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**DIVIDING RATIONAL POINTS ON ABELIAN
VARIETIES OF CM-TYPE**

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This note has to do with the general problem of Galois representations arising from abelian varieties of CM-type. More particularly, we wish to see what happens when one takes the ℓ^{th} roots (ℓ a varying prime) of a fixed set of rational points on a simple abelian variety A of CM-type. Provided that the rational points are independent over the endomorphism ring of A , the Galois groups that one obtains are as large as possible for all but finitely many ℓ . (See the theorem below for a precise statement.)

This result has recently been applied by Coates and Lang in a study involving diophantine approximation [4]. Similar results were previously obtained by Bašmakov [1, 2], who studied elliptic curves (both with and without complex multiplication). A special case was also discussed in [3].

1. Statement of the result, and beginning of the proof

Let A be an abelian variety over a number field K . We assume that all endomorphisms of A are defined over K and that the algebra

$$F = (\text{End } A) \otimes \mathbb{Q}$$

is a *field* of degree $2 \cdot \dim A$. Thus A is simple and of CM-type.

If ℓ is a prime, let

$$\rho_\ell : \text{Gal}(\bar{K}/K) \rightarrow \text{Aut } A_\ell$$

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be the character giving the action of $\text{Gal}(\bar{K}/K)$ on the group of ℓ -division points of A . Let $G_\ell \subseteq \text{Aut } A_\ell$ be the image of ρ_ℓ , and let $k_\ell = K(A_\ell)$ be the corresponding Galois extension of K .

Now let x_1, \dots, x_n be elements of the group $A(K)$ of K -rational points of A . Let K_ℓ be the extension of K obtained by adjoining to K all ℓ^{th} roots of all the points x_i . (These roots are taken in a fixed algebraic closure \bar{K} of K .) Then K_ℓ is a Galois extension of K which contains k_ℓ . Let G , H_ℓ , and C_ℓ be the Galois groups in the following diagram:

$$G \left(\begin{array}{c} \bar{K} \\ \downarrow \\ K_\ell \\ \downarrow \\ k_\ell \\ \downarrow \\ K \end{array} \right) \begin{array}{l} C_\ell \\ G_\ell \end{array} .$$

In view of the action of H_ℓ on the ℓ^{th} roots of the x_i , we may view C_ℓ as a subgroup of the abelian group

$$B_\ell = A_\ell \times \cdots \times A_\ell \text{ (} n \text{ times)}.$$

In fact, for any $x \in A(K)$, we define a continuous homomorphism

$$\varphi_x : H_\ell \rightarrow A_\ell$$

as follows: take any ℓ^{th} root r of x , and set $\varphi_x(\sigma) = \sigma r - r$ if $\sigma \in H_\ell$. It is immediate that φ_x is independent of the choice of r and that φ_x is a homomorphism which induces an isomorphism of the Galois group $\text{Gal}(k_\ell(\ell^{-1}x)/k_\ell)$ with a subgroup of A_ℓ . Set $\varphi_i = \varphi_{x_i}$ ($i = 1, \dots, n$), and put

$$\varphi = \varphi_1 \times \cdots \times \varphi_n.$$

Then φ is a continuous homomorphism $H_\ell \rightarrow B_\ell$ which induces an injection $C_\ell \hookrightarrow B_\ell$. It is sometimes useful to identify C_ℓ with its image in B_ℓ .

Before stating the theorem, we make one more remark on terminology. If M is a module over a ring R and if $m_1, \dots, m_n \in M$, we say that m_1, \dots, m_n are *linearly independent* (over R) if no non-trivial linear combination $\sum a_i m_i$ vanishes ($a_i \in R$).

THEOREM: *Assume that $x_1, \dots, x_n \in A(K)$ are linearly independent over $\text{End } A$. Then $C_\ell = B_\ell$ for all but finitely many primes ℓ .*

We shall show, first of all, that $B_\ell = C_\ell$ whenever ℓ satisfies a certain pair of conditions. Then, in the remaining two sections, we will show that each condition is satisfied provided that ℓ is sufficiently large.

Let O be the integer ring of F . One knows that $\text{End } A = \text{End}_K A$ is a subring of finite index in O . We shall always assume that our primes ℓ are unramified in F and prime to the index $(O : \text{End } A)$. This condition, satisfied by all but finitely many ℓ , implies that

$$(\text{End } A)/\ell(\text{End } A) = O/\ell O$$

is a product of fields and that A_ℓ is free of rank 1 over $(\text{End } A)/\ell(\text{End } A)$ [6, pp. 501–502]. Then we have

$$G_\ell \subseteq (O/\ell O)^* = \text{Aut}_{O/\ell O} A_\ell.$$

On the other hand, it is easy to see that C_ℓ is a G_ℓ -stable subgroup of B_ℓ . Indeed, this follows from the general formula

$$\varphi_x(\tau\sigma\tau^{-1}) = \tau \cdot \varphi_x(\sigma)$$

valid for $x \in A(K)$, $\tau \in G$, $\sigma \in H_\ell$.

LEMMA: *Let R be a product of fields, and let V be a free rank-1 module over R . Suppose that C is an R -submodule of $B = V \times \dots \times V$ (n times) which is strictly smaller than B . Then there are elements t_1, \dots, t_n of R , not all 0, such that*

$$\sum t_i v_i = 0$$

for all $(v_1, \dots, v_n) \in C$.

PROOF: Clear.

COROLLARY: *We have $C_\ell = B_\ell$ whenever the following two conditions are verified:*

- (i) *The subring $F_\ell[G_\ell]$ of $O/\ell O$ generated by the elements of G_ℓ is in fact all of $O/\ell O$.*
- (ii) *The homomorphisms $\varphi_1, \dots, \varphi_n : H_\ell \rightarrow A_\ell$ are linearly independent over $O/\ell O$.*

PROOF: Given condition (i), we apply the lemma with $R = O/\ell O$, $C = C_\ell$, $B = B_\ell$.

2. Galois action on points of finite order (verification of (i))

Let p be any rational prime which splits completely in the multiplication field F and such that A has good reduction at some prime of K lying over p . Let v be such a prime. Since the \mathbf{Q}_ℓ -adic Tate module V_ℓ of A is free of rank 1 over $F \otimes \mathbf{Q}_\ell$, and since all endomorphisms of A are defined over K , V_ℓ is the direct sum of $\text{Gal}(\bar{K}/K)$ -modules which are 1-dimensional over \mathbf{Q}_ℓ . By the Serre-Tate lifting theory, this implies that the endomorphism algebra $(\text{End } \tilde{A}_v) \otimes \mathbf{Q}$ of the reduction of A at v is precisely equal to $(\text{End } A) \otimes \mathbf{Q} = F$ [5, Theorem 2, p. IV-41; Cor., p. IV-42]. Since F is commutative, Tate's theorem says that $F = \mathbf{Q}(\pi_v)$, where $\pi_v \in O$ is the Frobenius endomorphism of \tilde{A}_v [9, Th. 2(a), p. 140]. This implies that the ring $\mathbf{Z}[\pi_v]$ has finite index in O .

PROPOSITION: *If ℓ is sufficiently large, then $F_\ell[G_\ell] = O/\ell O$.*

PROOF: From the above discussion we see that $F_\ell[\pi_v] = O/\ell O$ whenever ℓ is prime to the index of $\mathbf{Z}[\pi_v]$ in O . But if $\ell \neq p$ then π_v (or rather its image in $O/\ell O$) belongs to G_ℓ : it is the image in G_ℓ of any Frobenius element for v in $\text{Gal}(\bar{K}/K)$. We have then

$$O/\ell O = F_\ell[\pi_v] \subseteq F_\ell[G_\ell] \subseteq O/\ell O$$

if ℓ is prime to $(O : \mathbf{Z}[\pi_v])$ and different from p .

REMARK: Shimura has given an alternate proof of this proposition based on the theory of complex multiplication [8, Th. 1, p. 110], [7, Prop. 1.9]. As a compromise, one may obtain primes v for which $F = (\text{End } \tilde{A}_v) \otimes \mathbf{Q}$ by using [8, Th. 2, p. 114] and then employ Tate's Theorem as above.

3. Application of the Mordell-Weil theorem (verification of (ii))

We consider the sequence

$$A(K) \xrightarrow{\text{"}\ell\text{"}} A(K) \xrightarrow{\delta} H^1(G, A_\ell)$$

obtained by taking cohomology in the short exact sequence

$$0 \rightarrow A_\ell \rightarrow A(\bar{K}) \xrightarrow{\ell} A(\bar{K}) \rightarrow 0.$$

(“ ℓ ” is the map “multiplication by ℓ .”)

LEMMA:

1. *The map $h : A(K) \rightarrow \text{Hom}(H_\ell, A_\ell)$ defined by $x \mapsto \varphi_x$ is $(\text{End } A)$ -linear.*

2. *Further, h is the composition of δ with the restriction homomorphism*

$$\text{res} : H^1(G, A_\ell) \rightarrow H^1(H_\ell, A_\ell) = \text{Hom}(H_\ell, A_\ell).$$

3. *The map res is injective.*

PROOF: The first two statements are proved by a direct computation, which we omit. The third follows from the restriction-inflation sequence together with the vanishing of

$$H^1(G/H_\ell, A_\ell) = H^1(G_\ell, A_\ell).$$

This cohomology group vanishes because A_ℓ is an ℓ -group, whereas $G_\ell \subseteq (O/\ell O)^*$ has prime-to- ℓ order.

COROLLARY: *The map h induces an $(O/\ell O)$ -linear injection*

$$A(K)/\ell A(K) \hookrightarrow \text{Hom}(H_\ell, A_\ell).$$

Hence $\varphi_1, \dots, \varphi_n$ are linearly independent if and only if the images $\bar{x}_1, \dots, \bar{x}_n$ of x_1, \dots, x_n in $A(K)/\ell A(K)$ are linearly independent over $O/\ell O$.

PROOF: Clear.

PROPOSITION: *If ℓ is sufficiently large, then $\varphi_1, \dots, \varphi_n$ are linearly independent.*

PROOF: Because of the corollary, it suffices to prove that the map

$$\Gamma/\ell\Gamma \xrightarrow{i} A(K)/\ell A(K)$$

is injective, where Γ is the subgroup of $A(K)$ generated over O by x_1, \dots, x_n . Let

$$\Gamma' = \{y \in A(K) \mid my \in \Gamma \text{ for some } m \in \mathbf{Z}\}.$$

By the Mordell-Weil Theorem, Γ' is finitely generated, and hence the index $(\Gamma' : \Gamma)$ is finite. One sees that j is injective whenever ℓ is prime to $(\Gamma' : \Gamma)$.¹

As noted above, the theorem follows from the corollary of §1 together with the above proposition and the proposition of §2.

¹ Cassels remarks that one may avoid the use of the Mordell-Weil theorem here by using properties of heights and a trick from diophantine approximation.

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