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**DIVIDING RATIONAL POINTS ON ABELIAN  
VARIETIES OF CM-TYPE**

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This note has to do with the general problem of Galois representations arising from abelian varieties of CM-type. More particularly, we wish to see what happens when one takes the  $\ell^{\text{th}}$  roots ( $\ell$  a varying prime) of a fixed set of rational points on a simple abelian variety  $A$  of CM-type. Provided that the rational points are independent over the endomorphism ring of  $A$ , the Galois groups that one obtains are as large as possible for all but finitely many  $\ell$ . (See the theorem below for a precise statement.)

This result has recently been applied by Coates and Lang in a study involving diophantine approximation [4]. Similar results were previously obtained by Bašmakov [1, 2], who studied elliptic curves (both with and without complex multiplication). A special case was also discussed in [3].

**1. Statement of the result, and beginning of the proof**

Let  $A$  be an abelian variety over a number field  $K$ . We assume that all endomorphisms of  $A$  are defined over  $K$  and that the algebra

$$F = (\text{End } A) \otimes \mathbb{Q}$$

is a *field* of degree  $2 \cdot \dim A$ . Thus  $A$  is simple and of CM-type.

If  $\ell$  is a prime, let

$$\rho_\ell : \text{Gal}(\bar{K}/K) \rightarrow \text{Aut } A_\ell$$

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be the character giving the action of  $\text{Gal}(\bar{K}/K)$  on the group of  $\ell$ -division points of  $A$ . Let  $G_\ell \subseteq \text{Aut } A_\ell$  be the image of  $\rho_\ell$ , and let  $k_\ell = K(A_\ell)$  be the corresponding Galois extension of  $K$ .

Now let  $x_1, \dots, x_n$  be elements of the group  $A(K)$  of  $K$ -rational points of  $A$ . Let  $K_\ell$  be the extension of  $K$  obtained by adjoining to  $K$  all  $\ell^{\text{th}}$  roots of all the points  $x_i$ . (These roots are taken in a fixed algebraic closure  $\bar{K}$  of  $K$ .) Then  $K_\ell$  is a Galois extension of  $K$  which contains  $k_\ell$ . Let  $G$ ,  $H_\ell$ , and  $C_\ell$  be the Galois groups in the following diagram:

$$G \left( H_\ell \left( \begin{array}{c} \bar{K} \\ | \\ K_\ell \\ | \\ k_\ell \\ | \\ K \end{array} \right) C_\ell \right) G_\ell .$$

In view of the action of  $H_\ell$  on the  $\ell^{\text{th}}$  roots of the  $x_i$ , we may view  $C_\ell$  as a subgroup of the abelian group

$$B_\ell = A_\ell \times \cdots \times A_\ell \text{ (} n \text{ times)}.$$

In fact, for any  $x \in A(K)$ , we define a continuous homomorphism

$$\varphi_x : H_\ell \rightarrow A_\ell$$

as follows: take any  $\ell^{\text{th}}$  root  $r$  of  $x$ , and set  $\varphi_x(\sigma) = \sigma r - r$  if  $\sigma \in H_\ell$ . It is immediate that  $\varphi_x$  is independent of the choice of  $r$  and that  $\varphi_x$  is a homomorphism which induces an isomorphism of the Galois group  $\text{Gal}(k_\ell(\ell^{-1}x)/k_\ell)$  with a subgroup of  $A_\ell$ . Set  $\varphi_i = \varphi_{x_i}$  ( $i = 1, \dots, n$ ), and put

$$\varphi = \varphi_1 \times \cdots \times \varphi_n.$$

Then  $\varphi$  is a continuous homomorphism  $H_\ell \rightarrow B_\ell$  which induces an injection  $C_\ell \hookrightarrow B_\ell$ . It is sometimes useful to identify  $C_\ell$  with its image in  $B_\ell$ .

Before stating the theorem, we make one more remark on terminology. If  $M$  is a module over a ring  $R$  and if  $m_1, \dots, m_n \in M$ , we say that  $m_1, \dots, m_n$  are *linearly independent* (over  $R$ ) if no non-trivial linear combination  $\sum a_i m_i$  vanishes ( $a_i \in R$ ).

**THEOREM:** *Assume that  $x_1, \dots, x_n \in A(K)$  are linearly independent over  $\text{End } A$ . Then  $C_\ell = B_\ell$  for all but finitely many primes  $\ell$ .*

We shall show, first of all, that  $B_\ell = C_\ell$  whenever  $\ell$  satisfies a certain pair of conditions. Then, in the remaining two sections, we will show that each condition is satisfied provided that  $\ell$  is sufficiently large.

Let  $O$  be the integer ring of  $F$ . One knows that  $\text{End } A = \text{End}_K A$  is a subring of finite index in  $O$ . We shall always assume that our primes  $\ell$  are unramified in  $F$  and prime to the index  $(O : \text{End } A)$ . This condition, satisfied by all but finitely many  $\ell$ , implies that

$$(\text{End } A)/\ell(\text{End } A) = O/\ell O$$

is a product of fields and that  $A_\ell$  is free of rank 1 over  $(\text{End } A)/\ell(\text{End } A)$  [6, pp. 501–502]. Then we have

$$G_\ell \subseteq (O/\ell O)^* = \text{Aut}_{O/\ell O} A_\ell.$$

On the other hand, it is easy to see that  $C_\ell$  is a  $G_\ell$ -stable subgroup of  $B_\ell$ . Indeed, this follows from the general formula

$$\varphi_x(\tau\sigma\tau^{-1}) = \tau \cdot \varphi_x(\sigma)$$

valid for  $x \in A(K)$ ,  $\tau \in G$ ,  $\sigma \in H_\ell$ .

**LEMMA:** *Let  $R$  be a product of fields, and let  $V$  be a free rank-1 module over  $R$ . Suppose that  $C$  is an  $R$ -submodule of  $B = V \times \dots \times V$  ( $n$  times) which is strictly smaller than  $B$ . Then there are elements  $t_1, \dots, t_n$  of  $R$ , not all 0, such that*

$$\sum t_i v_i = 0$$

for all  $(v_1, \dots, v_n) \in C$ .

**PROOF:** Clear.

**COROLLARY:** *We have  $C_\ell = B_\ell$  whenever the following two conditions are verified:*

- (i) *The subring  $F_\ell[G_\ell]$  of  $O/\ell O$  generated by the elements of  $G_\ell$  is in fact all of  $O/\ell O$ .*
- (ii) *The homomorphisms  $\varphi_1, \dots, \varphi_n : H_\ell \rightarrow A_\ell$  are linearly independent over  $O/\ell O$ .*

PROOF: Given condition (i), we apply the lemma with  $R = O/\ell O$ ,  $C = C_\ell$ ,  $B = B_\ell$ .

## 2. Galois action on points of finite order (verification of (i))

Let  $p$  be any rational prime which splits completely in the multiplication field  $F$  and such that  $A$  has good reduction at some prime of  $K$  lying over  $p$ . Let  $v$  be such a prime. Since the  $\mathbf{Q}_\ell$ -adic Tate module  $V_\ell$  of  $A$  is free of rank 1 over  $F \otimes \mathbf{Q}_\ell$ , and since all endomorphisms of  $A$  are defined over  $K$ ,  $V_\ell$  is the direct sum of  $\text{Gal}(\bar{K}/K)$ -modules which are 1-dimensional over  $\mathbf{Q}_\ell$ . By the Serre-Tate lifting theory, this implies that the endomorphism algebra  $(\text{End } \tilde{A}_v) \otimes \mathbf{Q}$  of the reduction of  $A$  at  $v$  is precisely equal to  $(\text{End } A) \otimes \mathbf{Q} = F$  [5, Theorem 2, p. IV-41; Cor., p. IV-42]. Since  $F$  is commutative, Tate's theorem says that  $F = \mathbf{Q}(\pi_v)$ , where  $\pi_v \in O$  is the Frobenius endomorphism of  $\tilde{A}_v$  [9, Th. 2(a), p. 140]. This implies that the ring  $\mathbf{Z}[\pi_v]$  has finite index in  $O$ .

PROPOSITION: *If  $\ell$  is sufficiently large, then  $F_\ell[G_\ell] = O/\ell O$ .*

PROOF: From the above discussion we see that  $F_\ell[\pi_v] = O/\ell O$  whenever  $\ell$  is prime to the index of  $\mathbf{Z}[\pi_v]$  in  $O$ . But if  $\ell \neq p$  then  $\pi_v$  (or rather its image in  $O/\ell O$ ) belongs to  $G_\ell$ : it is the image in  $G_\ell$  of any Frobenius element for  $v$  in  $\text{Gal}(\bar{K}/K)$ . We have then

$$O/\ell O = F_\ell[\pi_v] \subseteq F_\ell[G_\ell] \subseteq O/\ell O$$

if  $\ell$  is prime to  $(O : \mathbf{Z}[\pi_v])$  and different from  $p$ .

REMARK: Shimura has given an alternate proof of this proposition based on the theory of complex multiplication [8, Th. 1, p. 110], [7, Prop. 1.9]. As a compromise, one may obtain primes  $v$  for which  $F = (\text{End } \tilde{A}_v) \otimes \mathbf{Q}$  by using [8, Th. 2, p. 114] and then employ Tate's Theorem as above.

## 3. Application of the Mordell-Weil theorem (verification of (ii))

We consider the sequence

$$A(K) \xrightarrow{\text{"}\ell\text{"}} A(K) \xrightarrow{\delta} H^1(G, A_\ell)$$

obtained by taking cohomology in the short exact sequence

$$0 \rightarrow A_\ell \rightarrow A(\bar{K}) \xrightarrow{\ell} A(\bar{K}) \rightarrow 0.$$

(“ $\ell$ ” is the map “multiplication by  $\ell$ .”)

LEMMA:

1. *The map  $h : A(K) \rightarrow \text{Hom}(H_\ell, A_\ell)$  defined by  $x \mapsto \varphi_x$  is  $(\text{End } A)$ -linear.*

2. *Further,  $h$  is the composition of  $\delta$  with the restriction homomorphism*

$$\text{res} : H^1(G, A_\ell) \rightarrow H^1(H_\ell, A_\ell) = \text{Hom}(H_\ell, A_\ell).$$

3. *The map  $\text{res}$  is injective.*

PROOF: The first two statements are proved by a direct computation, which we omit. The third follows from the restriction-inflation sequence together with the vanishing of

$$H^1(G/H_\ell, A_\ell) = H^1(G_\ell, A_\ell).$$

This cohomology group vanishes because  $A_\ell$  is an  $\ell$ -group, whereas  $G_\ell \subseteq (O/\ell O)^*$  has prime-to- $\ell$  order.

COROLLARY: *The map  $h$  induces an  $(O/\ell O)$ -linear injection*

$$A(K)/\ell A(K) \hookrightarrow \text{Hom}(H_\ell, A_\ell).$$

*Hence  $\varphi_1, \dots, \varphi_n$  are linearly independent if and only if the images  $\bar{x}_1, \dots, \bar{x}_n$  of  $x_1, \dots, x_n$  in  $A(K)/\ell A(K)$  are linearly independent over  $O/\ell O$ .*

PROOF: Clear.

PROPOSITION: *If  $\ell$  is sufficiently large, then  $\varphi_1, \dots, \varphi_n$  are linearly independent.*

PROOF: Because of the corollary, it suffices to prove that the map

$$\Gamma/\ell\Gamma \xrightarrow{i} A(K)/\ell A(K)$$

is injective, where  $\Gamma$  is the subgroup of  $A(K)$  generated over  $O$  by  $x_1, \dots, x_n$ . Let

$$\Gamma' = \{y \in A(K) \mid my \in \Gamma \text{ for some } m \in \mathbf{Z}\}.$$

By the Mordell-Weil Theorem,  $\Gamma'$  is finitely generated, and hence the index  $(\Gamma' : \Gamma)$  is finite. One sees that  $j$  is injective whenever  $\ell$  is prime to  $(\Gamma' : \Gamma)$ .<sup>1</sup>

As noted above, the theorem follows from the corollary of §1 together with the above proposition and the proposition of §2.

<sup>1</sup> Cassels remarks that one may avoid the use of the Mordell-Weil theorem here by using properties of heights and a trick from diophantine approximation.

#### REFERENCES

- [1] M. BAŠMAKOV: Un théorème de finitude sur la cohomologie des courbes elliptiques. *C.R. Acad. Sci. Paris Sér. A-B* 270, A 999-A 1000 (1970).
- [2] M. BAŠMAKOV: The cohomology of abelian varieties over a number field. *Russian Math. Surveys* 27 (6) (1972) 25–70.
- [3] J. COATES: An application of the division theory of elliptic functions to diophantine approximation. *Inventiones math.* 11 (1970) 167–182.
- [4] J. COATES and S. LANG: Diophantine approximation on abelian varieties with complex multiplication. (To appear).
- [5] J-P. SERRE: *Abelian  $\ell$ -adic representations and elliptic curves*. New York: Benjamin 1968.
- [6] J-P. SERRE and J. TATE: Good reduction of abelian varieties. *Ann of Math.* 88 (1968) 492–517.
- [7] G. SHIMURA: On canonical models of arithmetic quotients of bounded symmetric domains. *Ann. of Math.* 91 (1970) 144–222.
- [8] G. SHIMURA and Y. TANIYAMA: Complex multiplication of abelian varieties and its applications to number theory. *Publ. Math. Soc. Japan* 6 (1961).
- [9] J. TATE: Endomorphisms of abelian varieties over finite fields. *Inventiones Math.* 2 (1966) 134–144.

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