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A-SYSTEMS*

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1. Introduction

The axiomatic study of the substitutive algebra of functions has its roots in the works of Schonfinkel [16], Curry [2] and Menger [8]. In 1959, Menger [12] introduced a set of axioms designed to describe the algebra of ordinary functions under addition, multiplication or composition. During the 1960's this work was continued, notably, by Schweizer and Sklar [17, 18, 20, 21]. Their initial paper [17] discusses a set of five axioms which, together, are equivalent to the six axioms given by Menger [12]. Their later articles focus attention on the axiomatic study of composition. The algebra of functions III culminates in two representation theorems, one of which gives sufficient conditions for a function to be represented as a union of minimal functions called atoms [20]. The purpose of this paper is to axiomatically describe the substitutive or additive behavior of atoms.

2. Preliminaries

An *a-system* is an ordered triple $(A, \circ, ')$ such that:
A1. $(A, \circ, ')$ is an inverse semigroup with null element $\emptyset$.
A2. If $a, b \in A$ and $\emptyset \neq a \circ b$ then $b \circ b' = a' \circ a$.

EXAMPLE (2.1): Let A consist of all restrictions of the identity function on the set $S(|S| \geqq 2)$. If "$\circ$" represents composition and, for any $f \in S$, $f = f'$, then $(A, \circ, ')$ is an inverse semigroup (with null element $\emptyset$) violating axiom A2.

* Some of these results appeared in a thesis written by the author under the guidance of Prof. A. Sklar.

EXAMPLE (2.2): Let S be any non-empty set and let $A = S \times S \cup \{\emptyset\}$. Define $(a, b) \circ (c, d) = (c, b)$ if $a = d$; otherwise $(a, b) \circ (c, d) = \emptyset$. Then $(A, \circ, ')$ is an a-system where $(a, b)' = (b, a)$.

EXAMPLE (2.3): Let $(G, +)$ be any group. Let $A = G \times G \cup \{\emptyset\}$, where $(a, b) \circ (c, d) = (a, b + d)$ if $a = c$; otherwise $(a, b) \circ (c, d) = \emptyset$. Also, for any $(a, b) \in A$, $\emptyset \circ (a, b) = \emptyset = (a, b) \circ \emptyset$. Then $(A, \circ, ')$ is an a-system where $(a, b)' = (a, -b)$.

EXAMPLE (2.4): Let $(R, +, \cdot)$ be any division ring. Let

$$A = \{f: R \to R |\text{ for any } x \in R,\ f(x) \neq 0\} \cup \{\theta\}$$

where $\theta: R \to R$ is given by $\theta(x) = 0$ for all $x \in R$. Define $f \circ g$ by: $(f \circ g)(x) = f(x) \cdot g(x)$. Then $(A, \circ, ')$ is an a-system where $f'(x) = (f(x))^{-1}$ and $\emptyset = \theta$.

In the sequel, $(A, \circ, ')$ denotes an a-system.

LEMMA (2.5): *If $a \in A$, then the following are equivalent*:

(i) $a = \emptyset$.
(ii) $a \circ a' = \emptyset$.
(iii) $a' = \emptyset$.
(iv) $a' \circ a = \emptyset$.

LEMMA (2.6): *If $a, b \in A$, $\emptyset \neq a \circ b$ then $(a \circ b)' \circ (a \circ b) = b' \circ b$ and $(a \circ b) \circ (a \circ b)' = a \circ a'$.*

PROOF: $(a \circ b)' \circ (a \circ b) = b' \circ a' \circ a \circ b = b' \circ b \circ b' \circ b = b' \circ b$. The other identity is proved similarly.

LEMMA (2.7): *Let $a(\neq \emptyset)$, $b(\neq \emptyset) \in A$. Then either $a \circ b \circ b' = \emptyset$ or $a \circ b \circ b' = a$. The latter case occurs if and only if $a \circ b \neq \emptyset$.*

PROOF: If $a \circ b \circ b' \neq \emptyset$, then $a \circ b \neq \emptyset$ whence $b \circ b' = a' \circ a$. Thus $a \circ b \circ b' = a \circ a' \circ a = a$.

Dually, we have

LEMMA (2.8): *Let $a(\neq \emptyset)$, $b(\neq \emptyset) \in A$. Then either $a' \circ a \circ b = \emptyset$ or $a' \circ a \circ b = b$. The latter case occurs if and only if $a \circ b \neq \emptyset$.*

LEMMA (2.9): If *$a(\neq \emptyset)$, $b(\neq \emptyset) \in A$ and $b \circ b' = a' \circ a$ then $a \circ b \neq \emptyset$.*

PROOF: Suppose $a \circ b = \emptyset$. Then $a \circ b \circ b' = \emptyset$ whence $a = \emptyset$.

LEMMA (2.10): *If* $a, b \in A$ *and* $\emptyset \neq b = a \circ b$ *then* $a = b \circ b'$. *Dually, if* $\emptyset \neq b = b \circ a$ *then* $a = b' \circ b$.

PROOF: $\emptyset \neq b = a \circ b$ implies $a' \circ a = b \circ b'$ whence (from Lemma 2.7) $\emptyset \neq b \circ b' = a \circ b \circ b' = a$.

For any $a \in A$, let $La = a \circ a'$, $Ra = a' \circ a$.

THEOREM (2.11): $(A, \circ, L, R)$ *is a function system*; *i.e.*, $(A, \circ, L, R)$ *satisfies*:

1. $(A, \circ)$ *is a semigroup.*
2. *For all elements* $a \in A$,
   (a) $LRa = Ra$, $RLa = La$;
   (b) $La \circ a = a = a \circ Ra$.
3. *For all elements* $a, b \in A$,
   (a) $L(a \circ b) = L(a \circ Lb)$, $R(a \circ b) = R(Ra \circ b)$;
   (b) $La \circ Rb = Rb \circ La$;
   (c) $Ra \circ b = b \circ R(a \circ b)$[21].

PROOF: See [21; theorem 23].

3. Categorical semigroups and Brandt semigroups

If a, b are elements of any function system then $a \subseteq b$ means $a = b \circ Ra$[21].

THEOREM (3.1): *In any a-system,* "$\subseteq$" *is trivial*; *i.e.*, $a, b \in A$, $a \subseteq b$ *implies* $a = b$ *or* $a = \emptyset$.

PROOF: If $a \subseteq b$ then $a = b \circ a' \circ a$. If $a \neq \emptyset$ then, by Lemma 2.7, $a = b$.

COROLLARY (3.2): $(A, \circ, L, R)$ *is a categorical semigroup*; *i.e.*, $(A, \circ, L, R)$ *possesses a zero element* $\emptyset$ *satisfying* $R\emptyset = \emptyset$ *and*

1. $(A, \circ)$ *is a semigroup.*
2. *For all elements* $a \in A$,
   (a) $LRa = Ra$, $RLa = La$;
   (b) $La \circ a = a = a \circ Ra$.
3. *For all* a, b *in* A, $a \circ b \neq \emptyset$ *if and only if* $a \neq \emptyset, b \neq \emptyset$ *and* $Ra = Lb$[21].

PROOF: See [21; theorem 25].

EXAMPLE (3.3): Let

$$C = \{f\colon \mathfrak{R} \to \mathfrak{R} \mid \operatorname{dom} f = \mathfrak{R},\ f \text{ is constant, } f(x) \geqq 0\} \cup \{\emptyset\}.$$

For any $f(\neq \emptyset) \in C$, let Lf, $Rf \in C$ be defined by: $Lf(x) = Rf(x) = 0$ for all $x \in \mathfrak{R}$. Then $(C, +, L, R)$ is a categorical semigroup violating axiom A1.

LEMMA (3.4): *Let $(C, \circ, L, R)$ be a categorical semigroup. If, for any $a \in C$ there exists $x \in C$ such that $x \circ a = Ra$, then $a \circ x = La$.*

PROOF: Let $x \circ a = Ra \neq \emptyset$. Then $Rx = La$ and

$$a \circ x \circ a = a \circ Ra = a \neq \emptyset$$

whence $a \circ x \neq \emptyset$. Hence $Ra = Lx$. Now, there exists $y \in C$ such that $y \circ x = Rx$. Thus $Ry = Lx$ and $Ly = Rx$. Hence $y \circ x = La$ which implies that $y \circ x \circ a = La \circ a = a$. Thus $y \circ Ra = y \circ Ry = y = a$.

LEMMA (3.5): *If $(C, \circ, L, R)$ is a categorical semigroup such that for any $a \in C$ there exists $a' \in C$ such that $a' \circ a = Ra$ then C is cancellative; i.e.,*

$$\emptyset \neq a \circ b = a \circ c \quad \text{implies} \quad b = c$$

and

$$\emptyset \neq b \circ a = c \circ a \quad \textit{implies} \quad b = c.$$

PROOF: $\emptyset \neq b \circ a = c \circ a$ implies $Rb = La = Rc$ whence by Lemma 3.4, $b = b \circ Rb = b \circ La = b \circ a \circ a' = c \circ a \circ a' = c \circ La = c \circ Rc = c$.

THEOREM (3.6): *Let $(C, \circ, L, R)$ be a categorical semigroup. Then C is an a-system if and only if for any $a \in C$ there exists $a' \in C$ such that $a' \circ a = Ra$.*

PROOF: Suppose that C is a categorical semigroup having the above property. Then, for any $a \in C$, $a' \circ a \circ a' \circ a = a' \circ a \circ Ra = a' \circ a$. By cancelling, we get:

$$a \circ a' \circ a = a$$

and

$$a' \circ a \circ a' = a'.$$

Notice that a' must be unique (by the cancellative property). Thus $(C, \circ, ')$ is an inverse semigroup with null element $\emptyset$. Let $a, b \in C$. If $\emptyset \neq a \circ b$ then $Ra = Lb$; i.e., $a' \circ a = b \circ b'$.

Conversely, let $(C, \circ, L, R)$ be a categorical semigroup which is also an a-system. Let $a(\neq \emptyset) \in C$. Then $\emptyset \neq a = a \circ Ra$ and the result follows from Lemma 2.10.

THEOREM (3.7): *Let $(A, \circ, ')$ be an a-system. If, for any two idempotents $a, b \in A$ there exists $x \in A$ such that $a \circ x \circ b \neq \emptyset$ then A is a Brandt semigroup and conversely, where a Brandt semigroup is defined to be a semigroup $(B, \cdot)$ with zero element $\emptyset$ satisfying:*

1. *If $a, b, c \in B$ and, if $ac = bc \neq \emptyset$ or $ca = cb \neq \emptyset$ then $a = b$.*
2. *If $a, b, c \in B$ and if $ab \neq \emptyset$ and $bc \neq 0$ then $abc \neq \emptyset$.*
3. *For each $a(\neq \emptyset)$ in B there exists a unique $e \in B$ such that $ea = a$, a unique $f \in B$ such that $af = a$ and a unique $a' \in B$ such that $a'a = f$.*
4. *If e, f are non-zero idempotents of B, then there exists $a \in B$ such that $eaf \neq \emptyset$[1].*

PROOF: Let $(A, \circ, ')$ be an a-system. Let $a(\neq \emptyset) \in A$. Let $e = a \circ a'$ and $f = a' \circ a$. Then $e \circ a = a = a \circ f$. Moreover, $x \circ a = f$ implies $x = a'$ since A is cancellative.

Conversely, let $(B, \circ)$ be a Brandt semigroup. Then B is an inverse semigroup with null element $\emptyset$ [1; page 102]. Let $\emptyset \neq a \circ b$. Then there exists $e \in B$ such that $e \circ a \circ b = a \circ b$. Thus

$$e = a \circ a' = (a \circ b) \circ (a \circ b)' = a \circ b \circ b' \circ a'.$$

Hence

$$a' = a' \circ a \circ a' = a' \circ a \circ b \circ b' \circ a'.$$

Thus, $a' \circ a = a' \circ a \circ b \circ b'$; i.e., $a = a \circ b \circ b'$. But $a = a \circ a' \circ a$. Hence $a' \circ a = b \circ b'$.

COROLLARY (3.8): *Every Brandt semigroup is a categorical semigroup.*

EXAMPLE (3.9): Let $A = \{\emptyset, a, b\}$. Define

$$a \circ a = a; \quad b \circ b = b; \quad a \circ b = \emptyset = b \circ a; \quad a' = a; \quad b' = b.$$

Then $(A, \circ, ')$ is an a-system which is not a Brandt semigroup.

4. Functions over A

Let a, b be distinct elements of $(A, \circ, ')$. Then a and b are *inconsistent* if $a' \circ a = b' \circ b$ (cf. [11; page 169]).

If a and b are distinct elements of A and $a \circ Rb = b \circ Ra$ then either $\emptyset = a \circ Rb$ or $a = a \circ Rb$. In the former case $a' \circ a \neq b' \circ b$ and in the latter case $a = b$. In either case, a and b are consistent. Conversely, let a and b be distinct consistent elements. Then $a' \circ a \neq b' \circ b$. Hence $a \circ b' \circ b = a \circ Rb = \emptyset$. Similarly, $b \circ Ra = \emptyset$. Thus we have proved:

LEMMA (4.1): *Let $a, b \in A$. Then a and b are consistent if and only if $a \circ Rb = b \circ Ra$*[20].

LEMMA (4.2): *For any $a \in A$, a and $\emptyset$ are consistent.*

PROOF: $a \circ R\emptyset = \emptyset = \emptyset \circ Ra$.

LEMMA (4.3): *For any $a, b \in A$, $a \circ a'$ and $b \circ b'$ are consistent.*

PROOF: If $(a \circ a')' \circ (a \circ a') = (b \circ b')' \circ (b \circ b')$ then

$$a \circ a' \circ a \circ a' = a \circ a' = b \circ b' \circ b \circ b' = b \circ b'.$$

COROLLARY (4.4): *If $a, b \in A$, then $a \circ a'$, $a' \circ a$, $b' \circ b$, $b \circ b'$ are pairwise consistent.*

If $\emptyset \in f \subseteq A$ and $a, b \in f$ implies that a and b are consistent then f is a *function over A*. Let F denote the set of functions over A. For any $f, g \in F$, define:

$$f \circ g = \{a \circ b | a \in f, b \in g\};$$
$$Lf = \{a \circ a' | a \in f\};$$
$$Rf = \{a' \circ a | a \in f\}.$$

THEOREM (4.5): *$(F, \circ)$ is a semigroup.*

PROOF: Let $f, g \in F$, $a, c \in f$, $b, d \in g$. Suppose that $a \circ b$, $c \circ d$ are inconsistent. Then $a \circ b \neq \emptyset \neq c \circ d$ and $(a \circ b)' \circ (a \circ b) = (c \circ d)' \circ (c \circ d)$; i.e., $b' \circ b = d' \circ d$. But b and d are consistent whence $b = d$. Thus $a \neq c$. Since a and c are consistent, then $a' \circ a \neq c' \circ c$. Thus $a' \circ a = b \circ b'$ and $d \circ d' = c' \circ c$ whence $a' \circ a = c' \circ c$. This contradiction shows that $f \circ g \in F$. It is obvious that "$\circ$" is associative.

THEOREM (4.6): *$(F, \circ)$ contains an identity j.*

Proof: Let $j = \{a \circ a' | a \in A\}$; obviously, $j \in F$. Let $a \in f \in F$. Then $a' \circ a$, $a \circ a' \in j$ whence $a \circ (a' \circ a) = a \in f \circ j$. Hence $f \subseteq f \circ j$ and $f \subseteq j \circ f$.

In the other direction, let $a \in f$, $b \circ b' \in j$. Then either $a \circ b \circ b' = \emptyset$ or $a \circ b \circ b' = a$. In either case, $a \circ b \circ b' \in f$; i.e., $f \circ j \subseteq f$. Similarly, $j \circ f \subseteq f$.

Theorem (4.7): $(F, \circ, L, R)$ *is a function system.*

Proof:

(a) For any $f \in F$, $LRf = \{a \circ a' | a \in Rf\} = \{b' \circ b \circ b' \circ b | b \in f\} = Rf$. Similarly, $RLf = Lf$.

(b) For any $f \in F$, $Lf \circ f = \{a \circ a' \circ b | a, b \in f\}$. But $a \circ a' \circ b = \emptyset$ or $a \circ a' \circ b = b$. Hence $Lf \circ f \subseteq f$. But if $a \in f$ then $a \circ a' \circ a \in Lf \circ f$ whence $f \subseteq Lf \circ f$. Similarly, $f = f \circ Rf$.

(c) Let $f, g \in F$, $a \in f$, $b \in g$. Then

$$(a \circ b) \circ (a \circ b)' = (a \circ b \circ b') \circ (a \circ b \circ b')'$$

whence $L(f \circ g) = L(f \circ Lg)$. Similarly, $R(f \circ g) = R(Rf \circ g)$.

(d) Let $a \in f$, $b \in g$. Then

$$a \circ a' \circ b' \circ b = \emptyset \in Rg \circ Lf$$

or

$$a \circ a' \circ b' \circ b = b' \circ b \neq \emptyset$$

whence $a' \circ b' \neq \emptyset$ which implies $b \circ a \neq \emptyset$. Then

$$b' \circ b = b' \circ b \circ a \circ a' \in Rg \circ Lf.$$

Consequently $Lf \circ Rg \subseteq Rg \circ Lf$. The reverse inclusion is similarly established.

(e) Let $a \in f$, $b \in g$. Either $a' \circ a \circ b = \emptyset \in g \circ R(f \circ g)$ or

$$a' \circ a \circ b = b = b \circ b' \circ b = b \circ (a \circ b)' \circ (a \circ b) \in g \circ R(f \circ g).$$

Hence $Rf \circ g \subseteq g \circ R(f \circ g)$. In the opposite direction, let $a \in f$, $b, c \in g$. Suppose $c \circ (a \circ b)' \circ (a \circ b) \neq \emptyset$. Then $c \circ (a \circ b)' \circ (a \circ b) = c$ and $b' \circ b = c' \circ c$. Since b and c are consistent, $b = c$. Thus

$a \circ c \neq \emptyset$ whence, by lemma 2.8, $c = a' \circ a \circ c \in Rf \circ g$. Thus

$$g \circ R(f \circ g) = Rf \circ g.$$

THEOREM (4.8): *If "$\subseteq$" denotes set inclusion (not the partial order of section 3) then $(F, \circ, \subseteq)$ is a function semigroup; i.e.,*

1. *F is partially ordered by "$\subseteq$".*
2. *$(F, \circ)$ is a semigroup with identity j.*
3. (a) *For all $a, b \in F$, if $a \subseteq b$ then F contains an element $j_1 \subseteq j$ such that $a = b \circ j_1$.*
   (b) *If $j_2 (\subseteq j) \in F$, then for all $a \in F$, $a \circ j_2 \subseteq a$ and $j_2 \circ a \subseteq a$.*
4. *For every $a \in F$ there exist La and Ra in F such that*
   (a) *$La \circ a = a = a \circ Ra$.*
   (b) *$L(a \circ b) \subseteq La$, $R(a \circ b) \subseteq Rb$.*
   (c) *If $a \subseteq j$ then $La \subseteq a$ and $Ra \subseteq a$*[18].

PROOF: Let $f, g \in F$, $f \subseteq g$. If $a \in f$ then $a \circ a' \circ a = a \in g \circ Rf$; i.e., $f \subseteq g \circ Rf$. Conversely, if $\emptyset \neq b \circ a' \circ a \in g \circ Rf$ then $b' \circ b = a' \circ a$. Since $f \subseteq g$ then a and b are consistent whence $a = b$. Thus $b \circ a' \circ a = a \in f$. Hence $f \subseteq g$ implies $f = g \circ Rf$.

Conversely, let $a(\neq \emptyset) \in f$. Then there exist $b \in g$, $c \in f$ such that $a = b \circ c' \circ c$. Hence $a = b$. Thus $f \subseteq g$ if and only if $f = g \circ Rf$. The result now follows from [21; theorem 16].

5. Representations

Let S be any set. Then $\mathscr{A}_S$ will denote the *atomic semigroup* on S; i.e., $\mathscr{A}_S$ consists of the empty set and all functions $f: S \to S$ such that $|\text{dom } f| = |\text{ran } f| = 1$ where the semigroup operation is composition [20].

THEOREM (5.1): *Let $(A, \circ, ')$ be an a-system. Then $(A, \circ)$ can be homomorphically embedded in $(\mathscr{A}_A, \circ)$.*

PROOF: Let $f: A \to \mathscr{A}_A$ be defined by: $f(a) = (Ra, La)$ for any $a(\neq \emptyset) \in A$ and $f(\emptyset) = \emptyset$. Let $a, b \in A$. If $\emptyset \neq La \neq Rb \neq \emptyset$, then $b \circ a = \emptyset$. Hence $f(b \circ a) = \emptyset$. Also, $f(b) \circ f(a) = \emptyset$. On the other hand, if $La = Rb \neq \emptyset$ then $b \circ a \neq \emptyset$. Hence

$$f(b \circ a) = (R(b \circ a), L(b \circ a)) = (Ra, Lb) = f(b) \circ f(a).$$

Thus f is a semigroup homomorphism.

COROLLARY (5.2): *If* $Ra = Rb$, $La = Lb$ *imply* $a = b$ *then* $(A, \circ)$ *can be monomorphically embedded in* $\mathscr{A}_A$.

Let $(A, \circ, ')$ be an a-system. If, for any $a(\neq \emptyset)$, $b(\neq \emptyset) \in A$ there exists a unique $c \in A$ such that $Lc = La$, $Rc = Rb$ then A is a *composition system* (*c-system*).

THEOREM (5.3): *For any c-system* $(A, \circ, ')$ *there exists a set S such that* $(A, \circ)$ *is isomorphic to the atomic semigroup* $(\mathscr{A}_S, \circ)$.

PROOF: Let $S = \{Ra | a \in A\}$. Notice that $La = RLa \in S$ for all $a \in A$. By corollary 5.2, the mapping f defined in theorem 5.1 is a monomorphism. Let $(Ra, Rb) \in S \times S$. Then there exists $c \in A$ such that $Rc = Ra$, $Lc = LRb$, whence $f(c) = (Ra, Rb)$.

Let k be a function over A. Then k is a *constant* if for every function f over A, $k \circ f = k \circ Rf$ [17; page 380]. If k is a constant and $Rk = j$ then k is a *proper* constant.

Let $a, b \in A$. Then aLb means $La = Lb$. Obviously "L" is an equivalence relation on A. The L-equivalence class containing a will be denoted a_L.

THEOREM (5.4): *Let* $(A, \circ, ')$ *be a c-system. Then k is a proper constant if and only if there exists* $a(\neq \emptyset) \in A$ *such that* $k = a_L \cup \{\emptyset\}$.

PROOF: Let $k = a_L \cup \{\emptyset\}$. Let $b(\neq \emptyset)$, $c(\neq \emptyset) \in k$. Then $Lb = La = Lc$. Now if $Rb = Rc$, then, since A is a c-system, $b = c$. Thus b and c are consistent whence k is a function over A.

Next, let $c \in A$. Then there exists $x \in A$ such that $Rx = Rc$, $Lx = La$. Thus $x \in k$, whence $Rk = j$.

Now, let f be any function over A. Let $x \circ b \in k \circ f$. If $x \circ b \neq \emptyset$ then $L(x \circ b) = Lx$. Then there exists $y \in k$ such that $x \circ b = y$. Obviously, $x = y \circ b'$ whence $x \circ b = y \circ b' \circ b \in k \circ Rf$; i.e., $k \circ f \subseteq k \circ Rf$. In the opposite direction, let $\emptyset \neq x \circ (b' \circ b) \in k \circ Rf$. Then there exists $y \in A$ such that $Ry = RLb$, $Ly = Lx$; thus $y \circ b \neq \emptyset$. Then $R(y \circ b) = Rb = Rx$ since $x \circ b' \neq \emptyset$. Since $y \circ b$ and x are consistent, $y \circ b = x = x \circ b' \circ b$. Hence $k \circ Rf \subseteq k \circ f$.

Conversely, let k be a proper constant and let $a(\neq \emptyset) \in A$. Let $k_1 = a_L \cup \{\emptyset\}$. Then $Rk_1 = j$ and $k \circ k_1 = k \circ Rk_1 = k \circ j = k$. Now, let $x_1(\neq \emptyset)$, $x_2(\neq \emptyset) \in k$. Then there exist $x, y \in k_1, z, w \in k$ such that $z \circ x = x_1$ and $w \circ y = x_2$. Thus $Lx = Rz$ and $Ly = Rw$. However, $x, y \in k_1$; i.e., $Lx = Ly$. Hence $Rz = Rw$. Since z and w are consistent, $z = w$. Thus $z \circ x = x_1$; $z \circ y = x_2$. Therefore, $Lx_1 = Lz = Lx_2$; i.e., $x_1 L x_2$. Since $(A, \circ, ')$ is a c-system it follows that $k = x_{1L} \cup \{\emptyset\}$.

If Q is a collection of functions over A satisfying;

i) $f \in Q$ implies $Rf = j$;

ii) If $f, g \in Q$ and $f \neq g$ then $f \cap g = \{\emptyset\}$;

iii) $\bigcup_{f \in Q} f = A$

then Q will be called a *partition of A into proper functions*. Notice that for any c-system A there exists a partition Q of A into proper functions; to wit, the partition of A into proper constants.

THEOREM (5.5): *Let A be any c-system and let Q be a partition of A into proper functions. Then $(A, \circ)$ is isomorphic to $(\mathscr{A}_Q, \circ)$.*

PROOF: Let $r(\neq \emptyset) \in A$. For any $a(\neq \emptyset) \in A$ there exist unique elements $x, y \in A$ such that $Rx = Rr$, $Lx = La'$; $Ry = Rr$, $Ly = La$. Moreover there exist unique $f, g \in Q$ such that $x \in f$, $y \in g$. Define $h: A \to \mathscr{A}_Q$ by $h: a \mapsto (f, g)$ and $h(\emptyset) = \emptyset$. Notice first that $h(a') = (g, f)$. Next, let $a, b \in A$ and let $h(a) = (f_1, g_1)$, $h(b) = (f_2, g_2)$. There are two cases to consider:

Case 1: $a \circ b = \emptyset$. Then $Lb \neq Ra$. Suppose $g_2 = f_1$. Let y_2, x_1 be the unique elements satisfying $Rx_1 = Rr$, $Lx_1 = La'$; $Ry_2 = Rr$, $Ly_2 = Lb$. Then $x_1 \in f_1, y_2 \in g_2$, $Rx_1 = Ry_2$ imply (since x_1 and y_2 are consistent) that $x_1 = y_2$. Hence $Ra = Lb$. Thus $a \circ b \neq \emptyset$. This contradiction shows that $g_2 \neq f_1$ whence $h(a \circ b) = \emptyset = h(a) \circ h(b)$.

Case 2: $a \circ b \neq \emptyset$. Then $Lb = Ra = La'$ whence $g_2 = f_1$. Thus

$$h(a) \circ h(b) = (f_1, g_1)(f_2, g_2) = (f_2, g_1).$$

However, $L(a \circ b) = La$ and $R(a \circ b) = Rb$. Therefore $h(a \circ b) = (f_2, g_1)$. Thus h is a homomorphism.

Now, let $f, g \in Q$. Since f, g are proper functions, there exist unique elements $x \in f$, $y \in g$ such that $Rx = Rr$, $Ry = Rr$. Also, there exists a unique element $z \in A$ such that $Rz = Rx'$, $Lz = Ly$. Then $h(z) = (f, g)$; i.e., h is an epimorphism.

Finally, suppose $h(a) = h(b) \neq \emptyset$. Then there exist $x, y \in A$ such that $Rx = Rr$, $Lx = La' = Lb'$ and $Ry = Rr$, $Ly = La = Lb$. Hence $a = b$.

For any $a(\neq \emptyset) \in A$, let $k_a = a_L \cup \{\emptyset\}$.

COROLLARY (5.6): (Menger's representation by constant functions) [8; page 13]. *Let $(A, \circ, ')$ be a c-system. Let $Q = \{k_a | a \in A\}$. Then $h: A \to \mathscr{A}_Q$ given by $h: a \mapsto (k'_a, k_a)$ if $a \neq \emptyset$ and $h(\emptyset) = \emptyset$ is an isomorphism.*

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