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THE NUMBER OF EXTENSIONS OF AN INVARIANT MEAN

Joseph Max Rosenblatt

Abstract

In a non-discrete σ-compact locally compact metric group G, a Baire category argument gives a continuum of measurable sets $\{A_\gamma : \gamma \in \Gamma\}$ independent on the open sets. If G is amenable as a discrete group and $\epsilon : \Gamma \to \{0, 1\}$, then any invariant mean θ on $CB(G)$ can be extended to an invariant mean θ on $L_\infty(G)$ in such a way that for all $\gamma \in \Gamma$, $\theta(\chi_{A_\gamma}) = \epsilon(\gamma)$.

In a discrete group with card $(G) > \aleph_0$, if one is given an invariant subspace S of $\ell_\infty(G)$ with certain properties, then the axiom of choice gives a family $\{A_\gamma : \gamma \in \Gamma\}$ with card $(\Gamma) = 2^{\text{card}\,(G)}$ which is independent of S. If G is amenable as a discrete group and $\epsilon : \Gamma \to \{0, 1\}$, then any invariant mean θ on S can be extended to an invariant mean θ on $\ell_\infty(G)$ in such a way that for all $\gamma \in \Gamma$, $\theta(\chi_{A_\gamma}) = \epsilon(\gamma)$.

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Two different cases in which any invariant mean on an invariant subspace has many different extensions to an invariant mean on the whole space are considered here. In both cases, one constructs subsets in some sense independent of the subspace. Then if G is amenable, these independent sets can be used to get many different extensions of the invariant mean. The difference between the two cases is that in the one a Baire category argument gives the independent sets while in the other the axiom of choice gives the independent sets.

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In this section, G will be a non-discrete σ-compact locally compact metric group and λ will be a left-invariant Haar measure on β, the Lebesgue measurable sets in G. Assume when G is compact that $\lambda(G) = 1$. Let $CB(G)$ be the continuous bounded real-valued functions on G in the supremum norm and let $L_\infty(G)$ be the bounded measurable real-valued functions on G in the essential supremum norm $\|\cdot\|_\infty$. There is a natural inclusion of $CB(G)$ as a closed subspace of $L_\infty(G)$ which contains the constants and is invariant under the regular action of G on $L_\infty(G)$ defined by $gf(x) = f(g^{-1}x)$ for $g, x \in G$ and $f \in L_\infty(G)$.

The regular action of G on $L_\infty(G)$ induces an action of G on the dual space $L_\infty^*(G)$ defined by $g\theta(f) = \theta(g^{-1}f)$ for $g \in G$, $f \in L_\infty(G)$, and $\theta \in L_\infty^*(G)$. A *mean* θ on $L_\infty(G)$ is any $\theta \in L_\infty^*(G)$ with $\theta(1) = 1$ and $\theta(f) \geq 0$ if $f \geq 0$ a.e. $[\lambda]$. A *left-invariant mean* θ is a mean θ with $g\theta = \theta$ for all $g \in G$. The group G is an *amenable locally compact group* in case the set of left-invariant means LIM $\neq \emptyset$. Let $P(G)$ be the set of positive measurable functions h with $\int h d\lambda = 1$. Let $*$ be convolution of functions on G. A *topological left-invariant mean* is a mean θ on $L_\infty(G)$ with $\theta(h * f) = \theta(f)$ for all $h \in P(G)$ and $f \in L_\infty(G)$. The set TLIM of topological left-invariant means is contained in LIM and TLIM $\neq \emptyset$ if and only if LIM $\neq \emptyset$. See Greenleaf [5].

For any group, $\ell_\infty(G)$ denotes all bounded functions on G in the supremum norm $\|\cdot\|_\infty$. A group G is amenable (or amenable as discrete) if there is an invariant mean on $\ell_\infty(G)$. If G is amenable then in any suitable topology, G is an amenable locally compact group. If G is locally solvable then it will be amenable [5]. The set of left-invariant means on $\ell_\infty(G)$ is denoted lim. Chou [3] shows that if G is infinite and amenable, then the card (lim) $= 2^{2^{\operatorname{card}(G)}}$.

In the discrete case, every left-invariant mean is topological left-invariant but this is not true for the non-discrete case. Granirer [4] and Rudin [12] contain proofs that if G is a non-discrete locally compact group which is amenable as discrete, then LIM\TLIM $\neq \emptyset$. In [11], it is shown that if G is a non-discrete σ-compact locally compact metric group which is amenable as discrete, then there are 2^c mutually singular elements in LIM each of which is singular to TLIM. In this section, an improvement of the technique in [11] is used to show in the latter case that every invariant mean on $CB(G)$ has 2^c mutually singular extensions to $L_\infty(G)$ each of which is singular to TLIM.

It is convenient to have some of the notation from [11]. The mapping $\wedge : L_\infty(G) \to C(\mathscr{D})$ denotes the Gelfand isomorphism of

$L_\infty(G)$ with the continuous functions on its maximal ideal space $\mathscr{D}$. For any $A \in \beta$, $\hat{A}$ denotes the unique open closed set in $\mathscr{D}$ with the characteristic functions $\chi_{\hat{A}}$ and $\hat{\chi}_A$ equal. The mapping $\wedge$ on β is a set homomorphism with respect to finite unions and intersections. If $A \in \beta$, then $\mathscr{D} \backslash \hat{A} = (G \backslash A)\hat{}$ and $(gA)\hat{} = g\hat{A}$ for all $g \in G$. A family $\{A_\gamma : \gamma \in \Gamma\} \subseteq \beta$ has $\bigcap_{\gamma \in \Gamma} \hat{A}_\gamma = \emptyset$ if and only if there are $\gamma_1, \ldots, \gamma_m \in \Gamma$ with $\lambda(A_{\gamma_1} \cap \cdots \cap A_{\gamma_m}) = 0$.

Let $A^1 = A$ and $A^c = G \backslash A$.

1.1 Lemma: *Assume G is a non-discrete σ-compact locally compact metric group. Then there exists a continuum of measurable sets of finite measure $\{A_\gamma : \gamma \in \Gamma\}$ in G such that for any non-empty open set V and arbitrary subsets $F_\gamma \subseteq G$,*

$$\hat{V} \cap \left(\bigcap_{\gamma \in \Gamma} \left(\bigcap_{x \in F_\gamma} x\hat{A}_\gamma \cap \bigcap_{x \in F_\gamma^c} x\hat{A}_\gamma^c \right) \right) \neq \emptyset.$$

Proof: The proof is an elaboration of 2.2 [11]. It suffices to show that there is a continuum of distinct measurable sets of finite measure $\{A_\gamma : \gamma \in \Gamma\}$ such that for any $V \neq \emptyset$ from a countable basis in G, for any $m, n \geq 1$ and choices $x_{i1}, \ldots, x_{in}, y_{i1}, \ldots, y_{in} \in G$ distinct when $i = 1, \ldots, m$ is fixed, and for any distinct $\gamma_1, \ldots, \gamma_m \in \Gamma$,

$$\lambda \left(V \cap \left(\bigcap_{i=1}^{m} \left(\bigcap_{j=1}^{n} x_{ij} A_{\gamma_i} \cap \bigcap_{j=1}^{n} y_{ij} A_{\gamma_i}^c \right) \right) \right) > 0.$$

Since G is a σ-compact metric group, there is a sequence $\{C_j : j = 1, 2, 3, \ldots\}$ of compact sets in G such that for any distinct elements $z_1, \ldots, z_m \in G$, there are pairwise disjoint $C_{j_1}, \ldots, C_{j_m}$ in the sequence with $z_i \in C_{j_i}$, $i = 1, \ldots, m$. Let $\mathscr{S}$ denote all $2n$-tuples, $n \geq 1$, $(K_1, \ldots, K_n, L_1, \ldots, L_n)$ of pairwise disjoint elements in the sequence $\{C_j\}$. Let $\mathscr{S}_0$ be all finite sequences $S = (S_1, \ldots, S_m)$, $m \geq 1$, with each S_i a $2n$-tuple in $\mathscr{S}$. If $S_0 = (K_1, \ldots, K_n, L_1, \ldots, L_n) \in \mathscr{S}$ then $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in S_0$ means each $x_i \in K_i$ and each $y_i \in L_i$. There are a countable number of sequences in $\mathscr{S}_0$.

Let X be the measurable sets of finite measure in G modulo null sets. The metric ρ is defined on $E_1, E_2 \in X$ by $\rho(E_1, E_2) = \lambda(E_1 \backslash E_2 \cup E_2 \backslash E_1)$. The metric space (X, ρ) is a complete metric space dense in itself. Fix a countable basis $\mathscr{U}$ for the open sets in G and assume each $U \in \mathscr{U}$ is non-empty. For each $U \in \mathscr{U}$ and $S = (S_1, \ldots, S_m) \in \mathscr{S}_0$, the (U, S)-relation consists of all m-tuples $(A_1, \ldots, A_m) \in X^m$ for which there is $(x_{i1}, \ldots, x_{in}, y_{i1}, \ldots, y_{in}) \in S_i$ for

$i = 1, \ldots, m$ such that

$$\lambda\left(U \cap \left(\bigcap_{i=1}^{m}\left(\bigcap_{j=1}^{n} x_{ij}A_i \cap \bigcap_{j=1}^{n} y_{ij}A_i^c\right)\right)\right) = 0.$$

This gives a countable number of relations in X. If it is shown that each of these is closed and has no interior then as in [8, 9], there exists a continuum sized $\mathscr{A} \subseteq X$ independent of all the (U, S)-relations. Enumerate $\mathscr{A} = \{A_\gamma : \gamma \in \Gamma\}$. Since $\mathscr{U}$ is a basis and $\{C_j\}$ are sufficient to separate finite sets, $\{A_\gamma : \gamma \in \Gamma\}$ has the desired independence property.

The proof that each (U, S)-relation is closed follows exactly the proof of 2.2 [11]. To show that the relation has no interior, fix $x \in U$. An argument like the one in 2.2 [11] shows that for all $\epsilon > 0$ and $(A_1, \ldots, A_m)$ in X^m, there exists $(B_1, \ldots, B_m) \in X^m$ with $\rho(A_i, B_i) < \epsilon$, $i = 1, \ldots, m$, and such that for [all tuples $(x_{i1}, \ldots, x_{in}, y_{i1}, \ldots, y_{in}) \in S_i$, the set]

$$\bigcap_{i=1}^{m}\left(\bigcap_{j=1}^{n} x_{ij}B_i \cap \bigcap_{j=1}^{n} y_{ij}B_i^c\right)$$

contains an open set dense in a neighborhood of x. Thus, $(B_1, \ldots, B_m)$ is not (U, S)-related. Since ϵ and $(A_1, \ldots, A_m)$ were arbitrary, the (U, S)-relation has no interior. □

The *support of* $\theta \in \mathrm{LIM}$, denoted $\operatorname{supp} \theta$, is the support of θ interpreted as a regular Borel measure on $\mathscr{D}$ via the Gelfand isomorphism. Define θ_1 and θ_2 in LIM to be *mutually singular* if $\operatorname{supp} \theta_1 \cap \operatorname{supp} \theta_2 = \emptyset$. This differs from the definition in [10, 11]. The invariant means θ_1 and θ_2 are mutually singular if and only if there exists $A \in \beta$ with $\theta_1(\chi_A) = 1$ and $\theta_2(\chi_A) = 0$.

Given an invariant mean θ on $CB(G)$ which is bi-topological invariant as in [5], θ can be extended to a topological left-invariant mean on $L_\infty(G)$. Take $h \in P(G)$ and let $h^{\vee}(x) = h(x^{-1})$. Define an extension of θ by $\theta(f) = \theta(f * h^{\vee})$. This gives a topological left-invariant extension. The question is whether every left-invariant mean on $CB(G)$ can be extended to a left-invariant mean on $L_\infty(G)$? If G is compact or G is amenable as discrete, then this is the case. The next theorem shows that if G is amenable as discrete, then there is a wide degree of freedom in choosing the invariant extension. Let a family of measurable sets $\{A_\gamma : \gamma \in \Gamma\}$ be *independent on open sets* if it satisfies the property of 1.1.

1.2 THEOREM: *Let G be a non-discrete σ-compact locally compact metric group which is amenable as discrete. Then there exists a continuum of measurable sets $\{A_\gamma : \gamma \in \Gamma\}$ which is independent on the open sets. For any $\epsilon : \Gamma \to \{0, 1\}$, there is an extension of any left-invariant mean θ on $CB(G)$ to a left-invariant mean θ on $L_\infty(G)$ such that $\theta(\chi_{A_\gamma}) = \epsilon(\gamma)$ for all $\gamma \in \Gamma$. The extension can be chosen so that also $\theta(\chi_E) = 0$ if E is nowhere dense and measurable.*

PROOF: Let $\mathcal{N}$ be the span of all χ_E where $E \in \beta$ and E is nowhere dense. Let $\mathcal{A}$ be the span of $\{\chi_{gA_\gamma} : g \in G, \gamma \in \Gamma\}$. Define S to be $CB(G) + \mathcal{N} + \mathcal{A}$. Each $f \in S$ can be written as a sum $h + n + a$ where $h \in CB(G)$, $n \in \mathcal{N}$, and $a = \Sigma\{a(g, \gamma)x_{gA_\gamma} : g \in G, \ \gamma \in \Gamma\}$ with the scalar $a(g, \gamma) \neq 0$ at most finitely often. Choose $\epsilon : \Gamma \to \{0, 1\}$ and define an extension θ_ϵ of θ to S by

$$\theta_\epsilon(f) = \theta(h) + \Sigma\{a(g, \gamma)\epsilon(\gamma) : g \in G, \gamma \in \Gamma\}.$$

Once it is shown that θ_ϵ is well-defined and positive, it is an invariant extension of θ to S and can be in turn extended invariantly to $L_\infty(G)$ because G is amenable as discrete. This gives θ_ϵ on $L_\infty(G)$ with the desired properties.

To see that θ_ϵ is well-defined, assume $h_1 + n_1 + a_1 = h_2 + n_2 + a_2$ a.e. $[\lambda]$ with $h_1, h_2 \in CB(G)$, $n_1, n_2 \in \mathcal{N}$, and $a_1, a_2 \in \mathcal{A}$. Let W be an open dense set in which $n_1 = n_2 = 0$ a.e. $[\lambda]$. If $h_1 \neq h_2$ then $|h_1 - h_2| \geq \delta > 0$ on some non-empty open subset $U \subseteq W$. But the independence of $\{A_\gamma : \gamma \in \Gamma\}$ shows that the set

$$E = U \cap (\cap \{gA_\gamma^c : a_1(g, \gamma) \neq 0 \text{ or } a_2(g, \gamma) \neq 0\})$$

has positive measure. In this set $h_1 + n_1 + a_1 = h_2 + n_2 + a_2$ a.e. $[\lambda]$, so $h_1(x) = h_2(x)$ for some $x \in E \subseteq U$. This contradicts $|h_1 - h_2| \geq \delta > 0$ on U. Thus, $h_1 = h_2$ and $n_1 + a_1 = n_2 + a_2$ a.e. $[\lambda]$. Fix $(g_0, \gamma_0) \in G \times \Gamma$. The independence of $\{A_\gamma : \gamma \in \Gamma\}$ says that the set E defined as

$$W \cap (\cap \{gA_\gamma^c : (g, \gamma) \neq (g_0, \gamma_0) \text{ and } a_1(g, \gamma) \neq 0 \text{ or } a_2(g, \gamma) \neq 0\} \cap g_0 A_{\gamma_0})$$

has positive measure. In this set E, $n_1 + a_1 = n_2 + a_2$ a.e. $[\lambda]$ implies that $a_1(g_0, \gamma_0) = a_2(g_0, \gamma_0)$ a.e. $[\lambda]$ in E. Because these are constants and $\lambda(E) > 0$, this means $a_1(g_0, \gamma_0) = a_2(g_0, \gamma_0)$ for all $(g_0, \gamma_0) \in G \times \Gamma$.

The argument that θ_ϵ is positive on S is similar to this. If $f = h + n + a$ with $f \geq 0$ a.e. $[\lambda]$ and $h \in CB(G)$, $n \in \mathcal{N}$, $a \in \mathcal{A}$, then

$$h + \Sigma\{a(g, \gamma) : a(g, \gamma) < 0\} \geq 0.$$

Otherwise, on some non-empty open subset U there is $\delta < 0$ with $h + \Sigma\{a(g,\gamma): a(g,\gamma) < 0\} \leq \delta$ on U. Take a non-empty open subset $V \subseteq U$ on which $n = 0$ a.e. $[\lambda]$. By the independence of $\{A_\gamma : \gamma \in \Gamma\}$, the measurable set E defined as

$$V \cap (\cap \{gA_\gamma^c : a(g,\gamma) > 0\} \cap \cap \{gA_\gamma : a(g,\gamma) < 0\})$$

has positive measure. Since $h + n + a \geq 0$ a.e. $[\lambda]$ in E and $h + n + a = h + \Sigma\{a(g,\gamma): a(g,\gamma) < 0\}$ a.e. $[\lambda]$ in E, $\lambda(E) > 0$ implies that $h + \Sigma\{a(g,\gamma): a(g,\gamma) < 0\} \geq 0$ somewhere in E. But this contradicts the original assumption because $E \subseteq U$.

The conclusion is that $f \geq 0$ a.e. $[\lambda]$ implies that if $f = h + n + a$ with $h \in CB(G)$, $n \in \mathcal{N}$, and $a \in \mathcal{A}$, then $h + \Sigma\{a(g,\gamma): a(g,\gamma) < 0\} \geq 0$. So

$$\begin{aligned} 0 &\leq \theta(h + \Sigma\{a(g,\gamma): a(g,\gamma) < 0\}) \\ &= \theta(h) + \Sigma\{a(g,\gamma): a(g,\gamma) < 0\} \\ &\leq \theta(h) + \Sigma\{a(g,\gamma)\epsilon(\gamma): g \in G,\ \gamma \in \Gamma\} \\ &= \theta_\epsilon(h + n + a) = \theta_\epsilon(f). \end{aligned}$$

Hence, θ_ϵ is well-defined and positive and the proof is complete. □

REMARK: This construction actually proves that $\theta(\chi_{A_\gamma})$ can be arranged to be any number $\epsilon(\gamma) \in [0, 1]$. Also, the closure of S in $\|\cdot\|_\infty$ is isometrically isomorphic to the internal direct sum $CB(G) \oplus cl(\mathcal{N}) \oplus cl(\mathcal{A})$ and the elements $\{g\chi_{A_\gamma} : g \in G,\ \gamma \in \Gamma\}$ are linearly independent modulo the subspace $CB(G) + cl(\mathcal{N})$. In addition, the category argument used in constructing $\{A_\gamma : \gamma \in \Gamma\}$ shows that they can be chosen to be isomorphic to a Cantor discontinuum in (X, ρ). This means $\{\chi_{A_\gamma} : \gamma \in \Gamma\}$ can be chosen isomorphic to a Cantor discontinuum as a subset of the unit ball of $L_\infty(G)$ in the topology of pointwise convergence on the integrable functions.

1.3 COROLLARY: *Assume that G is a non-discrete σ-compact locally compact metric group which is amenable as discrete. Then every left-invariant mean on $CB(G)$ has 2^c mutually singular extensions to a left-invariant mean on $L_\infty(G)$ each of which is singular to any element of* TLIM.

PROOF: Take the extension θ_ϵ of θ to $L_\infty(G)$ as in 1.2. Since $\theta_\epsilon(\chi_V) = 1$ for every open dense set, θ_ϵ is mutually singular to any element of TLIM. See [10]. If $\epsilon_1 \neq \epsilon_2$, then there is some $\gamma \in \Gamma$ such that $\epsilon_1(\gamma) \neq \epsilon_2(\gamma)$. Hence, $\theta_{\epsilon_1}(\chi_{A_\gamma}) = 1$ and $\theta_{\epsilon_2}(\chi_{A_\gamma}) = 0$ or vice versa. It follows that θ_{ϵ_1} and θ_{ϵ_2} are mutually singular if $\epsilon_1 \neq \epsilon_2$. □

2

In this section, G will be a discrete group with card $(G) > \aleph_0$. Assume that $\mathscr{R}$ is a σ-algebra of subsets of G; $G \in \mathscr{R}$ and $\mathscr{R}$ is closed under taking countable unions, countable intersections, and complements. The *set* $\mathscr{R}$ *generates a subspace* $S(\mathscr{R}) =$ closure (span $\{\chi_E : E \in \mathscr{R}\}$) in $\ell_\infty(G)$. Assuming that $\mathscr{R}$ is invariant, $S(\mathscr{R})$ is a closed invariant subspace containing the constants. The purpose of this section is to show that with certain conditions on $\mathscr{R}$ and G, any invariant mean on $S(\mathscr{R})$ has many different extensions to an invariant mean on $\ell_\infty(G)$. The construction follows closely the one in Chou [3] which in turn was based on Kakutani-Oxtoby [7]. For this reason, some of the proofs of details will be incomplete. The most important aspect of the argument used here is that one gets sets with an independence property somewhat like the one in Section 1.

In order to have a more useful result, one takes into account an auxiliary set $\mathscr{N}$ of subsets of G which contains all the possibly small sets.

2.1 LEMMA: *Assume G is a group with* card $(G) > \aleph_0$. *Suppose $\mathscr{R}$ is a collection of subsets of G and that $\mathscr{N} \subset \mathscr{R}$. Assume* card $(\mathscr{R} \backslash \mathscr{N}) \leq$ card (G) *and that each* $E \in \mathscr{R} \backslash \mathscr{N}$, card $(E) =$ card (G). *Then there exists pairwise disjoint subsets $\{X_\alpha : \alpha \in A\}$ in G such that*

(1) card $(X_\alpha) =$ card (G),

(2) card $(A) =$ card (G),

(3) *for any* $E \in \mathscr{R} \backslash \mathscr{N}$, *any* $g_1, \ldots, g_m \in G$, *and any* $\alpha \in A$,

$$E \cap \left(\bigcap_{i=1}^{m} g_i X_\alpha \right) \neq \emptyset.$$

PROOF: Let A be the least ordinal with card $(A) =$ card (G). Enumerate $G = \{g_\alpha : \alpha \in A\}$. Assume $\mathscr{R} \backslash \mathscr{N} = \{E_\alpha : \alpha \in A\}$ with the convention that $E_\alpha = G$ for sufficiently large α in the case that card $(\mathscr{R} \backslash \mathscr{N}) <$ card (A). Let H_α be the subgroup of G generated by $\{g_\beta : \beta \leq \alpha\}$ for each $\alpha \in A$. So card $(H_\alpha) \leq$ max (card (α), $\aleph_0$). Choose subsets $F^\alpha_\beta \subset G$ with $\beta \leq \alpha \in A$ in a lexicographical ordering $(\alpha, \beta) \leq (\alpha', \beta')$ if $\alpha < \alpha'$ or $\alpha = \alpha'$ and $\beta \leq \beta'$ as follows. Let F^1_1 be any element of E_1. Assume $\{F^\alpha_\beta : (\alpha, \beta) < (\alpha_0, \beta_0)\}$ have been chosen so that the following hold:

(a) $F^\alpha_\beta \cap E_s \neq \emptyset$ if $s \leq \alpha$ and $(\alpha, \beta) < (\alpha_0, \beta_0)$.

(b) card $(F^\alpha_\beta) \leq$ card (α) if $(\alpha, \beta) < (\alpha_0, \beta_0)$.

(c) $\{H_\alpha F^\alpha_\beta : \beta \leq \alpha,\ (\alpha, \beta) < (\alpha_0, \beta_0)\}$ are pairwise disjoint.

Then for each $s \leq \alpha_0 \in A$, let $L_s = E_s \backslash \cup \{H_{\alpha_0} F^\alpha_\beta : (\alpha, \beta) < (\alpha_0, \beta_0)\}$. This L_s has cardinality the same as card (G) because of (b) and the definition of $\{H_\alpha : \alpha \in A\}$. So we can choose $F^{\alpha_0}_{\beta_0}$ by taking one point from each L_s with $s \leq \alpha_0$. Now properties (a), (b), (c) hold with (α_0, β_0) replaced by its successor in the lexicographical ordering. That is, we have the following:

(a′) $F^{\alpha}_{\beta} \cap E_s \neq \emptyset$ if $s \leq \alpha$ and $(\alpha, \beta) \leq (\alpha_0, \beta_0)$
(b′) $\operatorname{card}(F^{\alpha}_{\beta}) \leq \operatorname{card}(\alpha)$ if $(\alpha, \beta) \leq (\alpha_0, \beta_0)$
(c′) $\{H_\alpha F^{\alpha}_{\beta} : \beta \leq \alpha, (\alpha, \beta) \leq (\alpha_0, \beta_0)\}$ are pairwise disjoint.

Here (a′), (b′) follow from (a), (b) if $(\alpha, \beta) < (\alpha_0, \beta_0)$. If $(\alpha, \beta) = (\alpha_0, \beta_0)$, then the choice of $F^{\alpha_0}_{\beta_0}$ gives (a′) and (b′) immediately. If (c′) were not true, then (c) says that for some $(\alpha, \beta) < (\alpha_0, \beta_0)$, $H_\alpha F^{\alpha}_{\beta} \cap H_{\alpha_0} F^{\alpha_0}_{\beta_0} \neq \emptyset$. But then $H_{\alpha_0} F^{\alpha}_{\beta} \cap H_{\alpha_0} F^{\alpha_0}_{\beta_0} \neq \emptyset$ and so $H_{\alpha_0} F^{\alpha}_{\beta} \cap F^{\alpha_0}_{\beta_0} \neq \emptyset$ with $(\alpha, \beta) < (\alpha_0, \beta_0)$ contrary to the choice of $F^{\alpha_0}_{\beta_0}$. This completes the induction.

Define subsets X_β for $\beta \in A$ by letting $X_\beta = \cup\{H_\alpha F^{\alpha}_{\beta} : \beta \leq \alpha \in A\}$. By properties (a) and (c), (1) and (2) hold. Also, (c) implies that $\{X_\beta : \beta \in A\}$ are pairwise disjoint. To see that (3) holds, fix $\beta_0 \in A$ and $g_1, \ldots, g_m \in G$. Then any $E \in \mathscr{R}\backslash\mathscr{N}$, $E = E_s$ for some $s \in A$. But $\bigcap_{i=1}^m g_i X_{\beta_0}$ must contain a set of the form $H_{\alpha_0} F^{\alpha_0}_{\beta_0}$ where $\alpha_0 \geq \beta_0$ and $\alpha_0 \geq s$. Hence, $H_{\alpha_0} F^{\alpha_0}_{\beta_0} \cap E_s \supset F^{\alpha_0}_{\beta_0} \cap E_s \neq \emptyset$ implies property (3) holds for $\{X_\beta : \beta \in A\}$. □

The next lemma is given without proof. See [3, 6, 7].

2.2 Lemma: *Given any infinite set A, there exists a family of subsets $\{P_\gamma : \gamma \in \Gamma\}$ of A such that*

(1) $\operatorname{card}(\Gamma) = 2^{\operatorname{card}(A)}$
(2) *for all $\gamma_1, \ldots, \gamma_m$ distinct and $e_1, \ldots, e_m \in \{1, c\}$,*

$$\bigcap_{i=1}^{m} P^{e_i}_{\gamma_i} \neq \emptyset.$$

The mapping $\wedge : \ell_\infty(G) \to C(\Delta)$ denotes the Gelfand isomorphism of $\ell_\infty(G)$ with the continuous functions on its maximal ideal space Δ. For $A \subset G$, $\hat{A}$ denotes the unique open closed set with $\chi_{\hat{A}} = \hat{\chi}_A$. The mapping $\wedge$ is a ring homomorphism on subsets of G which commutes with taking complements and commutes with the action by G. A family $\{A_\gamma : \gamma \in \Gamma\}$ has $\bigcap_{\gamma \in \Gamma} \hat{A}_\gamma = \emptyset$ if and only if for some $\gamma_1, \ldots, \gamma_m \in \Gamma$, $A_{\gamma_1} \cap \cdots \cap A_{\gamma_m} = \emptyset$.

2.3 Proposition: *Assume G is a group with* $\operatorname{card}(G) > \aleph_0$. *Let $\mathscr{R}$ be a family of subsets of G and $\mathscr{N}$ a subset of $\mathscr{R}$ such that $E \in \mathscr{R}\backslash\mathscr{N}$ implies that* $\operatorname{card}(E) = \operatorname{card}(G)$. *Assume* $\operatorname{card}(\mathscr{R}\backslash\mathscr{N}) \leq \operatorname{card}(G)$. *Then there exists a family $\{A_\gamma : \gamma \in \Gamma\}$ of subsets of G with* $\operatorname{card}(\Gamma) =$

$2^{\text{card}(G)}$ *such that for all* $E \in \mathcal{R}\backslash\mathcal{N}$ *and all choices* $e_\gamma \in \{1, c\}$,

$$\hat{E} \cap \left(\bigcap_{\gamma \in \Gamma} \left(\bigcap_{x \in G} x\hat{A}_\gamma^{e_\gamma} \right) \right) \neq \emptyset.$$

Proof: Let $\{X_\alpha : \alpha \in A\}$ be as in Lemma 2.1 and let $\{P_\gamma : \gamma \in \Gamma\}$ be as in Lemma 2.2. Let $A_\gamma = \cup\{X_\alpha : \alpha \in P_\gamma\}$ for all $\gamma \in \Gamma$. Then for any choice of $\gamma_1, \ldots, \gamma_m$ distinct and any choice $e_1, \ldots, e_m \in \{1, c\}$, $\cap_{i=1}^m A_{\gamma_i}^{e_i}$ contains some X_{α_0} because the $\{X_\alpha : \alpha \in A\}$ are pairwise disjoint and the $\{P_\gamma : \gamma \in \Gamma\}$ have property (2) in 2.2. Thus, property (3) of X_{α_0} gives the independence property of the $\{A_\gamma : \gamma \in \Gamma\}$. □

This proposition shows that under certain conditions on $\mathcal{R}$ and $\mathcal{N}$, a large independent family exists. Together with the following theorem, it will give examples of when there are many extensions of an invariant mean.

Let $\mathcal{R}$ be a σ-algebra in G and at $\mathcal{N}$ be a ring contained in $\mathcal{R}$. One says $\mathcal{N}$ is *hereditary* if for every $E \in \mathcal{N}$ and $F \subset E$, $F \in \mathcal{N}$. Define an *equivalence relation* on $S(\mathcal{R})$ by $f_1, f_2 \in S(\mathcal{R})$ are *$\mathcal{N}$-equivalent* if $f_1 - f_2 \in S(\mathcal{N})$. If $\mathcal{N}$ is hereditary, then f_1 is $\mathcal{N}$-equivalent to f_2 if and only if for all $\delta > 0$, $\{x : |f_1 - f_2(x)| > \delta\} \in \mathcal{N}$. Define a *left-invariant mean* θ on $S(\mathcal{R})$ to be *independent of* $\mathcal{N}$ if it is zero on $S(\mathcal{N})$; then θ is independent of $\mathcal{N}$ if and only if it is constant on the $\mathcal{N}$-equivalence classes in $S(\mathcal{R})$.

A few simple facts will be needed in the next proof. If $f \in S(\mathcal{R})$, then $|f| \in S(\mathcal{R})$. If $f \in S(\mathcal{R})$ and δ is arbitrary, then $\{x : f(x) > \delta\} \in \mathcal{R}$. Also, if $f \in S(\mathcal{N})$ and $\delta > 0$, any subset F of $\{x : |f(x)| > \delta\}$ is a set in $\mathcal{N}$ when $\mathcal{N}$ is hereditary.

2.4 Theorem: *Suppose G is an amenable group and $\mathcal{R}$ is an invariant σ-algebra in G. Let $\mathcal{N}$ be a hereditary ring contained in $\mathcal{R}$ and $G \notin \mathcal{N}$. Suppose that there are subsets $\{A_\gamma : \gamma \in \Gamma\}$ of G such that for all $E \in \mathcal{R}\backslash\mathcal{N}$ and all choices $e_\gamma \in \{1, c\}$,*

$$\hat{E} \cap \left(\bigcap_{\gamma \in \Gamma} \left(\bigcap_{x \in G} x\hat{A}_\gamma^{e_\gamma} \right) \right) \neq \emptyset.$$

Then every left-invariant mean on $S(\mathcal{R})$ independent of $\mathcal{N}$ has $2^{\text{card}(\Gamma)}$ mutually singular extensions to a left-invariant mean on $\ell_\infty(G)$.

Proof: Fix $\epsilon : \Gamma \to \{0, 1\}$. Let A be the span $\{\chi_{gA_\gamma} : g \in G, \gamma \in \Gamma\}$ and let $S = S(\mathcal{R}) + A$. Any $f \in S$ is of the form $h + a$ where $h \in S(\mathcal{R})$ *and* $a = \Sigma\{a(g, \gamma)\chi_{gA_\gamma} : g \in G, \gamma \in \Gamma\}$ with the scalar $a(g, \gamma) \neq 0$ at most finitely often. Take any left-invariant mean θ on $S(\mathcal{R})$ independent of

$\mathcal{N}$ and define an extension θ_ϵ to S by

$$\theta_\epsilon(f) = \theta(h) + \Sigma\{a(g, \gamma)\epsilon(\gamma) : g \in G, \gamma \in \Gamma\}.$$

Once it is shown that θ_ϵ is well-defined and positive, θ_ϵ is a left-invariant mean on S and the amenability of G gives an extension of θ_ϵ to a left-invariant mean θ_ϵ on $\ell_\infty(G)$. Because $\epsilon_1 \neq \epsilon_2$ implies that for some γ, $\theta_{\epsilon_1}(\chi_{A_\gamma}) = 0$ and $\theta_{\epsilon_2}(\chi_{A_\gamma}) = 1$ or vice versa, the $\{\theta_\epsilon : \epsilon : \Gamma \to \{0, 1\}\}$ contains $2^{\text{card}(\Gamma)}$ mutually singular extensions of θ.

Suppose $h_1 + a_1 = h_2 + a_2$ with h_1, $h_2 \in S(\mathcal{R})$ and a_1, $a_2 \in A$. We claim then $h_1 - h_2 \in S(\mathcal{N})$ and for all $\gamma_0 \in \Gamma$, $\Sigma\{a_1(g, \gamma_0) : g \in G\} = \Sigma\{a_2(g, \gamma_0) : g \in G\}$. To prove this, suppose $h_1 - h_2 \notin S(\mathcal{N})$. Then there is $\delta > 0$ and a set $E \in \mathcal{R}\backslash\mathcal{N}$ on which $|h_1 - h_2| > \delta$. It follows by the independence of $\{A_\gamma : \gamma \in \Gamma\}$, there is $x \in E$ with $a_1(x) = a_2(x) = 0$. But then $h_1 + a_1 = h_2 + a_2$ implies $h_1(x) = h_2(x)$ contrary to $|h_1 - h_2| > \delta > 0$ on E. So $h_1 - h_2 \in S(\mathcal{N})$ and $a_1 - a_2 \in S(\mathcal{N})$. Now fix $\gamma_0 \in \Gamma$. Let F be $F_1 \cap F_2$ where

$$F_1 = \cap\{gA_{\gamma_0} : a_1(g, \gamma_0) \neq 0 \text{ or } a_2(g, \gamma_0) \neq 0\}$$

and

$$F_2 = \cap\{gA_\gamma^c : a_1(g, \gamma) \neq 0 \text{ or } a_2(g, \gamma) \neq 0 \text{ and } \gamma \neq \gamma_0\}.$$

Assume $\Sigma\{a_1(g, \gamma_0) : g \in G\} \neq \Sigma\{a_2(g, \gamma_0) : g \in G\}$. Then $F \in \mathcal{N}$ because $a_1 - a_2 \in S(\mathcal{N})$ and $a_1 - a_2 = \Sigma\{a_1(g, \gamma_0) : g \in G\} - \Sigma\{a_2(g, \gamma_0) : g \in G\}$ on F. This means $F^c \in \mathcal{R}\backslash\mathcal{N}$ and then the independence of $\{A_\gamma : \gamma \in \Gamma\}$ implies $F^c \cap F \neq \emptyset$ which is impossible. Thus, $h_1 + a_1 = h_2 + a_2$ says that $h_1 - h_2 \in S(\mathcal{N})$ and for each $\gamma_0 \in \Gamma$, $\Sigma\{a_1(g, \gamma_0) : g \in G\} = \Sigma\{a_2(g, \gamma_0) : g \in G\}$. Because θ is independent of $\mathcal{N}$, $\theta(h_1) = \theta(h_2)$ if $h_1 - h_2 \in S(\mathcal{N})$. Hence, θ_ϵ is well-defined.

Similarly, suppose $h + a \geq 0$ where $h \in S(\mathcal{R})$ and $a \in A$. Let Γ_0 be all $\gamma_0 \in \Gamma$ with $s(\gamma_0) = \Sigma\{a(g, \gamma_0) : g \in G\} < 0$. We claim that $\theta(h) + \Sigma\{s(\gamma_0) : \gamma_0 \in \Gamma_0\} \geq 0$. It would be enough to show that for all $\delta < 0$, $h + \Sigma\{s(\gamma_0) : \gamma_0 \in \Gamma_0\} \geq \delta$ except on a set $N_\delta \in \mathcal{N}$. For then for all $\delta < 0$,

$$\theta(h) + \Sigma\{s(\gamma_0) : \gamma_0 \in \Gamma_0\} \geq \delta$$

because θ is independent of $\mathcal{N}$. But if this is not the case for some $\delta < 0$, then $E = \{x : h(x) + \Sigma\{s(\gamma_0) : \gamma_0 \in \Gamma_0\} < \delta\}$ is not in $\mathcal{N}$. The independence of $\{A_\gamma : \gamma \in \Gamma\}$ says then that for some $x_0 \in E$, $a(x_0) = \Sigma\{s(\gamma_0) : \gamma_0 \in \Gamma_0\}$. So we have $0 \leq (h + a)(x_0) = h(x_0) + \Sigma\{s(\gamma_0) : \gamma_0 \in \Gamma_0\}$ which contradicts $x_0 \in E$. The conclusion is that

$h + a \geq 0$ implies

$$\begin{aligned} 0 &\leq \theta(h) + \Sigma\{s(\gamma_0) : \gamma_0 \in \Gamma_0\} \\ &\leq \theta(h) + \Sigma\{s(\gamma_0)\epsilon(\gamma_0) : \gamma_0 \in \Gamma_0\} \\ &\leq \theta(h) + \Sigma\{a(g, \gamma)\epsilon(\gamma) : g \in G,\ \gamma \in \Gamma\} \\ &= \theta_\epsilon(h + a). \end{aligned}$$

Hence, $\theta_\epsilon \geq 0$ and the proof is complete. □

REMARK: The independence property of $\{A_\gamma : \gamma \in \Gamma\}$ that is used in the proof of 2.5 is weaker than the one used in Theorem 1.2. But it is sufficient to show that the initial extension θ_e of θ is well-defined and positive. The independence property of $\{A_\gamma : \gamma \in \Gamma\}$ which is assumed in 2.5 guarantees that for any $\gamma_1 \neq \gamma_2$, $\chi_{A_{\gamma 1}}$ and $\chi_{A_{\gamma 2}}$ are linearly independent modulo $S(\mathscr{R})$. But it does not follow necessarily that for $g_1 \neq g_2$, $\chi_{g_1 A_\gamma}$ and $\chi_{g_2 A_\gamma}$ are linearly independent modulo $S(\mathscr{R})$.

EXAMPLE 1: Let G be a non-discrete σ-compact locally compact metric group. Let $\mathscr{R}$ be the Borel sets and $\mathcal{N}$ the countable sets, then card $(\mathscr{R})$ = card (G) = card $(E) = c$ for all $E \in \mathscr{R} \backslash \mathcal{N}$. So we can choose $\{A_\gamma : \gamma \in \Gamma\}$ as in 2.3 and apply 2.4. Because any invariant mean on $S(\mathscr{R})$ is independent of $\mathcal{N}$, when G is amenable as discrete, any invariant mean on the bounded Borel measurable functions has 2^{2^c} mutually singular extensions to an invariant mean of $\ell_\infty(G)$.

EXAMPLE 2. Let G again be a non-discrete σ-compact locally compact metric group. Let β be the Lebesgue measurable sets and $\mathcal{N}$ the λ-null sets in β. Choose $\{A_\gamma : \gamma \in \Gamma\}$ as in Example 1. Because any $E \in \beta$ with $\lambda(E) > 0$ contains an uncountable Borel set, $\{A_\gamma : \gamma \in \Gamma\}$ satisfies the hypotheses of 2.4 with $\mathscr{R} = \beta$ and $\mathcal{N}$ the λ-null sets. Thus, if G is amenable as discrete, any invariant mean on the bounded Lebesgue measurable functions which is independent of λ-null sets has 2^{2^c} mutually singular extensions to an invariant mean on $\ell_\infty(G)$.

EXAMPLE 3: Let G be any amenable group with card $(G) > \aleph_0$. Let $\mathscr{R}$ be the trivial σ-algebra and $\mathcal{N} = \{\emptyset\}$. Then Proposition 2.3 and Theorem 2.4 prove that there are $2^{2^{\operatorname{card}(G)}}$ left-invariant means on $\ell_\infty(G)$. This proves a result in Chou [3] by essentially the same method. A question asked in Chou [3] is if there are 2^c mutually singular bi-invariant and inversion invariant means on $\ell_\infty(G)$ when G is a countably infinite amenable group. Here is an argument which answers this question for all infinite amenable groups simultaneously. Let G be any infinite group and let A be the least ordinal with card (A) = card (G). Enumerate $G = \{g_\alpha : \alpha \in A\}$. For each $\beta \in A$, let $F_\beta = \{g_\alpha : \alpha \leq \beta\}$. Fix a set of elements $\{x_\beta : \beta \in A\} \subset G$. Let $Z_\beta = F_\beta x_\beta \cup x_\beta F_\beta$ and let $Y_\beta = Z_\beta \cup Z_\beta^{-1}$. A transfinite induction argument shows that one can choose $\{x_\beta : \beta \in A\}$ so that $\{Y_\beta : \beta \in A\}$ are pairwise disjoint. Let $\{\mathscr{E}_\alpha : \alpha \in A\}$ be a partition of the set $\{Y_\beta : \beta \in A\}$ so that card ($\mathscr{E}_\alpha$) = card (A) for each $\alpha \in A$. Then for each $\alpha \in A$ and $\beta_0 \in A$, there is $\beta \geq \beta_0$ with $Y_\beta \in \mathscr{E}_\alpha$. Let $X_\alpha = \cup \{Y_\beta : Y_\beta \in \mathscr{E}_\alpha\}$ for each $\alpha \in A$. This gives pairwise disjoint sets $\{X_\alpha : \alpha \in A\}$ which are symmetric subsets of G. Also, because any finite number of elements of G are in F_β for β sufficiently large, one has for any $g_1, \ldots, g_m$, $h_1, \ldots, h_m \in G$ and $\alpha \in A \cap_{i=1}^m g_i X_\alpha \cap \cap_{i=1}^m X_\alpha h_i \neq \phi$. One now uses Boolean independent subsets of the index A as in Lemma 2.2 to get $2^{\text{card}(G)}$ symmetric subsets $\{A_\gamma : \gamma \in \Gamma\}$ of G such that for any $e : \Gamma \to \{l, c\}$,

$$\bigcap_\gamma \left(\bigcap_{x \in G} x \hat{A}_\gamma^{e(\gamma)} \cap \bigcap_{y \in G} \hat{A}_\gamma^{e(\gamma)} y \right) \neq \phi.$$

But each of these sets is a symmetric bi-invariant set in βG. This proves that in any infinite group G, there are $2^{2^{\text{card}(G)}}$ pairwise disjoint nonempty closed symmetric bi-invariant subsets of βG. Hence, if G is an infinite amenable group, then there are $2^{2^{\text{card}(G)}}$ mutually singular bi-invariant and inversion invariant means on $\ell_\infty(G)$.

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