n = p_1 \cdots p_t = z_1 \cdots z_s$, and let w be the word $p_1 \cdots p_t z_s^{-1} \cdots z_1^{-1}$. After deletion and insertion of products pp^{-1} and $p^{-1}p$, we obtain a word u which is a product of conjugates of relators and for which $v(y, u) = v(y, w)$. Deleting the relators from u gives a word m with $v(y, m) = v(y, u)$ and the invariance of v under deletion of products pp^{-1} and $p^{-1}p$ implies that $v(y, m) = 1$. Hence $v(y, w) = 1$ and $v(y, p_1 \cdots p_t) = v(y, z_1 \cdots z_s)$. For $y \in Y$ and $n \in N$, with $n = p_1 \cdots p_t$, we put $(y, n)\phi = (y, p_1)\theta \cdots (y p_1 \cdots p_{t-1}, p_t)\theta$. This gives a well defined map from $Y \times N$ to A , which is clearly a homomorphism. We extend ϕ to a map from $X \times C$ by taking $(x, c)\phi = (x, c)\theta$ for (x, c) outside $Y \times N$. Now $(x, p)\phi = (x, p)\theta$ for all $p \in P \cup P^{-1}$ and all $x \in X$. Suppose $m, n \in N$ and $(x, m)\phi = (x, m)\theta$, $(x, n)\phi = (x, n)\theta$, for almost all $x \in X$. Then, for almost all $y \in Y$, $(y, mn)\phi = (y, m)\phi(y, n)\phi = (y, m)\theta(y, n)\theta = (y, mn)\theta$ and so $(x, mn)\phi = (x, mn)\theta$, for almost all $x \in X$. Thus, for all $n \in N$, we have $(x, n)\phi = (x, n)\theta$, for almost all $x \in X$, and so ϕ is an almost homomorphism equivalent to θ . From Lemma 3, ϕ is equivalent to an almost homomorphism ψ , trivial on $Y \times N$.

LEMMA (8): *Suppose C has a finitely presented normal subgroup N of infinite index. Every almost homomorphism from $X \times C$ to A is equivalent to an almost homomorphism ψ trivial on all N -sheets of $X \times C$. For such a ψ , if $xf_c = (x, c)\psi$, then f_c is almost N -invariant, for all $c \in C$.*

PROOF: From Lemma 7, any almost homomorphism is equivalent to an almost homomorphism θ trivial on almost all N -sheets. Let T be a finite subset of X with one vertex in each exceptional N -sheet. Since C/N is infinite, for any N -sheet with vertex set yN , $y \in T$, there exists $d \in C$ such that θ is trivial on the sheet with vertex set $yNd = ydN$. If $n \in N$, then $d^{-1}nd \in N$ and, for almost all $m \in N$,

$$\begin{aligned} (ym, n)\theta &= (ym, d)\theta(ymd, d^{-1}nd)\theta((ymn, d)\theta)^{-1} \\ &= (ym, d)\theta((ymn, d)\theta)^{-1}. \end{aligned}$$

We now define $\gamma: X \rightarrow A$ by putting $(ym)\gamma = (ym, d)\theta$, for $y \in T$, $m \in N$, and $d = d(y)$ chosen as above; we take $x\gamma$ trivial on all other

$x \in X$. If $(x, c)\phi = (x\gamma)^{-1}(x, c)\theta(xc)\gamma$, for $(x, c) \in X \times C$, then ϕ is equivalent to θ . For fixed $n \in N$, $(x, n)\phi$ is non-trivial only for $x = ym$, where $y \in T$, $m \in N$, and for each y the exceptions m form a finite set. Thus $(x, n)\phi$ is trivial for almost all $x \in X$ and so ϕ is equivalent to an almost homomorphism ψ trivial on all N -sheets.

Let $c \in C$ be fixed. For $x \in X$ and $n \in N$, we have $(xn, c) = (xn, n^{-1})(x, c)(xc, c^{-1}nc)$ so, for fixed n , $(xn, c)\psi = (x, c)\psi$, for almost all $x \in X$. Thus ${}^nf_c = {}^af_c$, for all $n \in N$, and f_c is almost N -invariant.

LEMMA (9): *Suppose C has a finitely generated normal subgroup N . Let U be a transversal of the orbits of X under C and T a transversal of the cosets of N in C . If $f: X \rightarrow A$ is almost N -invariant, there exists a partition $\{N_1, \dots, N_r\}$ of N into infinite almost invariant subsets such that f is almost equal to a function constant on almost all xN and on all uNt , for $u \in U$, $t \in T$.*

PROOF: Let P be a finite generating set for N and let V be the intersection of UT with $(\cup_{p \in P} \sigma(f^{-1}pf))N$. Then V is finite and, for $x \in X \setminus VN$, $(xp)f = xf$, for $p \in P$, and so f is constant on xN . For $v = ut \in V$ with $u \in U$, $t \in T$, let $h_v = h: N \rightarrow A$ be defined by $mh = (umt)f$, for $m \in N$. If $m, n \in N$, then

$$m({}^nh) = (mn)h = (umnt)f = (umt(t^{-1}nt))f = (umt){}^yf,$$

where $y = t^{-1}nt$. But ${}^yf = {}^af$ so, for almost all $m \in N$, $m({}^nh) = (umt)f = mh$, and so h is almost invariant. If S_i , $i \in I$, are the constancy sets of h and h is not constant on N , then, for each $i \in I$, $S_iN \neq S_i$ and so $S_ip \cap (N \setminus S_i)$ is non-empty, for some $p \in P$. But $\cup_{i \in I} S_ip \cap (N \setminus S_i)$ is finite for all $p \in P$ and so I is finite. Consider all possible subsets of N of the form $\cap_{v \in V} R_v$, where R_v is a constancy set of h_v . These form a finite partition of N into almost invariant subsets on which all h_v are constant. Incorporating all the finite parts in one of the infinite parts, we have a partition $\{N_1, \dots, N_r\}$ of N into infinite almost invariant subsets such that, for $v \in V$, $h_v = {}^ak_v$, where k_v is a function constant on all N_i . Then f is almost equal to a function constant on almost all xN and on all uNt , with $u \in U$, $t \in T$.

LEMMA (10): *Suppose C has a finitely generated normal subgroup N with a partition $\{N_1, \dots, N_r\}$ into infinite almost invariant subsets. Let U be a transversal of the orbits of X under C and T a transversal of the cosets of N in C and put $Y = X/N$, $D = C/N$. Let Z be the set of equivalence classes of almost homomorphisms θ from $X \times C$ to A which are trivial on all N -sheets and such that, if $xf_c = (x, c)\theta$, then f_c is*

constant on almost all xN and constant on all uN_it , with $u \in U$, $t \in T$. Then Z is trivial if $r = 1$. If $r \geq 2$, then Z has a subset bijective with the set of equivalence classes of almost trivial homomorphisms from $Y \times D$ to A and this is the whole of Z if $r = 2$.

PROOF: Let θ be an almost homomorphism satisfying the given conditions. If $n \in N$, $c \in C$, then $f_{nc} = {}^a f_n {}^c f_c = {}^a f_c$. But f_c and f_{nc} are constant on all the infinite sets uN_it and so $f_c = f_{nc}$.

Let ρ denote the natural maps from X to Y and D to C . For fixed i , there is a well defined map $\phi = \phi_i$ from $Y \times D$ to A given by $((ut)\rho, c\rho)\phi = (uN_it)f_c$, for $u \in U$, $t \in T$, $c \in C$. If $c, e \in C$, then

$$(((ut)\rho, c\rho)((ut)\rho, e\rho))\phi = ((ut)\rho, (ce)\rho)\phi = (uN_it)f_{ce}.$$

Since $f_{ce} = {}^a f_c {}^e f_e$, we have $(unt)f_{ce} = (unt)f_c(untc)f_e$, for almost all $n \in N_i$. For fixed t, c , we have $tc = ms$, for some $m \in N$, $s \in T$. Since $N_im \cap (N \setminus N_i)$ is finite, $uN_im s \cap u(N \setminus N_i)s = uN_it c \cap u(N \setminus N_i)s$ is finite and hence $(untc)f_e = (uN_is)f_e$, for almost all $n \in N_i$. Thus

$$\begin{aligned} (uN_it)f_{ce} &= (uN_it)f_c(uN_is)f_e = ((ut)\rho, c\rho)\phi((us)\rho, e\rho)\phi \\ &= ((ut)\rho, c\rho)\phi((ut)\rho, e\rho)\phi. \end{aligned}$$

So $\phi = \phi_i$ is a homomorphism from $Y \times D$ to A .

We take $i = 1$ and note from Lemma 3 that there is a map $\gamma : Y \rightarrow A$ such that $(y, d)\phi_1 = (y\gamma)^{-1}(yd)\gamma$ for $y \in Y$, $d \in D$. Let $g : X \rightarrow A$ be given by $xg = x\rho\gamma$, for $x \in X$, and put $(x, c)\psi = (xg)(x, c)\theta((xc)g)^{-1}$, for $x \in X$, $c \in C$. Then ψ is equivalent to θ and, since g is constant on each xN , ψ satisfies the conditions given for θ . Let $(x, c)\psi = xh_c$ and, for each i , let λ_i denote the homomorphism from $Y \times D$ to A associated with ψ and N_i . For $u \in U$, $t \in T$, $c \in C$, we have

$$\begin{aligned} ((ut)\rho, c\rho)\lambda_1 &= (uN_1t)h_c = (uN_1t)g(uN_1t, c)\theta(uN_1tc)g \\ &= (ut)\rho\gamma((ut)\rho, c\rho)\phi_1((ut)\rho\gamma)^{-1} = 1. \end{aligned}$$

Thus λ_1 is trivial and h_c is trivial on UN_1T . Since h_c is constant on almost all xN , it is trivial on almost all xN and so each λ_i is almost trivial. Thus each almost homomorphism θ is equivalent to an almost homomorphism ψ given by $(unt, c)\psi = ((ut)\rho, c\rho)\lambda_i$, for $u \in U$, $t \in T$, $c \in C$ and $n \in N_i$, where $\lambda_1 = 1$, $\lambda_2, \dots, \lambda_r$ are almost trivial homomorphisms from $Y \times D$ to A .

Conversely, suppose $\lambda_1 = 1$, $\lambda_2, \dots, \lambda_r$ are almost trivial homomorphisms from $Y \times D$ to A and $\psi : X \times C \rightarrow A$ is given by $(unt, c)\psi =$

$((ut)\rho, c\rho)\lambda_i$, for $u \in U$, $t \in T$, $c \in C$ and $n \in N_i$. If $e \in C$ and $n \in N_i$,

$$(unt, ce)\psi = ((ut)\rho, (ce)\rho)\lambda_i = ((ut)\rho, c\rho)\lambda_i((utc)\rho, e\rho)\lambda_i.$$

Now $(unt, c)\psi = ((ut)\rho, c\rho)\lambda_i$ and, if $tc = ms$ with $m \in N$, $s \in T$, then $untc = unms \in uN_i s$, for almost all $n \in N_i$, and so $(untc, e)\psi = ((utc)\rho, e\rho)\lambda_i$, for almost all $n \in N_i$. So, for fixed u , t , c , e , we have $(unt, ce)\psi = (unt, c)\psi(untc, e)\psi$, for almost all $n \in N$. Since the λ_i are almost trivial, $(x, c)\psi$ is trivial on almost all xN and hence $(x, ce)\psi = (x, c)\psi(xc, e)\psi$, for almost all $x \in X$. Thus ψ is an almost homomorphism.

Next suppose $\phi : X \times C \rightarrow A$ corresponds to the almost trivial homomorphisms $\mu_1 = 1, \mu_2, \dots, \mu_r$ from $Y \times D$ to A . If ϕ is equivalent to ψ there is a map $\gamma : X \rightarrow A$ such that, for $c \in C$, $(x, c)\phi = (x\gamma)^{-1}(x, c)\psi(xc)\gamma$, for almost all $x \in X$. Since ϕ and ψ are trivial on all N -sheets, $\gamma = {}^a n \gamma$, for $n \in N$, and so γ is almost N -invariant. From Lemma 9, $\gamma = {}^a \delta$, for some δ constant on almost all xN and on all uSt , where S runs through the sets in a finite partition of N into infinite almost invariant subsets. For each i , there is some such S with $M_i = N_i \cap S$ infinite. Then M_i is almost invariant and both δ and all f_c are constant on all $uM_i t$. Since $\delta = {}^a \gamma$, $(x, c)\phi = (x\delta)^{-1}(x, c)\psi(xc)\delta$, for almost all $x \in X$. Now $(uN_1 t, c)\phi = 1 = (uN_1 t, c)\psi$ so, for fixed t , $s \in T$ with $tc = ms$, we have $(unt)\delta = (untc)\delta = (unms)\delta$, for almost all $n \in M_1$. Since $M_1 \cap M_1 m$ is infinite, $(uM_1 t)\delta = (uM_1 s)\delta$, and so δ is constant on each $uM_1 T$. For $u \in U$, $t \in T$, $c \in C$, with $tc \in Ns$, $s \in T$, we have

$$\begin{aligned} ((ut)\rho, c\rho)\mu_i &= (uN_i t, c)\phi = (uM_i t, c)\phi \\ &= ((uM_i t)\delta)^{-1}(uM_i t, c)\psi(uM_i s)\delta \\ &= ((uM_i t)\delta)^{-1}((ut)\rho, c\rho)\lambda_i(uM_i s)\delta. \end{aligned}$$

Putting $(ut)\rho\epsilon_i = (uM_i t)\delta$, we have

$$((ut)\rho, c\rho)\mu_i = ((ut)\rho\epsilon_i)^{-1}((ut)\rho, c\rho)\lambda_i(utc)\rho\epsilon_i.$$

Now δ is constant on each $uM_i T$ and on almost all xN . Thus $(uM_i t)\delta = (uM_1 t)\delta$, for almost all $ut \in UT$ and ϵ_i is almost equal to the function $\epsilon = \epsilon_1$, which is constant on each $u\rho D$. If $\beta_i = \epsilon_i \epsilon^{-1}$, we have a function ϵ , constant on each yD , and functions $\beta_i = {}^a 1$ such that $(y, d)\mu_i = (y\beta_i \epsilon)^{-1}(y, d)\lambda_i(yd)\beta_i \epsilon$. In this situation, we say that $(\mu_1, \dots, \mu_r)$ is equivalent to $(\lambda_1, \dots, \lambda_r)$. This implies, but is not implied by, the equivalence of μ_i and λ_i for each i .

Finally, we show that if $(\mu_1, \dots, \mu_r)$ is equivalent to $(\lambda_1, \dots, \lambda_r)$, then ϕ is equivalent to ψ . We have functions β_i, ϵ from Y to A , with $\beta_i = {}^a 1$

and ϵ constant on each yD , such that

$$((ut)\rho, c\rho)\mu_i = ((ut)\rho\beta_i\epsilon)^{-1}((ut)\rho, c\rho)\lambda_i(utc)\rho\beta_i\epsilon.$$

Define $\delta, \nu : X \rightarrow A$ by $(uN_it)\delta = (ut)\rho\beta_i$, $(uC)\nu = u\rho\epsilon$. Let $c \in C$, $u \in U$, $t \in T$, be fixed. If $tc = ms$, with $m \in M$, $s \in T$, then

$$(uN_it, c)\phi = ((ut)\rho, c\rho)\mu_i = ((uN_it)\delta\nu)^{-1}(uN_it, c)\psi(uN_is)\delta\nu.$$

Now ν is constant on all xC , so $(uns)\nu = (untc)\nu$, for $n \in N$. Also, $(uns)\delta = (untc)\delta$ unless, for some i , we have $(utc)\rho \in \sigma(\beta_i)$ and $n \in N_i$, $nm \notin N_i$, that is, $n \in N_i \cap (N \setminus N_i)m^{-1}$. For each i , β_i has finite support and N_i is almost invariant. Thus, for fixed c , only a finite number of exceptions occur and so, for almost all $x \in X$, $(x, c)\phi = (x\delta\nu)^{-1}(x, c)\psi(xc)\delta\nu$. Hence ϕ is equivalent to ψ .

We have shown that Z is bijective with the set of equivalence classes of r -tuples $(1, \lambda_2, \dots, \lambda_r)$ and so Z is trivial if $r = 1$. For $r > 1$, $(1, \lambda_2, 1, \dots, 1)$ is equivalent to $(1, \mu_2, 1, \dots, 1)$ if and only if λ_2 and μ_2 are equivalent. So Z has a subset bijective with the set of equivalence classes of almost trivial homomorphisms from $Y \times D$ to A , which is the whole of Z when $r = 2$.

THEOREM (11): *Let C act semiregularly on X . If C has a finitely presented normal subgroup N of infinite index such that $e(N) = 1$, then all almost homomorphisms from $X \times C$ to A are equivalent.*

PROOF: Since $e(N) = 1$, the only partition of N into infinite almost invariant subsets is the trivial partition $\{N\}$. From Lemmas 8 and 9, any almost homomorphism is equivalent to an almost homomorphism θ satisfying the conditions of Lemma 10, with $r = 1$. Then θ is equivalent to the trivial homomorphism.

THEOREM (12): *Let C act semiregularly on X and suppose C has a normal subgroup N such that $e(N) = 2$. If $e(C/N) = 1$, all almost homomorphisms from $X \times C$ to A are equivalent. If $e(C/N) > 1$, the set Z of equivalence classes of almost homomorphisms contains a subset bijective with $A_0^{(X/C)}$, where A_0 is the set of conjugacy classes of A ; if $e(C/N) = 2$, this subset is the whole of Z .*

PROOF: Let S be an infinite almost invariant subset of N with infinite complement R . From Lemma 8, we need only consider almost homomorphisms ψ trivial on all N -sheets and with f_c almost N -invariant, where $xf_c = (x, c)\psi$. Since $e(N) = 2$, Lemma 9 implies that each f_c is almost equal to a function constant on all uSt and uRt . Thus

ψ is equivalent to an almost homomorphism satisfying the conditions of Lemma 10 with $r = 2$. The result now follows from Lemma 6.

THEOREM (13): *Let C act semiregularly on X . If C has a finitely generated free subgroup of finite index, all almost homomorphisms from $X \times C$ to A are equivalent. If C has a finitely presented normal subgroup N of infinite index with $e(N) = \infty$, then all almost homomorphisms from $X \times C$ to A are equivalent if C is finitely generated and $e(C/N) = 1$. If $e(C/N) > 1$, there are inequivalent almost homomorphisms.*

PROOF: If C is finitely generated free by finite, it has a finitely generated free normal subgroup N of finite index. Using Lemmas 7 and 8, in both cases we need only consider almost homomorphisms ψ such that ψ is trivial on all N -sheets and f_c is almost N -invariant, where $xf_c = (x, c)\psi$. If C is finitely generated, then $C = \langle c_1, \dots, c_v, N \rangle$ for some $c_1, \dots, c_v \in C \setminus N$. From Lemma 9, for each c_i there is a finite partition of N into almost invariant subsets S such that f_{c_i} is almost equal to a function constant on almost all xN and on all uSt . Taking the intersections of all such S that arise, over all j , we obtain a finite partition of N into almost invariant subsets. If we incorporate the finite parts in one of the infinite parts, we have a partition $\{N_1, \dots, N_r\}$ of N into infinite almost invariant subsets so that each f_{c_i} is almost equal to a function g_{c_i} constant on all uN_it and on almost all xN . Now $f_n = 1$, for $n \in N$, and $C = \langle c_1, \dots, c_v, N \rangle$. Suppose for some $c, d \in C$, we have f_c, f_d almost equal to functions g_c, g_d constant on all uN_it and on almost all xN . Then ${}^c g_d$ is constant on almost all xN . Let $xN = uNt$ be an exception and suppose $tc = ms$, with $m \in N, s \in T$. For $n \in N_i$,

$$(unt) {}^c g_d = (untc)g_d = (unms)g_d = (uN_is)g_d,$$

unless $n \in N_i \cap (N \setminus N_i)m^{-1}$. Thus ${}^c g_d$ is almost equal to a function constant on all uN_it and on almost all xN . Now $f_{c^{-1}} = {}^a ({}^{c^{-1}} f_c)^{-1}$ and $f_{ca} = {}^a f_c {}^c f_d$, so it follows by induction that, for all $c \in C$, f_c is almost equal to a function g_c , constant on all uN_it and almost all xN . Putting $(x, c)\phi = xg_c$, we obtain an almost homomorphism equivalent to ψ and satisfying the conditions of Lemma 10. The first two statements of the theorem now follow from Lemma 6. If $e(C/N) > 1$, we take an arbitrary partition of N into two infinite almost invariant subsets. Applying Lemmas 10 and 6, we know that there are almost homomorphisms not equivalent to the trivial homomorphism.

Theorem B follows from Theorems 1, 11, 12 and 13, together with the next result.

COROLLARY (14): *Suppose C is a polycyclic by finite group acting semiregularly on X . All almost homomorphisms from $X \times C$ to A are equivalent unless C has Hirsch number 2.*

PROOF: Let h be the Hirsch number of C . The result is trivial for $h = 0$, since C is then finite. Otherwise, C has a non-trivial poly-(infinite cyclic) normal subgroup N of finite index. If $h = 1$, then N is infinite cyclic and the result follows from Theorem 13. If $h > 2$, then C has a normal series $C = C_0 \geq N = C_1 > C_2 > 1$, with all C_i finitely presented, C/N finite, C_1/C_2 infinite, and C_2 poly-(infinite cyclic) with Hirsch number > 1 . From the remarks preceding Lemma 6, $e(C_2) = 1$ and so Theorem 11 implies that an almost homomorphism from $X \times C$ to A is equivalent to one θ which is trivial on all N -sheets. Now N also has 1 end and so Lemmas 8 and 9 show that θ is equivalent to an almost homomorphism satisfying the conditions of Lemma 10, with $r = 1$. So θ is equivalent to the trivial homomorphism.

Finally, suppose $h = 2$. Then C has a normal subgroup N of finite index which is infinite cyclic by infinite cyclic. From Theorem 12, given $x \in X$, there is an almost homomorphism ϕ from $xN \times N$ to A which is not equivalent to the trivial homomorphism. If we can extend ϕ to $X \times C$, the extension will not be equivalent to the trivial homomorphism. Let T be a transversal of the cosets of N in C , with $\tau : C \rightarrow T$ the transversal map and $1 \in T$. For $t \in T$, $n \in N$, $c \in C$, put $(xnt, c)\theta = (xn, tc((tc)\tau)^{-1})\phi$. Then θ extends ϕ to $xC \times C$ and if $d \in C$,

$$\begin{aligned} (xnt, c)\theta(xntc, d)\theta \\ = (xn, tc((tc)\tau)^{-1})\phi(xntc((tc)\tau)^{-1}, (tc)\tau d((tcd)\tau)^{-1})\phi. \end{aligned}$$

For fixed t, c, d , this equals $(xn, tcd((tcd)\tau)^{-1})\phi = (xnt, cd)\theta$, for almost all $n \in N$. Now T is finite, so $(xe, c)\theta(xec, d)\theta = (xe, cd)\theta$, for almost all $e \in C$. Thus θ is an almost homomorphism from $xC \times C$ to A extending ϕ . Defining θ to be trivial on all other C -sheets, we have an almost homomorphism from $X \times C$ to A which is not equivalent to the trivial one.

REFERENCES

- [1] R. BIERI: Normal subgroups in duality groups and in groups of cohomological dimension 2. *J. Pure and Applied Algebra* 7 (1976) 35–51.
- [2] D. E. COHEN: Groups of cohomological dimension one. *Lecture Notes in Mathematics* 245. (Springer-Verlag, Berlin, 1972).

- [3] F. T. FARRELL: The second cohomology group of G with Z_2G coefficients. *Topology* 13 (1974) 313–326.
- [4] F. T. FARRELL: Poincaré duality and groups of type (FP) . *Comm. Math. Helv.* 50 (1975) 187–195.
- [5] B. HARTLEY: Complements, baseless subgroups and Sylow subgroups of infinite wreath products. *Compositio Mathematica* 26 (1973) 3–30.
- [6] P. J. HIGGINS: *Notes on categories and groupoids*. Van Nostrand Reinhold, London, 1971.
- [7] C. H. HOUGHTON: Ends of groups and the associated first cohomology groups. *J. London Math. Soc.* 6 (1972) 81–92.
- [8] C. H. HOUGHTON: Ends of groups and baseless subgroups of wreath products. *Compositio Mathematica* 27 (1973) 205–211.
- [9] C. H. HOUGHTON and D. SEGAL: Some sufficient conditions for groups to have one end. *J. London Math. Soc.* 10 (1975) 89–96.

(Oblatum 3-X-1975)

Department of Pure Mathematics
University College
P.O. Box 78
Cardiff CF1 1XL
Wales, U.K.