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## A NUMERICAL CRITERION FOR THE PERMISSIBILITY OF A BLOWING-UP

Balwant Singh

### Introduction

Let  $\mathcal{O}$  be a noetherian local ring and  $\mathfrak{p}$  a proper ideal of  $\mathcal{O}$ . The concept of the permissibility of  $\mathfrak{p}$  in  $\mathcal{O}$  (more precisely, of  $\text{Spec}(\mathcal{O}/\mathfrak{p})$  in  $\text{Spec} \mathcal{O}$  at the closed point) as a center for blowing-up was introduced by Hironaka in his paper [3] on the resolution of singularities. If the center of a blowing-up  $\mathcal{O} \rightarrow \mathcal{O}'$  is permissible in  $\mathcal{O}$  then the singularity of  $\mathcal{O}'$  is no worse than that of  $\mathcal{O}$ . Here, as a measure of singularity, we may take either the characters  $\nu^*$ ,  $\tau^*$  defined by Hironaka in [3] in case  $\mathcal{O}$  is given as the quotient of a regular local ring, or the Hilbert functions of  $\mathcal{O}$  and  $\mathcal{O}'$  (See [1], [4], [6]). In this note we give a numerical criterion for the permissibility of a blowing-up, i.e. of its center (Theorem 1) and study the effect of an arbitrary blowing-up on the Hilbert function of a local ring (Theorems 2 and 3). As a corollary to Theorem 1, we get yet another criterion for the permissibility of a blowing-up (Corollary (1.4)). The criterion in Theorem 1 leads to the definition of a numerical function  $D_{\mathfrak{p}}$  such that  $\mathfrak{p}$  is permissible in  $\mathcal{O}$  if and only if  $D_{\mathfrak{p}} = 0$ . (See Remark 2.) A significance of this function  $D_{\mathfrak{p}}$  is that it appears explicitly in a comparison between the Hilbert functions of  $\mathcal{O}$  and  $\mathcal{O}'$ , where  $\mathcal{O} \rightarrow \mathcal{O}'$  is a blowing-up of  $\mathcal{O}$  with center  $\mathfrak{p}$ . (See Theorems 2 and 3.) In Remark 3 below we indicate how the criterion in Theorem 1 compares with a numerical criterion for normal flatness given by Bennett [1].

In order to state our results more precisely, we need some notation. By a *numerical function*  $H$  we mean a map  $H: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ . If  $H$  is a numerical function, we get from  $H$  a sequence  $\{H^{(r)}\}_{r \geq 0}$  of numerical functions by successive 'integration' as follows:  $H^{(0)} = H$  and, for  $r \geq 1$ ,

$$H^{(r)}(n) = \sum_{i=0}^n H^{(r-1)}(i).$$

If  $H_1, H_2$  are numerical functions, then by  $H_1 \geq H_2$  we shall always mean the total order inequality, i.e.  $H_1(n) \geq H_2(n)$  for every  $n \in \mathbb{Z}^+$ .

Let  $\mathcal{O}$  be a noetherian local ring. For a *proper* ideal  $\mathfrak{p}$  of  $\mathcal{O}$  we define a numerical function  $H_{\mathfrak{p}}$  by

$$H_{\mathfrak{p}}(n) = \dim_{\mathcal{O}/\mathfrak{m}} \mathfrak{p}^n / \mathfrak{m}\mathfrak{p}^n,$$

where  $\mathfrak{m}$  is the maximal ideal of  $\mathcal{O}$ . This gives us a sequence  $\{H_{\mathfrak{p}}^{(r)}\}_{r \geq 0}$  of numerical functions. We write  $H_{\mathcal{O}}^{(r)}$  for  $H_{\mathfrak{m}}^{(r)}$ , so that  $\{H_{\mathcal{O}}^{(r)}\}_{r \geq 0}$  is the usual sequence of the Hilbert functions of  $\mathcal{O}$ .

We denote by  $\dim \mathcal{O}$  the Krull dimension of  $\mathcal{O}$  and by  $\text{emdim } \mathcal{O}$  the embedding dimension of  $\mathcal{O}$ , i.e.  $\text{emdim } \mathcal{O} = H_{\mathcal{O}}^{(0)}(1)$ .

Recall that a proper ideal  $\mathfrak{p}$  of  $\mathcal{O}$  is said to be *permissible* in  $\mathcal{O}$  (as a center for a blowing-up) if the following two conditions are satisfied:

- (i) *regularity*:  $\mathcal{O}/\mathfrak{p}$  is regular
- (ii) *normal flatness*:  $\mathcal{O}$  is normally flat along  $\mathfrak{p}$ , i.e. the graded  $\mathcal{O}/\mathfrak{p}$ -algebra  $\text{gr}_{\mathfrak{p}}(\mathcal{O}) = \bigoplus_{n \geq 0} \mathfrak{p}^n / \mathfrak{p}^{n+1}$  is  $\mathcal{O}/\mathfrak{p}$ -flat.

**THEOREM 1:** *Let  $\mathcal{O}$  be a noetherian local ring and  $\mathfrak{p}$  a proper ideal of  $\mathcal{O}$ . Let  $d = \dim \mathcal{O}/\mathfrak{p}$  and  $e = \text{emdim } \mathcal{O}/\mathfrak{p}$ . Then we have  $H_{\mathcal{O}}^{(0)} \leq H_{\mathfrak{p}}^{(e)}$ . Further, the following three conditions are equivalent:*

- (i)  $\mathfrak{p}$  is permissible in  $\mathcal{O}$
- (ii)  $\mathcal{O}/\mathfrak{p}$  is regular and  $H_{\mathcal{O}}^{(0)} = H_{\mathfrak{p}}^{(d)}$
- (iii)  $H_{\mathcal{O}}^{(0)} = H_{\mathfrak{p}}^{(e)}$ .

We prove this theorem in § 1.

**REMARK 1:** For the implication (i)  $\Rightarrow$  (ii), cf. [3, Chapter II, Proposition 1].

**REMARK 2:** For a proper ideal  $\mathfrak{p}$  of  $\mathcal{O}$ , let us define  $D_{\mathfrak{p}} = H_{\mathfrak{p}}^{(e)} - H_{\mathcal{O}}^{(0)}$ , where  $e = \text{emdim } \mathcal{O}$ . Theorem 1 shows that  $D_{\mathfrak{p}}$  is a numerical function, and  $\mathfrak{p}$  is permissible in  $\mathcal{O}$  if and only if  $D_{\mathfrak{p}} = 0$ . We may therefore call  $D_{\mathfrak{p}}$  the *permissibility defect* of  $\mathfrak{p}$ . Another justification for the use of this term is provided by Theorem 2, which states, roughly, that if  $\mathcal{O} \rightarrow \mathcal{O}'$  is a blowing-up of  $\mathcal{O}$  with center  $\mathfrak{p}$ , then  $H_{\mathcal{O}}^{(0)} - H_{\mathcal{O}'}^{(\delta)} \geq -D_{\mathfrak{p}}$ , where  $\delta$  is the residue transcendence degree of  $\mathcal{O}'$  over  $\mathcal{O}$ . In the case when  $\mathfrak{p}$  is permissible in  $\mathcal{O}$ , the inequality  $H_{\mathcal{O}}^{(0)} - H_{\mathcal{O}'}^{(\delta)} \geq 0$  is already known [6]. One can thus say that under a blowing-up the singularity of  $\mathcal{O}$  can become worse only to the extent that the blowing-up is non-permissible, this non-permissibility being measured by the numerical function  $D_{\mathfrak{p}}$ .

**REMARK 3:** Bennett has given a numerical criterion for the permissibility of  $\mathfrak{p}$  in  $\mathcal{O}$  in the case when  $\mathcal{O}/\mathfrak{p}$  is regular [1, Theorem (3) and Q(2.1.2)]. He has shown that if  $\mathcal{O}/\mathfrak{p}$  is regular of dimension  $d$  then  $\mathfrak{p}$  is permissible in  $\mathcal{O}$  if and only if  $H_{\mathcal{O}}^{(0)} = H_{\mathcal{O}/\mathfrak{p}}^{(d)}$ . Let us compare this criterion with the one given in Theorem 1 above. Suppose that  $\mathcal{O}$  is excellent. Then we have  $H_{\mathcal{O}/\mathfrak{p}}^{(d)} \leq H_{\mathcal{O}}^{(0)}$ , where  $d = \dim \mathcal{O}/\mathfrak{p}$ . (See [1, Theorem (2)] and [6, page 202].) In this case, therefore, the difference  $D'_{\mathfrak{p}} = H_{\mathcal{O}}^{(0)} - H_{\mathcal{O}/\mathfrak{p}}^{(d)}$  is a numerical function, and  $\mathfrak{p}$  is permissible in  $\mathcal{O}$  if and only if  $D'_{\mathfrak{p}} = 0$ . However, the definition of this measure  $D'_{\mathfrak{p}}$  of the deviation of  $\mathfrak{p}$  from being permissible requires, in the first place, that  $\mathfrak{p}$  be a prime ideal. Even then it is apparently defined (i.e. is non-negative) only for  $\mathcal{O}$  excellent, it being not known whether the inequality  $H_{\mathcal{O}/\mathfrak{p}}^{(d)} \leq H_{\mathcal{O}}^{(0)}$  holds for non-excellent  $\mathcal{O}$ . Moreover, in order that  $D'_{\mathfrak{p}} = 0$  imply the permissibility of  $\mathfrak{p}$  in  $\mathcal{O}$ , we have to assume already that  $\mathcal{O}/\mathfrak{p}$  is regular. Finally,  $D'_{\mathfrak{p}}$  does not seem to intervene directly in a formula for the difference  $H_{\mathcal{O}}^{(0)} - H_{\mathcal{O}}^{(d)}$  as  $D_{\mathfrak{p}}$  does. (Here  $\mathcal{O} \rightarrow \mathcal{O}'$  is a blowing-up as in Remark 2.) It is interesting, however, to note that if  $\mathcal{O}$  is excellent and  $\mathcal{O}/\mathfrak{p}$  is regular of dimension  $d$  then we have

$$(*) \quad H_{\mathcal{O}/\mathfrak{p}}^{(d)} \leq H_{\mathcal{O}}^{(0)} \leq H_{\mathfrak{p}}^{(d)}$$

and one of these inequalities is an equality if and only if the other is. One may therefore ask: What is the relationship, in this case, between  $D_{\mathfrak{p}} = H_{\mathfrak{p}}^{(d)} - H_{\mathcal{O}}^{(0)}$  and  $D'_{\mathfrak{p}} = H_{\mathcal{O}}^{(0)} - H_{\mathcal{O}/\mathfrak{p}}^{(d)}$ ?

**REMARK 4:** The inequalities (\*) of Remark 3 yield another interesting criterion for the permissibility of  $\mathfrak{p}$  in  $\mathcal{O}$ . (See Corollary (1.4) in § 1.)

**REMARK 5:** With the notation of Theorem 1, we do not, in general, have the inequality  $H_{\mathcal{O}}^{(0)} \leq H_{\mathfrak{p}}^{(d)}$ . *Example:* Let  $\mathcal{O}$  be a non-regular Cohen-Macaulay local ring of dimension 1 (e.g.,  $\mathcal{O} = k[[X, Y]]/(Y^2 - X^3)$ , where  $k$  is a field). Choose any non-zero divisor  $x$  in the maximal ideal of  $\mathcal{O}$ , and let  $\mathfrak{p} = \mathcal{O}x$ . Then  $d = 0$ ,  $H_{\mathfrak{p}}^{(0)}(n) = 1$  for every  $n$ , but  $H_{\mathcal{O}}^{(0)}(1) \geq 2$ .

**REMARK 6:** With the notation of Theorem 1, the equality  $H_{\mathcal{O}}^{(0)} = H_{\mathfrak{p}}^{(d)}$  alone does not imply that  $\mathfrak{p}$  is permissible in  $\mathcal{O}$ . *Example:* Let  $\mathcal{O}$  be a regular local ring of dimension 1. Let  $x$  be any non-zero element in the square of the maximal ideal of  $\mathcal{O}$  and let  $\mathfrak{p} = x\mathcal{O}$ .

We now proceed to state Theorems 2 and 3. Let  $\mathcal{O} \rightarrow \mathcal{O}'$  be a blowing-up of  $\mathcal{O}$  with center a proper ideal  $\mathfrak{p}$  of  $\mathcal{O}$ . Let  $e = \text{emdim } \mathcal{O}/\mathfrak{p}$ . Choose  $t_1, \dots, t_e$  in the maximal ideal  $\mathfrak{m}$  of  $\mathcal{O}$  such that  $\mathfrak{m} = \mathfrak{p} + \sum_{i=1}^e t_i \mathcal{O}$ . Let

$t_0 \in \mathfrak{p}$  be such that  $\mathfrak{p}\mathcal{O}' = t_0\mathcal{O}'$ . For such a choice of  $t = (t_0, t_1, \dots, t_e)$  we define, for every  $i$ ,  $0 \leq i \leq e$ , a sequence  $\{\alpha_{t,i}(n)\}_{n \geq 0}$  of ideals of  $\mathcal{O}'$  as follows:

$$\alpha_{t,i}(n) = \{f \in \mathcal{O}' \mid t_i f \in \mathfrak{m}'^{n+1} + \sum_{j=0}^{i-1} t_j \mathcal{O}'\},$$

where  $\mathfrak{m}'$  is the maximal ideal of  $\mathcal{O}'$ . Clearly,  $\alpha_{t,i}(n) \supset \mathfrak{m}'^n + \sum_{j=0}^{i-1} t_j \mathcal{O}'$  for every  $i$  and  $n$ . Let  $L_{t,i}$ ,  $0 \leq i \leq e$ , be the numerical functions defined by

$$L_{t,i}(n) = \text{length}_{\mathcal{O}'} \alpha_{t,i}(n) / (\mathfrak{m}'^n + \sum_{j=0}^{i-1} t_j \mathcal{O}').$$

**THEOREM 2:** *Let  $\mathfrak{p}$  be a proper ideal of a noetherian local ring  $\mathcal{O}$  and let  $e = \text{emdim } \mathcal{O}/\mathfrak{p}$ . Let  $\mathcal{O} \rightarrow \mathcal{O}'$  be a blowing-up of  $\mathcal{O}$  with center  $\mathfrak{p}$  and let  $\delta$  be the residue transcendence degree of  $\mathcal{O}'$  over  $\mathcal{O}$ . Then, for any choice of  $t = (t_0, t_1, \dots, t_e)$  as above, we have*

$$H_{\mathcal{O}}^{(0)} - H_{\mathcal{O}'}^{(\delta)} \geq \sum_{i=0}^e L_{t,i}^{(i+\delta)} - D_{\mathfrak{p}} \geq -D_{\mathfrak{p}}.$$

*In particular, if  $\mathfrak{p}$  is permissible in  $\mathcal{O}$ , then*

$$H_{\mathcal{O}}^{(0)} - H_{\mathcal{O}'}^{(\delta)} \geq \sum_{i=0}^e L_{t,i}^{(i+\delta)} \geq 0.$$

In the case when  $\mathcal{O} \rightarrow \mathcal{O}'$  is residually rational, we can give a more precise formula for the difference  $H_{\mathcal{O}}^{(0)} - H_{\mathcal{O}'}^{(0)}$ . As above, let  $t_0 \in \mathfrak{p}$  be such that  $\mathfrak{p}\mathcal{O}' = t_0\mathcal{O}'$ . Then  $\mathcal{O}'$  is obtained as a localization of the subring  $\{f/t_0^n \mid n \geq 0, f \in \mathfrak{p}^n\}$  of  $\mathcal{O}_{t_0}$ . We define a sequence  $\{\mathfrak{b}_{t_0}(n)\}_{n \geq 0}$  of ideals of  $\mathcal{O}$  by

$$\mathfrak{b}_{t_0}(n) = \{f \in \mathfrak{p}^n \mid f/t_0^n \in \mathfrak{m}'^{n+1} + \mathfrak{m}\mathcal{O}'\},$$

where  $\mathfrak{m}, \mathfrak{m}'$  are the maximal ideals of  $\mathcal{O}, \mathcal{O}'$ , respectively. Clearly,  $\mathfrak{b}_{t_0}(n) \supset \mathfrak{m}\mathfrak{p}^n$  for every  $n$ . Let  $L_{t_0}$  be the numerical function defined by

$$L_{t_0}(n) = \text{length}_{\mathcal{O}} \mathfrak{b}_{t_0}(n) / \mathfrak{m}\mathfrak{p}^n.$$

**THEOREM 3:** *Let the notation be as in Theorem 2. Assume, moreover, that  $\mathcal{O} \rightarrow \mathcal{O}'$  is residually rational. Then for any choice of  $t = (t_0, t_1, \dots, t_e)$  as above, we have*

$$H_{\mathcal{O}}^{(0)} - H_{\mathcal{O}'}^{(0)} = L_{t_0}^{(e)} + \sum_{i=0}^e L_{t,i}^{(i)} - D_{\mathfrak{p}}.$$

In particular, if  $\mathfrak{p}$  is permissible in  $\mathcal{O}$ , then

$$H_{\mathcal{O}}^{(0)} - H_{\mathcal{O}'}^{(0)} = L_{t_0}^{(e)} + \sum_{i=0}^e L_{t,i}^{(i)}.$$

Theorems 2 and 3 are proved in § 2.

### 1. Proof of Theorem 1

(1.1) Let  $\mathcal{O}$  be a noetherian local ring with maximal ideal  $\mathfrak{m}$ . For any ideal  $\mathfrak{p}$  of  $\mathcal{O}$  we define

$$\mu(\mathfrak{p}) = \dim_{\mathcal{O}/\mathfrak{m}} \mathfrak{p}/\mathfrak{m}\mathfrak{p},$$

so that  $\mu(\mathfrak{p})$  is the cardinality of a minimal set of generators of  $\mathfrak{p}$ . Note that, if  $\mathfrak{p}$  is a proper ideal of  $\mathcal{O}$ , then  $\mu(\mathfrak{p}^n) = H_{\mathfrak{p}}^{(0)}(n)$  for every  $n$ .

(1.2) LEMMA :

(1) Let  $\alpha_i$ ,  $1 \leq i \leq r$ , be ideals of  $\mathcal{O}$  such that  $\mu(\sum_i \alpha_i) = \sum_i \mu(\alpha_i)$ . If  $S_i$  is a minimal set of generators of  $\alpha_i$ , then  $\bigcup_i S_i$  is a minimal set of generators of  $\sum_i \alpha_i$ . In particular, for every  $j$ ,  $1 \leq j \leq r$ , we have

$$S_j \cap (\mathfrak{m}(\sum_i \alpha_i) + \sum_{i \neq j} \alpha_i) = \emptyset.$$

(2) Let  $\mathfrak{p}, \mathfrak{q}$  be proper ideals of  $\mathcal{O}$  and let  $\alpha = \mathfrak{p} + \mathfrak{q}$ . Let  $e = \mu(\mathfrak{q})$ . Then  $H_{\alpha}^{(0)} \leq H_{\mathfrak{p}}^{(e)}$ .

(3) With the notation of (2), suppose that  $H_{\alpha}^{(0)} = H_{\mathfrak{p}}^{(e)}$ . Then, for every  $m, n \geq 0$ , we have

$$(a) \mu(\mathfrak{q}^n) = \binom{n+e-1}{e-1}$$

$$(b) \mu(\mathfrak{p}^m \mathfrak{q}^n) = \mu(\mathfrak{p}^m) \mu(\mathfrak{q}^n)$$

$$(c) \mu(\alpha^{n+1}) = \mu(\mathfrak{q}^{n+1}) + \mu(\alpha^n \mathfrak{p}).$$

PROOF: (1) is immediate. To prove (2) and (3), we have only to observe the following easily verified facts:

$$(i) \mu(\alpha^n) \leq \sum_{i=0}^n \mu(\mathfrak{p}^{n-i} \mathfrak{q}^i) \leq \sum_{i=0}^n \mu(\mathfrak{p}^{n-i}) \mu(\mathfrak{q}^i).$$

$$(ii) \mu(\mathfrak{q}^n) \leq \binom{n+e-1}{e-1}.$$

(iii) For any numerical function  $H = H^{(0)}$  we have

$$H^{(e)}(n) = \sum_{i=0}^n \binom{i+e-1}{e-1} H^{(0)}(n-i).$$

(1.3) LEMMA : (Bennett). *Let  $\mathcal{O}$  be a noetherian local ring and  $\mathfrak{p}$  an ideal of  $\mathcal{O}$  such that  $\mathcal{O}/\mathfrak{p}$  is regular. Let  $d = \dim \mathcal{O}/\mathfrak{p}$ . Then  $\mathfrak{p}$  is permissible in  $\mathcal{O}$  if and only if  $H_{\mathcal{O}}^{(0)} = H_{\mathcal{O}/\mathfrak{p}}^{(d)}$ .*

For a proof of this lemma, see [1, Theorem (3) and 0(2.1.2)].

Before coming to the proof of Theorem 1, we note the following corollary to Theorem 1:

(1.4) COROLLARY : *Suppose  $\mathcal{O}$  is excellent<sup>1</sup> and  $\mathcal{O}/\mathfrak{p}$  is regular. Then  $\mathfrak{p}$  is permissible in  $\mathcal{O}$  if and only if  $\mu(\mathfrak{p}^n) = \mu(\mathfrak{p}^n \mathcal{O}_{\mathfrak{p}})$  for every  $n \geq 0$ .*

PROOF: As mentioned in Remark 3 of the Introduction, we have

$$H_{\mathcal{O}/\mathfrak{p}}^{(d)} \leq H_{\mathcal{O}}^{(0)} \leq H_{\mathfrak{p}}^{(d)}.$$

(The second inequality follows from Theorem 1 and the first from [1, Theorem (2)] and [6, page 202].) By Theorem 1,  $H_{\mathcal{O}}^{(0)} = H_{\mathfrak{p}}^{(d)}$  if and only if  $\mathfrak{p}$  is permissible in  $\mathcal{O}$ . By Lemma (1.3),  $\mathfrak{p}$  is permissible in  $\mathcal{O}$  if and only if  $H_{\mathcal{O}/\mathfrak{p}}^{(d)} = H_{\mathcal{O}}^{(0)}$ . Therefore,  $\mathfrak{p}$  is permissible in  $\mathcal{O}$  if and only if  $H_{\mathcal{O}/\mathfrak{p}}^{(d)} = H_{\mathfrak{p}}^{(d)}$ . Now, clearly,  $H_{\mathcal{O}/\mathfrak{p}}^{(d)} = H_{\mathfrak{p}}^{(d)}$  if and only if  $H_{\mathcal{O}/\mathfrak{p}}^{(0)} = H_{\mathfrak{p}}^{(0)}$ . This proves the corollary, since  $\mu(\mathfrak{p}^n) = H_{\mathfrak{p}}^{(0)}(n)$  and  $\mu(\mathfrak{p}^n \mathcal{O}_{\mathfrak{p}}) = H_{\mathcal{O}/\mathfrak{p}}^{(0)}(n)$ .

PROOF OF THEOREM 1: Let  $\mathfrak{m}$  be the maximal ideal of  $\mathcal{O}$  and let  $k = \mathcal{O}/\mathfrak{m}$ .

Since  $e = \text{emdim } \mathcal{O}/\mathfrak{p}$ , there exists an ideal  $\mathfrak{q}$  of  $\mathcal{O}$  such that  $\mathfrak{m} = \mathfrak{p} + \mathfrak{q}$  and  $\mu(\mathfrak{q}) = e$ . Therefore, the inequality  $H_{\mathcal{O}}^{(0)} \leq H_{\mathfrak{p}}^{(e)}$  follows from Lemma (1.2)(2).

<sup>1</sup> It was pointed out by W. Vogel that the proof of this corollary goes through also for non-excellent  $\mathcal{O}$ . For it follows, from Lemma 1 of [A. Ljungström, "An inequality between Hilbert functions of certain prime ideals one of which is immediately included in the other", Preprint, University of Stockholm, 1975] that  $H_{\mathcal{O}/\mathfrak{p}}^{(d)} \leq H_{\mathcal{O}}^{(0)}$  for arbitrary  $\mathcal{O}$  if  $\mathcal{O}/\mathfrak{p}$  is regular of dimension  $d$ . It was precisely for this inequality that we assumed the excellence of  $\mathcal{O}$ . For a more direct proof of this corollary, see [R. Achilles, P. Schenzel and W. Vogel, "Einige Anwendungen der normalen Flachheit", Preprint, Martin-Luther-Universität, 1975].

We now proceed to show that conditions (i), (ii) and (iii) of Theorem 1 are equivalent.

(i)  $\Rightarrow$  (ii). Since  $\mathfrak{p}$  is permissible in  $\mathcal{O}$ , we have  $d = e$ , and for every  $n \geq 0$ ,  $\mathfrak{p}^n/\mathfrak{p}^{n+1}$  is  $\mathcal{O}/\mathfrak{p}$ -flat, hence  $\mathcal{O}/\mathfrak{p}$ -free. Therefore, we have

$$\begin{aligned} H_{\mathfrak{p}}^{(0)}(n) &= \dim_k \mathfrak{p}^n / \mathfrak{m} \mathfrak{p}^n \\ &= \dim_k \mathfrak{p}^n / \mathfrak{p}^{n+1} \otimes_{\mathcal{O}/\mathfrak{p}} k \\ &= \text{rank}_{\mathcal{O}/\mathfrak{p}} \mathfrak{p}^n / \mathfrak{p}^{n+1} \\ &= \dim_{\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathfrak{p}}} \mathfrak{p}^n \mathcal{O}_{\mathfrak{p}} / \mathfrak{p}^{n+1} \mathcal{O}_{\mathfrak{p}} \\ &= H_{\mathcal{O}_{\mathfrak{p}}}^{(0)}(n). \end{aligned}$$

Thus  $H_{\mathfrak{p}}^{(0)} = H_{\mathcal{O}_{\mathfrak{p}}}^{(0)}$ , so that  $H_{\mathfrak{p}}^{(d)} = H_{\mathcal{O}_{\mathfrak{p}}}^{(d)} = H_{\mathcal{O}}^{(0)}$ , the last equality by Lemma (1.3).

(ii)  $\Rightarrow$  (iii). Since  $\mathcal{O}/\mathfrak{p}$  is regular, we have  $d = e$ .

(iii)  $\Rightarrow$  (ii). We have only to show that  $\mathcal{O}/\mathfrak{p}$  is regular. Choose  $t_1, \dots, t_e \in \mathfrak{m}$  such that their canonical images  $\bar{t}_1, \dots, \bar{t}_e$  in  $\bar{\mathcal{O}} = \mathcal{O}/\mathfrak{p}$  form a (necessarily minimal) set of generators of  $\bar{\mathfrak{m}} = \mathfrak{m}/\mathfrak{p}$ . Let  $\mathfrak{q} = \sum_{i=1}^e t_i \mathcal{O}$ . Then  $\mathfrak{m} = \mathfrak{p} + \mathfrak{q}$  and  $e = \mu(\mathfrak{q})$ . Therefore, the assumption  $H_{\mathcal{O}}^{(0)} = H_{\mathfrak{p}}^{(e)}$  implies, by Lemma (1.2)(3), that we have

$$\mu(\mathfrak{q}^n) = \binom{n+e-1}{e-1}$$

(\*)

$$\mu(\mathfrak{m}^{n+1}) = \mu(\mathfrak{q}^{n+1}) + \mu(\mathfrak{m}^n \mathfrak{p})$$

for every  $n \geq 0$ . Let  $S_n = \{t^\alpha \mid |\alpha| = n\}$ . (Here we have used the standard notation:  $t^\alpha = t_1^{\alpha_1} \dots t_e^{\alpha_e}$  and  $|\alpha| = \alpha_1 + \dots + \alpha_e$  for  $\alpha = (\alpha_1, \dots, \alpha_e) \in (\mathbb{Z}^+)^e$ .) It follows from (\*) and Lemma (1.2)(1) that the following two statements are true for every  $n \geq 0$ .

- (1) $_n$   $S_n$  is a minimal set of generators of  $\mathfrak{q}^n$ .
- (2) $_n$  If  $T_n$  is any minimal set of generators of  $\mathfrak{m}^n \mathfrak{p}$ , then  $T_n \cup S_{n+1}$  is a minimal set of generators of  $\mathfrak{m}^{n+1}$ .

Suppose now that  $\mathcal{O}/\mathfrak{p}$  is not regular. Then there exists  $r \in \mathbb{Z}^+$  and  $\alpha = (\alpha_1, \dots, \alpha_e) \in (\mathbb{Z}^+)^e$  with  $|\alpha| = r$  such that

$$\bar{t}^\alpha \in \sum_{\substack{|\beta|=r \\ \beta \neq \alpha}} \bar{t}^\beta \bar{\mathcal{O}} + \bar{\mathfrak{m}}^{r+1}.$$

This means that



$$t^\alpha \in \sum_{x \in \mathcal{S}_r - \{t^\alpha\}} x\mathcal{O} + \mathfrak{m}^{r+1} + \mathfrak{p}.$$

We can therefore write  $t^\alpha = y + p$  with  $p \in \mathfrak{p}$  and

$$y \in \sum_{x \in \mathcal{S}_r - \{t^\alpha\}} x\mathcal{O} + \mathfrak{m}^{r+1}.$$

If  $p \neq 0$ , let  $s \in \mathbb{Z}^+$  be such that  $p \in \mathfrak{m}^s \mathfrak{p} - \mathfrak{m}^{s+1} \mathfrak{p}$ . Then there exists a minimal set  $T$  of generators of  $\mathfrak{m}^s \mathfrak{p}$  such that  $p \in T$ . If  $p = 0$ , we put  $s = \infty$ . Now consider the three cases  $s+1 < r$ ,  $s+1 = r$  and  $s+1 > r$ .

*Case (1).*  $s+1 < r$ . Then  $p = t^\alpha - y \in \mathfrak{m}^r \subset \mathfrak{m}^{s+2}$ . This contradicts (2)<sub>s</sub>, since we may take  $T_s = T$ , so that  $p \in T_s$ .

*Case (2).*  $s+1 = r$ . In this case we have

$$t^\alpha = y + p \in \sum_{x \in \mathcal{S}_r - \{t^\alpha\}} x\mathcal{O} + p\mathcal{O} + \mathfrak{m}^{s+2},$$

which again contradicts (2)<sub>s</sub>, by taking  $T_s = T$ .

*Case (3).*  $s+1 > r$ . In this case  $p \in \mathfrak{m}^s \mathfrak{p} \subset \mathfrak{m}^{r+1}$ , so that we have

$$t^\alpha = y + p \in \sum_{x \in \mathcal{S}_r - \{t^\alpha\}} x\mathcal{O} + \mathfrak{m}^{r+1},$$

which contradicts (2)<sub>r-1</sub>.

This shows that  $\mathcal{O}/\mathfrak{p}$  is regular and  $d = e$ , which proves (ii).

(ii)  $\Rightarrow$  (i). We prove this implication by induction on  $d$ . The case  $d = 0$  is trivial. We shall now prove:

(A) *The implication (ii)  $\Rightarrow$  (i) for  $d = 1$ .*

(B) *The inductive step from  $d-1$  to  $d$ , assuming (A).*

We first prove (B). Let  $d \geq 1$  and let  $t_1, \dots, t_d \in \mathfrak{m}$  be such that  $\mathfrak{m} = \mathfrak{p} + \sum_{i=1}^d t_i \mathcal{O}$ . Let  $\mathfrak{n} = \mathfrak{p} + \sum_{i=1}^{d-1} t_i \mathcal{O}$ . Then  $\mathfrak{m} = \mathfrak{n} + t_d \mathcal{O}$ . Therefore  $H_{\mathcal{O}}^{(0)} \leq H_{\mathfrak{n}}^{(1)}$ , by Lemma (1.2)(2). Also  $H_{\mathfrak{n}}^{(0)} \leq H_{\mathfrak{p}}^{(d-1)}$ , by Lemma (1.2)(2). Therefore  $H_{\mathcal{O}}^{(0)} \leq H_{\mathfrak{n}}^{(1)} \leq H_{\mathfrak{p}}^{(d)}$ . Since  $H_{\mathcal{O}}^{(0)} = H_{\mathfrak{p}}^{(d)}$ , we get  $H_{\mathcal{O}}^{(0)} = H_{\mathfrak{n}}^{(1)}$ . Now  $\mathcal{O}/\mathfrak{n}$  is regular of dimension 1. Therefore, by (A),  $H_{\mathcal{O}}^{(0)} = H_{\mathfrak{n}}^{(1)}$  implies that  $\mathfrak{n}$  is permissible in  $\mathcal{O}$ . Hence

$$(*) \quad H_{\mathcal{O}}^{(0)} = H_{\mathcal{O}_{\mathfrak{n}}}^{(1)}$$

by Lemma (1.3). Thus  $H_{\mathcal{O}_{\mathfrak{n}}}^{(1)} = H_{\mathfrak{p}}^{(d)}$ , which gives  $H_{\mathcal{O}_{\mathfrak{n}}}^{(0)} = H_{\mathfrak{p}}^{(d-1)}$ . This implies that  $H_{\mathcal{O}_{\mathfrak{n}}}^{(0)} \geq H_{\mathfrak{p}\mathcal{O}_{\mathfrak{n}}}^{(d-1)}$ , since  $\mu(\mathfrak{p}^n \mathcal{O}_{\mathfrak{n}}) \leq \mu(\mathfrak{p}^n)$  for every  $n$ . On the other hand, by Lemma (1.2)(2), we have  $H_{\mathcal{O}_{\mathfrak{n}}}^{(0)} \leq H_{\mathfrak{p}\mathcal{O}_{\mathfrak{n}}}^{(d-1)}$ , since

$$\mathfrak{n}\mathcal{O}_{\mathfrak{n}} = \mathfrak{p}\mathcal{O}_{\mathfrak{n}} + \sum_{i=1}^{d-1} t_i \mathcal{O}_{\mathfrak{n}}.$$

Thus  $H_{\mathcal{O}_n}^{(0)} = H_{\mathfrak{p}\mathcal{O}_n}^{(d-1)}$ . This implies, by induction hypothesis, that  $\mathfrak{p}\mathcal{O}_n$  is permissible in  $\mathcal{O}_n$ , since  $\mathcal{O}_n/\mathfrak{p}\mathcal{O}_n$  is regular of dimension  $d-1$ . Therefore  $H_{\mathcal{O}_n}^{(d-1)} = H_{\mathcal{O}_n}^{(0)}$ , by Lemma (1.3). This gives  $H_{\mathcal{O}_n}^{(d)} = H_{\mathcal{O}_n}^{(1)} = H_{\mathcal{O}_n}^{(0)}$ , by (\*). Therefore, by Lemma (1.3),  $\mathfrak{p}$  is permissible in  $\mathcal{O}$ , and (B) is proved.

We now turn to the proof of (A). We are given that  $\mathcal{O}/\mathfrak{p}$  is a discrete valuation ring and  $H_{\mathcal{O}}^{(0)} = H_{\mathfrak{p}}^{(1)}$ . We have to show that  $\text{gr}_{\mathfrak{p}}(\mathcal{O})$  is  $\mathcal{O}/\mathfrak{p}$ -flat or, equivalently, that  $\mathfrak{p}^n/\mathfrak{p}^{n+1}$  is  $\mathcal{O}/\mathfrak{p}$ -free for every  $n \geq 0$ . Choose  $t \in \mathfrak{m}$  such that its image  $\bar{t}$  in  $\mathcal{O}/\mathfrak{p}$  is a uniformising parameter for  $\mathcal{O}/\mathfrak{p}$ . It is then enough to show that  $\bar{t}$  is a non-zero divisor in  $\mathfrak{p}^n/\mathfrak{p}^{n+1}$  for every  $n \geq 0$ .

By the choice of  $t$ , we have  $\mathfrak{m} = \mathfrak{p} + t\mathcal{O}$ . Therefore the equality  $H_{\mathcal{O}}^{(0)} = H_{\mathfrak{p}}^{(1)}$  implies, by Lemma (1.2)(3), that  $\mu(t^m\mathfrak{p}^n) = \mu(\mathfrak{p}^n)$  for all  $m, n \geq 0$ , so that  $\mu(\mathfrak{m}^n) = \sum_{i=0}^n \mu(t^i\mathfrak{p}^{n-i})$ .

Suppose now that there exists  $n \geq 0$  such that  $\bar{t}$  is a zero-divisor in  $\mathfrak{p}^n/\mathfrak{p}^{n+1}$ . Then there exists  $p \in \mathfrak{p}^n - \mathfrak{p}^{n+1}$  such that  $tp \in \mathfrak{p}^{n+1}$ . We consider the two cases  $p \notin \mathfrak{m}\mathfrak{p}^n$  and  $p \in \mathfrak{m}\mathfrak{p}^n$ .

*Case (1).*  $p \notin \mathfrak{m}\mathfrak{p}^n$ . In this case  $p$  can be completed to a minimal set, say  $S$ , of generators of  $\mathfrak{p}^n$ . Then  $tS = \{tx|x \in S\}$  is a minimal set of generators of  $t\mathfrak{p}^n$ , since  $\mu(t\mathfrak{p}^n) = \mu(\mathfrak{p}^n)$ , as noted above. But this is a contradiction, by Lemma (1.2)(1), of the equality

$$\mu(\mathfrak{m}^{n+1}) = \sum_{i=0}^{n+1} \mu(t^i\mathfrak{p}^{n+1-i}),$$

since  $tp \in tS \cap \mathfrak{p}^{n+1}$ .

*Case (2)*<sup>2</sup>  $p \in \mathfrak{m}\mathfrak{p}^n$ . Since  $\mathfrak{m}\mathfrak{p}^n = (\mathfrak{p} + t\mathcal{O})\mathfrak{p}^n = \mathfrak{p}^{n+1} + t\mathfrak{p}^n$ , we can write  $p = q'_{n+1} + t^{\alpha_0-1}p_n$  with  $q'_{n+1} \in \mathfrak{p}^{n+1}$ ,  $p_n \in \mathfrak{p}^n$  and  $\alpha_0$  an integer  $\geq 2$ . Since  $p \notin \mathfrak{p}^{n+1}$ , we may choose  $q'_{n+1}$ ,  $\alpha_0$  and  $p_n$  to be such that  $p_n \in \mathfrak{p}^n - \mathfrak{m}\mathfrak{p}^n$ . Now  $tp = tq'_{n+1} + t^{\alpha_0}p_n$ . Put  $q_{n+1} = t^{\alpha_0}p_n = tp - tq'_{n+1}$ . Then  $q_{n+1} \in \mathfrak{p}^{n+1}$ . Suppose  $q_{n+1} \in \mathfrak{m}\mathfrak{p}^{n+1} = \mathfrak{p}^{n+2} + t\mathfrak{p}^{n+1}$ . Then we can write  $q_{n+1} = q_{n+2} - t^{\alpha_1}p_{n+1}$  with  $q_{n+2} \in \mathfrak{p}^{n+2}$ ,  $\alpha_1 \geq 1$  and  $p_{n+1} \in \mathfrak{p}^{n+1}$ . Now, if  $q_{n+1} \notin \mathfrak{p}^{n+2}$ , we may assume (by choosing  $q_{n+2}$ ,  $\alpha_1$ ,  $p_{n+1}$  suitably) that  $p_{n+1} \in \mathfrak{p}^{n+1} - \mathfrak{m}\mathfrak{p}^{n+1}$ . If  $q_{n+1} \in \mathfrak{p}^{n+2}$ , then we put  $q_{n+2} = q_{n+1}$ ,  $p_{n+1} = 0$  and  $\alpha_1 = \alpha_0 + 1$ . We get  $q_{n+2} = t^{\alpha_0}p_n + t^{\alpha_1}p_{n+1}$ . Proceeding thus, we write

$$(**) \quad q_{n+r+1} = t^{\alpha_0}p_n + t^{\alpha_1}p_{n+1} + \dots + t^{\alpha_r}p_{n+r},$$

<sup>2</sup> The author wishes to express his thanks to the referee for pointing out a correction in the proof of this case.

where  $q_{n+r+1} \in \mathfrak{p}^{n+r+1}$  and for every  $i, 0 \leq i \leq r$ , either  $p_{n+i} \in \mathfrak{p}^{n+i} - m\mathfrak{p}^{n+i}$  and  $\alpha_i \geq 1$  or  $p_{n+i} = 0$  and  $\alpha_i = \alpha_0 + 1$ . Now suppose we have obtained  $q_{n+r+1}$  for a given  $r \geq 0$ . For this  $r$ , let

$$\alpha = \inf \{\alpha_0, \alpha_1 + 1, \dots, \alpha_r + r\}$$

and let

$$J = \{j | 0 \leq j \leq r \text{ and } \alpha = \alpha_j + j\}.$$

Then  $J$  is not empty,  $\alpha_j = \alpha - j$  for every  $j$  in  $J$  and from (\*\*\*) we get

$$(***) \quad q_{n+r+1} \equiv \sum_{j \in J} t^{\alpha-j} p_{n+j} \pmod{m^{n+\alpha+1}}.$$

Now, since  $p_{n+j} \in \mathfrak{p}^{n+j} - m\mathfrak{p}^{n+j}$  for every  $j \in J$ , we can complete  $p_{n+j}$  to a minimal set of generators of  $\mathfrak{p}^{n+j}$ . Therefore, since we have

$$\mu(m^{n+\alpha}) = \sum_{i=0}^{n+\alpha} \mu(t^{n+\alpha-i} \mathfrak{p}^i),$$

we see by Lemma (1.2) that the set  $\{t^{\alpha-j} p_{n+j} | j \in J\}$  can be completed to a minimal set of generators of  $m^{n+\alpha}$ . In particular,  $\sum_{j \in J} t^{\alpha-j} p_{n+j}$  is not in  $m^{n+\alpha+1}$ , since  $J$  is non-empty. Therefore, by (\*\*\*),  $q_{n+r+1}$  is not in  $m^{n+\alpha+1}$ . Therefore, since  $q_{n+r+1} \in \mathfrak{p}^{n+r+1}$ , we conclude that  $n+r+1 < n+\alpha+1$ , so that  $r < \alpha \leq \alpha_0$ .

This shows that the process of generating the  $q_{n+r+1}$  cannot go on indefinitely, i.e. we must eventually come to an  $r$  for which  $q_{n+r+1}$  is not in  $m\mathfrak{p}^{n+r+1}$ . For this  $r$ ,  $q_{n+r+1}$  can be completed to a minimal set of generators of  $\mathfrak{p}^{n+r+1}$  and hence of  $m^{n+r+1}$  by Lemma (1.2), since by hypothesis

$$\mu(m^{n+r+1}) = \sum_{i=0}^{n+r+1} \mu(t^i \mathfrak{p}^{n+r+1-i}).$$

Now if  $\alpha > r+1$  then (\*\*\*) shows that  $q_{n+r+1} \in m^{n+r+2}$ , which is a contradiction. If  $\alpha = r+1$  then, by Lemma (1.2), the set

$$\{q_{n+r+1}\} \cup \{t^{\alpha-j} p_{n+j} | j \in J\}$$

can be completed to a minimal set of generators of  $m^{n+\alpha}$ . This contradicts (\*\*\*) .

Thus (A) is proved, and the proof of the theorem is complete.

## 2. Proof of Theorems 2 and 3

(2.1) The proof of Theorems 2 and 3 is contained essentially in the proof of the Main Theorem in [6]. What is needed is elaboration of certain points. We do this in the proof below, referring frequently to [6].

(2.2) We have the following situation:  $\mathfrak{p}$  is a proper ideal of  $\mathcal{O}$ , and  $\mathcal{O} \xrightarrow{h} \mathcal{O}'$  is a blowing-up of  $\mathcal{O}$  with center  $\mathfrak{p}$ . We have  $e = \text{emdim } \mathcal{O}/\mathfrak{p}$  and  $\delta = \text{tr.deg}_k k'$ , where  $k \rightarrow k'$  is the residue field extension induced by  $h$ . We are given  $t = (t_0, t_1, \dots, t_e)$  with  $t_0 \in \mathfrak{p}$  such that  $\mathfrak{p}\mathcal{O}' = t_0\mathcal{O}'$  and  $t_i \in \mathfrak{m}$ ,  $1 \leq i \leq e$ , such that  $\mathfrak{m} = \mathfrak{p} + \sum_{i=1}^e t_i\mathcal{O}$ . The ideals  $\mathfrak{a}_{t,i}(n)$  of  $\mathcal{O}'$  and  $\mathfrak{b}_{t_0}(n)$  of  $\mathcal{O}$  and the numerical functions  $L_{t,i}$ ,  $1 \leq i \leq e$ , and  $L_{t_0}$  are defined as in the Introduction. Let  $\mathcal{O}'' = \mathcal{O}'/\mathfrak{m}\mathcal{O}'$ .

With the notation of (2.2) we shall prove the following three lemmas:

(2.3) LEMMA:

$$H_{\mathcal{O}'}^{(0)} = H_{\mathcal{O}''}^{(e+1)} - \sum_{i=0}^e L_{t,i}^{(i)}.$$

(2.4) LEMMA: If  $k = k'$  then  $H_{\mathfrak{p}}^{(0)} = H_{\mathcal{O}''}^{(1)} + L_{t_0}$ .

(2.5) LEMMA:  $H_{\mathfrak{p}}^{(0)} \geq H_{\mathcal{O}''}^{(1+\delta)}$ .

Assume these three lemmas for the moment. Then we get an immediate

PROOF OF THEOREMS 2 AND 3: Since  $D_{\mathfrak{p}} = H_{\mathfrak{p}}^{(e)} - H_{\mathcal{O}'}^{(0)}$ , we have

$$\begin{aligned} H_{\mathcal{O}'}^{(0)} - H_{\mathcal{O}'}^{(\delta)} &= H_{\mathfrak{p}}^{(e)} - H_{\mathcal{O}'}^{(\delta)} - D_{\mathfrak{p}} \\ &= H_{\mathfrak{p}}^{(e)} - H_{\mathcal{O}''}^{(e+1+\delta)} + \sum_{i=0}^e L_{t,i}^{(i+\delta)} - D_{\mathfrak{p}} \end{aligned} \quad (\text{Lemma (2.3)})$$

$$\geq \sum_{i=0}^e L_{t,i}^{(i+\delta)} - D_{\mathfrak{p}} \quad (\text{Lemma (2.5)}).$$

This proves Theorem 2. Now, if  $k = k'$ , then

$$\begin{aligned} H_{\mathcal{O}'}^{(0)} - H_{\mathcal{O}'}^{(0)} &= H_{\mathfrak{p}}^{(e)} - H_{\mathcal{O}''}^{(e+1)} + \sum_{i=0}^e L_{t,i}^{(i)} - D_{\mathfrak{p}} \quad (\text{as above, since } \delta = 0) \\ &= L_{t_0}^{(e)} + \sum_{i=0}^e L_{t,i}^{(i)} - D_{\mathfrak{p}} \end{aligned} \quad (\text{Lemma (2.4)}).$$

This proves Theorem 3.

PROOF OF LEMMA (2.3): Since  $\mathcal{O}'' = \mathcal{O}' / \sum_{i=0}^e t_i \mathcal{O}'$ , the lemma follows from [6, Theorem 1] and a straightforward induction on  $e$ .

PROOF OF LEMMA (2.4): Let  $\mathfrak{m}''$  be the maximal ideal of  $\mathcal{O}''$ . It is enough to show that there exists an exact sequence

$$(*) \quad 0 \rightarrow \mathfrak{b}_{t_0}(n)/\mathfrak{m}\mathfrak{p}^n \rightarrow \mathfrak{p}^n/\mathfrak{m}\mathfrak{p}^n \xrightarrow{\varphi} \mathcal{O}''/\mathfrak{m}''^{n+1} \rightarrow 0$$

of  $k$ -vector spaces. For we have

$$H_{\mathfrak{p}}^{(0)}(n) = \dim_k \mathfrak{p}^n/\mathfrak{m}\mathfrak{p}^n, \quad H_{\mathcal{O}''}^{(1)}(n) = \dim_k \mathcal{O}''/\mathfrak{m}''^{n+1}$$

and

$$L_{t_0}(n) = \text{length}_{\mathcal{O}} \mathfrak{b}_{t_0}(n)/\mathfrak{m}\mathfrak{p}^n = \dim_k \mathfrak{b}_{t_0}(n)/\mathfrak{m}\mathfrak{p}^n.$$

To show the existence of (\*) we have only to define  $\varphi$  suitably. Since  $\mathfrak{p}\mathcal{O}' = t_0\mathcal{O}'$ , we can identify  $\mathcal{O}'$  with a localization of the subring  $\{f/t_0^n | n \geq 0, f \in \mathfrak{p}^n\}$  of  $\mathcal{O}_{t_0}$ . Define  $\psi: \mathfrak{p}^n \rightarrow \mathcal{O}'$  by  $\psi(f) = \eta(f/t_0^n)$ , where  $\eta: \mathcal{O}' \rightarrow \mathcal{O}''$  is the canonical homomorphism. Then  $\psi$  induces a  $k$ -homomorphism  $\bar{\psi}: \mathfrak{p}^n/\mathfrak{m}\mathfrak{p}^n \rightarrow \mathcal{O}''$ . We define  $\varphi$  to be the composite of  $\bar{\psi}$  and the canonical homomorphism  $\mathcal{O}'' \rightarrow \mathcal{O}''/\mathfrak{m}''^{n+1}$ . It was proved in [6, (3.3), Proof of Lemma 2] that  $\varphi$  is surjective if  $k = k'$ . Also, it is clear from the definition of  $\mathfrak{b}_{t_0}(n)$  that  $\ker \varphi = \mathfrak{b}_{t_0}(n)/\mathfrak{m}\mathfrak{p}^n$ . Thus (\*) is exact and the lemma is proved.

PROOF OF LEMMA (2.5): By Lemma (2.4), we already have the inequality  $H_{\mathfrak{p}}^{(0)} \geq H_{\mathcal{O}''}^{(1+\delta)}$  in the case  $k = k'$ . The inequality in the general case can now be proved by a standard inductive procedure used in [1], [4] and [6]. What we do is the following: Choose an element  $\alpha \in k' - k$ . If  $\delta \geq 1$ , we assume that  $\alpha$  is transcendental. If  $\delta = 0$ , we assume that  $\alpha$  is either separable or purely inseparable. Let  $\bar{f}(Z) \in k[Z]$  be the minimal monic polynomial of  $\alpha$  over  $k$ . (If  $\alpha$  is transcendental, we take  $\bar{f}(Z) = 0$ .) Let  $f(Z) \in \mathcal{O}[Z]$  be a monic lift of  $\bar{f}(Z)$  such that, for every  $i \geq 0$ , if the coefficient of  $Z^i$  in  $\bar{f}(Z)$  is 0 then the coefficient of  $Z^i$  in  $f(Z)$  is also 0. Let  $\tilde{\mathcal{O}}$  be the localization of  $0[Z]/f(Z)\mathcal{O}[Z]$  at the prime ideal  $\mathfrak{n} = (\mathfrak{m}[Z] + f(Z)\mathcal{O}[Z])/f(Z)\mathcal{O}[Z]$ , where  $\mathfrak{m}$  is the maximal ideal of  $\mathcal{O}$ . Let  $\eta: \mathcal{O} \rightarrow \tilde{\mathcal{O}}$  be the canonical homomorphism. Let  $a$  be a lift of  $\alpha$  to  $\mathcal{O}'$  and let  $\tilde{\mathcal{O}}'$  be the localization of  $\mathcal{O}'[Z]/f(Z)\mathcal{O}'[Z]$  at the maximal ideal

$$\mathfrak{n}' = (\mathfrak{m}'[Z] + (Z - a)\mathcal{O}'[Z])/f(Z)\mathcal{O}'[Z],$$

where  $\mathfrak{m}'$  is the maximal ideal of  $\mathcal{O}'$ . Let  $\eta': \mathcal{O}' \rightarrow \tilde{\mathcal{O}}'$  be the canonical homomorphism. Then there exists a commutative diagram

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{h} & \mathcal{O}' \\ \eta \downarrow & & \downarrow \eta' \\ \tilde{\mathcal{O}} & \xrightarrow{\tilde{h}} & \tilde{\mathcal{O}}' \end{array}$$

such that

(i)  $\tilde{h}$  is a blowing-up of  $\tilde{\mathcal{O}}$  with center  $\tilde{\mathfrak{p}} = \mathfrak{p}\tilde{\mathcal{O}}$ ;  
(ii) the residue field extension induced by  $\tilde{h}$  is the  $k$ -inclusion  $k(\alpha) \rightarrow k'$ . (See [6, (4.3), (4.6)].) Let  $\tilde{\delta} = \text{tr.deg}_{k(\alpha)} k'$ . If  $\delta \geq 1$ , then  $\tilde{\delta} = \delta - 1$ . If  $\delta = 0$ , then  $[k': k(\alpha)] < [k': k]$ . Therefore, by an obvious induction, we may assume that  $H_{\tilde{\mathfrak{p}}}^{(0)} \cong H_{\tilde{\mathcal{O}}''}^{(1+\tilde{\delta})}$ , where  $\tilde{\mathcal{O}}'' = \tilde{\mathcal{O}}'/\tilde{\mathfrak{m}}\tilde{\mathcal{O}}'$ ,  $\tilde{\mathfrak{m}}$  being the maximal ideal of  $\tilde{\mathcal{O}}$ . Now, in order to complete the proof of the lemma, it is clearly enough to prove the following three statements:

- (1)  $H_{\mathfrak{p}}^{(0)} = H_{\tilde{\mathfrak{p}}}^{(0)}$ .
- (2)  $H_{\tilde{\mathcal{O}}''}^{(0)} \cong H_{\mathcal{O}''}^{(0)}$  if  $\tilde{\delta} = \delta = 0$ .
- (3)  $H_{\tilde{\mathcal{O}}''}^{(0)} = H_{\mathcal{O}''}^{(1)}$  if  $\tilde{\delta} = \delta - 1$ .

PROOF OF (1): Let  $\tilde{k} = k(\alpha)$  be the residue field of  $\tilde{\mathcal{O}}$ . For every  $n \geq 0$ , we have  $H_{\tilde{\mathfrak{p}}}^{(0)}(n) = \dim_{\tilde{k}} \tilde{\mathfrak{p}}^n \otimes_{\tilde{\mathcal{O}}} \tilde{k} = \dim_{\tilde{k}} \mathfrak{p}^n \otimes_{\mathcal{O}} \tilde{k}$ , since,  $\tilde{\mathcal{O}}$  being  $\mathcal{O}$ -flat, we have  $\tilde{\mathfrak{p}}^n \approx \mathfrak{p}^n \otimes_{\mathcal{O}} \tilde{\mathcal{O}}$ . Now  $\mathfrak{p}^n \otimes_{\mathcal{O}} \tilde{k} \approx (\mathfrak{p}^n \otimes_{\mathcal{O}} k) \otimes_k \tilde{k}$ . Therefore,

$$\dim_{\tilde{k}} \mathfrak{p}^n \otimes_{\mathcal{O}} \tilde{k} = \dim_k \mathfrak{p}^n \otimes_{\mathcal{O}} k = H_{\mathfrak{p}}^{(0)}(n).$$

PROOF OF (2) AND (3): Let  $\mathfrak{m}''$  be the maximal ideal of  $\mathcal{O}''$ . Then  $\tilde{\mathcal{O}}'' = \tilde{\mathcal{O}}'/\tilde{\mathfrak{m}}\tilde{\mathcal{O}}' = \tilde{\mathcal{O}}'/\mathfrak{m}\tilde{\mathcal{O}}' = (\mathcal{O}''[Z]/f(Z)\mathcal{O}''[Z])_{\mathfrak{m}''}$ , where

$$\mathfrak{m}'' = (\mathfrak{m}''[Z] + (Z - a)\mathcal{O}''[Z])/f(Z)\mathcal{O}''[Z].$$

Now, if  $\tilde{\delta} = \delta - 1$ , then  $\alpha$  is transcendental and  $f(Z) = 0$ . Therefore the equality  $H_{\tilde{\mathcal{O}}''}^{(0)} = H_{\mathcal{O}''}^{(1)}$  is clear in this case. This proves (3). If  $\delta = 0$  and  $\alpha$  is separable then  $\tilde{f}(Z)$  being a separable polynomial,  $\mathcal{O}'' \rightarrow \tilde{\mathcal{O}}''$  is etale, so that in this case we have, in fact,  $H_{\tilde{\mathcal{O}}''}^{(0)} = H_{\mathcal{O}''}^{(0)}$ . Now suppose  $\delta = 0$  and  $\alpha$  is purely inseparable. Then  $\tilde{f}(Z) = Z^q - \beta$ , where  $q$  is a power of char  $k$  and  $\beta = \alpha^q \in k$ . This implies that  $f(Z) = Z^q - b$ , where  $b \in \mathcal{O}$  is some lift of  $\beta$ . Let  $\bar{b}$  be the canonical image of  $b$  in  $\mathcal{O}''$ . Since  $\mathcal{O}''[Z]/(Z^q - \bar{b})\mathcal{O}''[Z]$  is already a local ring, we have

$$\tilde{\mathcal{O}}'' = \mathcal{O}''[Z]/(Z^q - \bar{b})\mathcal{O}''[Z].$$

Let  $\bar{a}$  be the canonical image of  $a$  in  $\mathcal{O}''$  and let  $t = \bar{b} - \bar{a}^q$ . Then  $t \in \mathfrak{m}''$ . Let  $Y = Z - \bar{a}$ . Then  $\tilde{\mathcal{O}}'' = \mathcal{O}''[Y]/(Y^q - t)\mathcal{O}''[Y]$ . Now, the inequality  $H_{\tilde{\mathcal{O}}''}^{(0)} \geq H_{\mathcal{O}''}^{(0)}$  follows from [6, Lemma (4.5)]. This proves (2).

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