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THE RATIONALITY OF THE FOURIER COEFFICIENTS OF CERTAIN EISENSTEIN SERIES ON TUBE DOMAINS (I)

Liang-Chi Tsao*

Contents

Introduction

In this paper we wish to prove that under certain conditions the Fourier coefficients of certain Eisenstein series for an arithmetic group acting on a tube domain are rational numbers. The conditions are that the domain be equivalent to a bounded symmetric domain having a 0-dimensional boundary component (with respect to the arithmetic group), and that the arithmetic group be a subgroup of a special arithmetic group (cf. § 7.1).

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As in [4], let G be a connected, semi-simple, $\mathbb{Q}$-simple, linear algebraic group defined over the rational number field $\mathbb{Q}$. Let $\mathbb{R}$ be the real number field. We assume that $G^0_{\mathbb{R}}$ is centerless and has no compact simple factors, and that if K is a maximal compact subgroup of $G^0_{\mathbb{R}}$ then $\mathfrak{X} = K\backslash G^0_{\mathbb{R}}$ is a Hermitian symmetric space. We may write $G = R_{\mathbb{K}/\mathbb{Q}}G'$, where G' is absolutely simple, and defined over a totally real algebraic number field $\mathbb{K}$ and $R_{\mathbb{K}/\mathbb{Q}}$ is the ground field reduction functor [21, Chapt. 1]. We assume also that $\mathfrak{X}$ is isomorphic to a tube domain

$$\mathfrak{T} = \{X + iY \in \mathbb{C}^{\mathbb{M}} | Y \in \mathfrak{R}\}$$

where $\mathbb{C}$ is the complex number field and $\mathfrak{R}$ is a homogeneous, self-adjoint cone in $\mathbb{R}^{\mathbb{M}}$. It follows that the relative $\mathbb{R}$-root system of G is a sum of simple root systems of type C. We assume further that, when $\mathfrak{X}$ is realized as a bounded symmetric domain D in $\mathbb{C}^{\mathbb{M}}$, then D has a 0-dimensional boundary component F_0. Then $\mathfrak{X}$ may be identified with the tube domain $\mathfrak{T}$ in such a way that every element of $N(F_0) = \{g \in G^0_{\mathbb{R}} | F_0 \cdot g = F_0\}$ acts on $\mathfrak{T}$ by a linear affine transformation of the ambient vector space $\mathbb{C}^{\mathbb{M}}$ of $\mathfrak{T}$, that every element of the unipotent radical $U_{\mathbb{R}} = U(F_0)$ of $N(F_0)$ acts by a real translation, and that $N(F_0)$ is a $\mathbb{Q}$-parabolic subgroup of $G^0_{\mathbb{R}}$. Let N_h be the normalizer of $N(F_0)$ in the group G_h of all holomorphic automorphisms of $\mathfrak{T}$, and Γ be an arithmetic subgroup of G_h.

Let $\Gamma' = G^0_{\mathbb{R}} \cap \Gamma$, and $\Gamma_0 = \Gamma \cap N_h$. If $g \in G_h$ and $Z \in \mathfrak{T}$, let $j(Z, g)$ be the functional determinant of g at Z. Let $G'_{\mathbb{Q}} = G_{\mathbb{Q}} \cap G^0_{\mathbb{R}}$; and if $a \in G'_{\mathbb{Q}}$, let $\Gamma_{0,a} = \Gamma \cap aN_h a^{-1}$. Then, if l is a large positive rational number (with conditions on its denominator to be indicated later), we form the following Eisenstein series at the cusp a:

$$E_{l,a}(Z) = \sum_{\gamma \in \Gamma/\Gamma_{0,a}} j(Z, \gamma a)^l,$$

which converges absolutely and uniformly on compact subsets of $\mathfrak{T}$ and represents there an automorphic form with respect to Γ.

Let $G^*_{\mathbb{Q}}$ be the normalizer of $G'_{\mathbb{Q}}$ in G_h, and let $N_{\mathbb{Q}} = N_h \cap G^*_{\mathbb{Q}}$. Then $\Gamma \subset G^*_{\mathbb{Q}}$, and we can write $G^*_{\mathbb{Q}} = \bigcup_{a \in A} \Gamma a N_{\mathbb{Q}}$, a disjoint union of a finite number of double cosets with $A \subset G'_{\mathbb{Q}}$. We have also $G_{\mathbb{Q}} = \bigcup_{a \in A_1} \Gamma' a P_{\mathbb{Q}}$ for some finite set A_1, where P is a maximal $\mathbb{Q}$-parabolic subgroup of G such that $P \cap G^0_{\mathbb{R}} = N(F_0)$.

For each $a \in A$, let $c(a)$ be a complex number, and define an Eisenstein series

$$E_l = E_{l,c} = \sum_{a \in A} E_{l,a} c(a)^l.$$

Let $\Lambda' = \Gamma \cap U_{\mathbb{Q}}$, then Λ' is a lattice in $U_{\mathbb{Q}}$; let Λ be its dual lattice with respect to the non-degenerate symmetric bilinear form induced by the trace function on $U_{\mathbb{R}} = \mathbb{R}^{\mathbb{M}}$, when the latter is realized as a real Jordan algebra.

Since $E_l(Z + \mathbb{S}) = E_l(Z)$ for all $\mathbb{S} \in \Lambda'$, E_l has a Fourier expansion:

$$E_l(Z) = \sum_{\mathbb{T} \in \Lambda} a_l(\mathbb{T}) \varepsilon((\mathbb{T}, Z)),$$

where $\varepsilon(\ ) = e^{2\pi i(\ )}$, and $(\ ,\ )$ is the induced symmetric bilinear form on $\mathbb{C}^{\mathbb{M}}$.

In this paper, we restrict ourselves to the case when $\Gamma \subset G^0_{\mathbb{R}}$ and Γ is a subgroup of a special arithmetic group. Then we are able to prove that the Fourier coefficients $a_l(\mathbb{T})$ are rational numbers for suitably chosen $c(a)$.

Our methods have been adapted largely from the proof of this result in a special case in [5].

For the historical background, see [6, Introduction].

We sketch the contents of each section of this paper as follows:

In Section 1, we describe the relationships between tube domains and Jordan algebras. The next two sections are then devoted to Jordan algebras. Some technical lemmas, which are important for the later calculation of exponential sums, are proved. Sections 4, 5 and 6 are devoted to parabolic subgroups, functional determinants and boundary components, respectively, and the relationships among them, which enable us in Section 7 to reduce our treatment of the Fourier coefficients, by induction, to those of the 'biggest cell'. Also, in Section 7, we describe the way of choosing $c(a)$. In Section 8, we apply Gamma integral and Poisson summation formula to the biggest cell, and express the Fourier coefficients in terms of exponential sums and the volume $v(\Lambda)$ of a fundamental period parallelogram of the lattice Λ. In Section 9, by applying Hensel's lemma [3], we further reduce the calculations of the exponential sums to those over finite Jordan algebras. Then we devote Section 10 to the explicit calculations of the exponential sums, which turn out to be products of Euler factors; and then devote Section 11 to the calculation of the volume $v(\Lambda)$. Finally, in the last section, by using the values of L-functions (which come from the product of exponential sums), we are able to prove the rationality of the Fourier coefficients.

In June, 1972, when the author submitted part of this paper as the dissertation (see [19] for the announcement) at the University of Chicago, he was not able to calculate the value of $v(\Lambda)$ for certain tube domains of type A, thus left the work uncompleted. In the summer of that year, Baily [6] carried out this calculation via theory of quadratic forms while the author did it in a rather straightforward way and was able to dispose the type A case in full generality. We combine this calculation with the dissertation and present here the whole story.

The author would like to express his appreciation and thanks to his thesis advisor, Professor W. L. Baily, Jr., without whose advice and encouragement this work would not have been possible. Thanks also are due to Dr. M. Karel, with whom the author has had several fruitful talks.

0. Notation

0.1. As usual, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote, respectively, the ring of integers, and the fields of rational, real and complex numbers. $\mathbb{F}_{p^\alpha}$ denotes the finite field of p^α elements, where p is a prime and α is a positive integer.

0.2. If $\mathscr{A}$ is an algebra, then $\mathscr{A}^*$ denotes the set of invertible elements of $\mathscr{A}$.

0.3. Let $\mathbb{E} \subset \mathbb{F}$ be two field, then $N_{\mathbb{F}/\mathbb{E}}(x)$ or $N(x)$ (resp. $\mathrm{Tr}_{\mathbb{F}/\mathbb{E}}(x)$ or $\mathrm{Tr}\,(x)$) denotes the norm (resp. trace) of $x \in \mathbb{F}$ in $\mathbb{E}$.

0.4. Let $\mathbb{K}$ be an algebraic number field. Then $\mathfrak{p}$ denotes a prime of $\mathbb{K}$ dividing a prime p of $\mathbb{Q}$. $\mathbb{K}_\mathfrak{p}$ (resp. $\mathbb{Q}_p$) is the local field at $\mathfrak{p}$ (resp. p), and $\mathbb{Z}_\mathfrak{p}$ (resp. $\mathbb{Z}_p$) is the $\mathfrak{p}$-adic (resp. p-adic) integers of $\mathbb{K}_\mathfrak{p}$ (resp. $\mathbb{Q}_p$). Let π be a local parameter of $\mathbb{K}_\mathfrak{p}$. Let $|\ |_\mathfrak{p}$ (resp. $|\ |_p$) be the standard norm on $\mathbb{K}_\mathfrak{p}$ (resp. $\mathbb{Q}_p$) (so that the product formula is true).

0.5. Let $\mathscr{A}$ be an algebra with an involution $J: a \to a^*$. We write $(\mathscr{A}, J)$ for such an algebra, and $\mathscr{H}(\mathscr{A}, J)$ for the set of all symmetric elements in this algebra; if there is no confusion we write $\mathscr{H}(\mathscr{A}) = \mathscr{H}(\mathscr{A}, J)$. Let $M(n, \mathscr{A})$ denote the set of all $n \times n$ matrices over $\mathscr{A}$. Denote the identity matrix by I (or I_n), and the matrix with 1 on the (i, j)-th entry and 0 elsewhere by e_{ij}. If X is a matrix (not necessarily a square matrix), then X^* denotes the transpose conjugate (w.r.t. $a \to a^*$) of X. Given an invertible element A in $M(n, s)$ such that $A = A^*$, define an involution J_A on $M(n, s)$ by $X \to AX^*A^{-1}$. Write

$$(\mathscr{A}_n, J_A) = (M(n, \mathscr{A}), J_A) \quad \text{and} \quad \mathscr{H}(\mathscr{A}_n, J_A) = \mathscr{H}(M(n, \mathscr{A}), J_A).$$

Write $\mathrm{diag}\,(a_1, \ldots, a_n)$ for a diagonal matrix in $M(n, \mathscr{A})$.

0.6. Let H be a non-empty subset of a group G, then $N(H)$ (resp. $Z(H)$) denotes the normalizer (resp. centralizer) of H in G. If G is an algebraic group defined over an algebraic number field $\mathbb{K}$, we write $G_\mathfrak{p}$ for $G_{\mathbb{K}_\mathfrak{p}}$ for a prime $\mathfrak{p}$. (Also applied to an algebra $\mathscr{A}$ defined over $\mathbb{K}$: $\mathscr{A}_\mathfrak{p} = \mathscr{A}_{\mathbb{K}_\mathfrak{p}}$, if there is no confusion.)

Part I. TUBE DOMAINS

1. Tube domains and Jordan algebras

We summarize the relationships between tube domains and Jordan algebras as stated in [4, §§ 2, 3].

1.1. Let G be a centerless, connected, simple, linear algebraic group defined over $\mathbb{R}$. If K is a maximal compact subgroup of $G^0_{\mathbb{R}}$, we assume that $\mathfrak{X} = K\backslash G^0_{\mathbb{R}}$ is isomorphic to a bounded hermitian symmetric domain D. Assume further that the relative $\mathbb{R}$-root system of G is of type C, then $\mathfrak{X}$ is isomorphic to a tube domain

$$\mathfrak{T} = \{X + iY \in \mathbb{C}^{\mathbb{M}} | Y \in \mathfrak{K}\},$$

where $\mathfrak{K}$ is a homogeneous irreducible self-adjoint cone in $\mathbb{R}^{\mathbb{M}}$.

Let $S = {}_{\mathbb{R}}T$ be a maximal $\mathbb{R}$-trivial torus of G with simple $\mathbb{R}$-root system ${}_{\mathbb{R}}\Delta$. Then precisely one simple root α is non-compact, since ${}_{\mathbb{R}}\Delta$ is of type C. Let S_0 be the 1-dimensional subtorus of S on which all simple $\mathbb{R}$-roots vanish except α. The centralizer $Z(S_0)$ of S_0 and the positive $\mathbb{R}$-root subgroups of G generate a maximal $\mathbb{R}$-parabolic subgroup P of G. Then $P \cap G^0_{\mathbb{R}} = N(F_0) = \{g \in G^0_{\mathbb{R}} | F_0 \cdot g = F_0\}$ for some 0-dimensional boundary component F_0 of D, and $\mathfrak{X}$ may be identified with $\mathfrak{T}$ in such a way that every element of $N(F_0)$ acts on $\mathfrak{T}$ by a linear affine transformation on the ambient vector space $\mathbb{C}^{\mathbb{M}}$ of $\mathfrak{T}$, and every element of the unipotent radical $U_{\mathbb{R}}$ $(= U \cap G^0_{\mathbb{R}}$, U being the unipotent radical of $P)$ of $N(F_0)$ by a real translation. Then $U_{\mathbb{R}}$ may be identified with $\mathbb{R}^{\mathbb{M}}$, and will have a simple compact real Jordan algebra structure such that the cone $\mathfrak{K}$ may be described as the interior of the set of all squares of this Jordan algebra [13; 20].

1.2. Let G be taken subject to the general assumption of the Introduction; then $G = R_{\mathbb{K}/\mathbb{Q}}G'$ for some absolutely simple linear algebraic group G' defined over a totally real number field $\mathbb{K}$. We may choose a maximal torus T, a maximal $\mathbb{R}$-trivial torus ${}_{\mathbb{R}}T$, and a maximal $\mathbb{Q}$-trivial torus ${}_{\mathbb{Q}}T$ in G such that T is defined over $\mathbb{Q}$ and ${}_{\mathbb{Q}}T \subset {}_{\mathbb{R}}T \subset T$. Then for a suitable maximal $\mathbb{K}$-trivial torus ${}_{\mathbb{K}}T'$, a maximal $\mathbb{R}$-trivial torus ${}_{\mathbb{R}}T'$ and a maximal $\mathbb{K}$-torus T' in G', we have ${}_{\mathbb{K}}T' \subset {}_{\mathbb{R}}T' \subset T'$ and ${}_{\mathbb{Q}}T \subset R_{\mathbb{K}/\mathbb{Q}}({}_{\mathbb{K}}T')$, $T = R_{\mathbb{K}/\mathbb{Q}}T'$. We take compatible orderings on all the root systems.

Assume that D has a 0-dimensional rational boundary component F_0. Let $P = N(F_0)_{\mathbb{C}}$, and let U be the unipotent radical of P. The $\mathbb{R}$-root system of each simple factor of G is of type C; and since $\dim F_0 = 0$, we know that the non-compact simple root in each simple factor of G

is critical and that the simple $\mathbb{Q}$-root system ${}_{\mathbb{Q}}\Delta$ of G is of type C. Let S_0 be the subtorus of ${}_{\mathbb{Q}}T$ on which all simple $\mathbb{Q}$-roots vanish except the non-compact one. Then P is the maximal $\mathbb{Q}$-parabolic subgroup of G generated by the centralizer $Z(S_0)$ of S_0 and the positive $\mathbb{Q}$-root subgroups of G, and then $P = R_{\mathbb{K}/\mathbb{Q}}P'$, $U = R_{\mathbb{K}/\mathbb{Q}}U'$, where P' is the maximal $\mathbb{K}$-parabolic (and hence $\mathbb{R}$-parabolic) subgroup of G' of § 1.1 (in which they were denoted by P and G, respectively), and U' is its unipotent radical. Then $U_{\mathbb{R}}$ is a semi-simple compact real Jordan algebra with $\mathbb{Q}$-structure $U_{\mathbb{Q}}$ induced (via the ground field reduction functor $R_{\mathbb{K}/\mathbb{Q}}$) by the simple real compact Jordan algebra $U'_{\mathbb{R}}$ defined over the totally real algebraic number field $\mathbb{K}$.

The tube domain $\mathfrak{T}$ is then a direct product of irreducible tube domains ${}^{\sigma}\mathfrak{T}'$ corresponding to simple factors ${}^{\sigma}G'$, where $G = R_{\mathbb{K}/\mathbb{Q}}G' = \prod_{\sigma\in\Sigma}{}^{\sigma}G'$, and Σ is the set of all isomorphisms of $\mathbb{K}$ into $\mathbb{R}$.

2. Jordan algebras

In this section, we discuss the compact real Jordan algebras and their $\mathbb{K}$-structures, where $\mathbb{K}$ is a totally real algebraic number field.

2.1. In this article, we intend to describe all the simple compact real Jordan algebras [20, Chapt. 2].

Let V be a vector space of dimension $\geqq 1$ over $\mathbb{R}$, provided with a negative definite bilinear form (,). Consider the vector space $\mathbb{R} \oplus V$ with the product of two of its elements defined by

$$(a+u)(b+v) = (ab+(u,v))+(av+bu),$$

for $a, b \in \mathbb{R}$, $u, v \in V$. Then $\mathbb{R} \oplus V$ becomes an algebra with involution $a+u \to (a+u)^* = a-u$.

Let $\mathscr{C}$ be any one of the real, the complex, the quaternion, the Cayley or the above-constructed algebra, then $\mathscr{C}$ is a real algebra with involution $c \to c^*$ such that

(1) trace of $c \in \mathscr{C}: tr(c) = c+c^* \in \mathbb{R}$,

(2) norm of $c \in \mathscr{C}: n(c) = cc^* = c^*c \in \mathbb{R}$, and

(3) the norm form is positive definite on the real vector space $\mathscr{C}$.

Let $\dim \mathscr{C} = n_0$, and denote $\mathscr{C}_n^{n_0}$ the space of all $n \times n$ hermitian (with respect to the involution *) symmetric matrices with entries in $\mathscr{C}$. Then any simple compact real Jordan algebra $\mathscr{J}$ is isomorphic to one of the following five types:

$$\mathscr{C}_n^1(n \geqq 1), \mathscr{C}_n^2(n \geqq 2), \mathscr{C}_n^4(n \geqq 3), \mathscr{C}_3^8, \mathscr{C}_2^{n_0}(n_0 \geqq 3),$$

provided with the Jordan multiplication $A \circ B = \frac{1}{2}(AB + BA)$, where AB is the usual matrix multiplication.

Note: The restrictions on n for the first three types and n_0 for the last type are not necessary; but, by such restrictions, no two of the Jordan algebras listed above are isomorphic.

Now, the irreducible homogeneous self-adjoint cone $\mathfrak{R}$ may be described as $\mathfrak{R} = \{T^*T | T$ is an $n \times n$ upper triangular matrix with entries in $\mathscr{C}$ and with positive real numbers on the diagonal$\}$, and $\mathfrak{T} = \mathscr{J} + i\mathfrak{R}$ is the irreducible tube domain.

2.2. Let us collect some well-known facts about the general structure theory of Jordan algebras over any field Φ of characteristic $\neq 2$ [12, Chapt. 4, 5].

THEOREM 2.2.1: *A finite-dimensional central simple Jordan algebra is one of the following types*:

(1) *a Jordan algebra of a non-degenerate symmetric bilinear form* $\Phi \oplus V$, $\dim V \geqq 2$;

(2) *a Jordan algebra* $\mathscr{H}(\mathscr{A}, J)$ *of symmetric elements of a finite-dimensional central simple associative algebra with involution* $(\mathscr{A}, J)$;

(3) *a Jordan algebra* $\mathscr{J}$, *such that* $\mathscr{J}_\Omega$, *for* Ω *the algebraic closure of* Φ, *is isomorphic to* $\mathscr{H}(\mathbb{O}_3, \mathscr{J}_I)$ *of* 3×3 *octonion symmetric matrices over the octonion algebra* $\mathbb{O}$ *over* Ω.

THEOREM 2.2.2: [12, p. 209]. *If* $(\mathscr{A}, J)$ *is a finite-dimensional central simple associative algebra with involution, then* $(\mathscr{A}_\Omega, J)$ *is isomorphic to* $(\mathbb{D}_n, J_I)$, $n \geqq 1$, *the* $n \times n$ *matrices over the split composition algebra* $\mathbb{D}$ *over* Ω, *of dimension* 2, 1 *or* 4.

The algebra $(\mathscr{A}, J)$ is then said to be of type A, B or C, respectively, and n is called the degree of $(\mathscr{A}, J)$.

THEOREM 2.2.3: [12, p. 209]. *Two finite-dimensional central simple associative algebras with involutions* $(\mathscr{A}_\Omega, J_I)$ *and* $(\mathscr{B}_\Omega, J_I)$ *are isomorphic if and only if they are of the same type and of the same degree.*

THEOREM 2.2.4: [12, p 210]. *A Jordan algebra is finite-dimensional, special, and central simple of degree* $\geqq 3$ *if and only if it is isomorphic to a Jordan algebra* $\mathscr{H}(\mathscr{A}, J)$, *where* $(\mathscr{A}, J)$ *is a finite-dimensional simple associative algebra with involution of degree n. If* $(\mathscr{A}, J)$ *and* $(\mathscr{B}, J')$ *are* *two finite-dimensional central simple algebras with involutions of degree* $\geqq 3$, *then they are isomorphic if and only if the Jordan algebras* $\mathscr{H}(\mathscr{A}, J)$ *and* $\mathscr{H}(\mathscr{B}, J')$ *are isomorphic.*

A Jordan algebra $\mathscr{H}(\mathscr{A}, J)$ is of type A, B or C if and only if $(\mathscr{A}, J)$ is of the corresponding type. And the Jordan algebras of (1) and (3) in Theorem 2.2.1 are said to be of types D and E, respectively.

THEOREM 2.2.5: [12, p. 208]. *There are three types of finite-dimensional central simple associative algebras with involutions*:

(1) $(\mathscr{A}, J)$, *where* $\mathscr{A}$ *is central simple and* J *is an involution of the first kind*;

(2) $(\mathscr{A}, J)$, *where* $\mathscr{A}$ *is central simple, and* J *is an involution of the second kind*;

(3) $(\mathscr{A}, J) = (\mathscr{B} \oplus \mathscr{B}, J)$, *where* $\mathscr{B}$ *is central simple, and* J *is the interchange involution.*

Now we restrict ourselves to the case when $\Phi = \mathbb{K}$, and consider the Jordan algebra $\mathscr{J}$ defined over $\mathbb{K}$ such that $\mathscr{J}_{\mathbb{R}}$ is a compact real Jordan algebra of § 2.1.

The third case of the above theorem is then ruled out, because $\mathscr{B} \oplus \mathscr{B}$ has a two-dimensional split center which cannot occur in the associative algebra corresponding to a simple compact real Jordan algebra. We shall discuss (1) and (2) over a totally real algebraic number field in the next two articles.

THEOREM 2.3: *If* $(\mathscr{A}_{\mathbb{K}}, J)$ *is a finite-dimensional central simple associative algebra with involution of the first kind, then* $(\mathscr{A}_{\mathbb{K}}, J) = (\mathbb{D}_n, J_A)$, *where* $\mathbb{D}$ *is* $\mathbb{K}$ *or a quaternion division algebra over* $\mathbb{K}$, *and*

$$A = diag\,(a_1, \ldots, a_n),$$

with $a_i \in \mathbb{K}^*$.

PROOF: $\mathscr{A}_{\mathbb{K}}$ is central simple, hence is a matrix algebra over a division algebra over the field $\mathbb{K}$. When J is restricted to this division algebra, it is an involution of the first kind. It is known [1, Chapt. 10, Th. 20] that a division algebra with involution of the first kind over an algebraic number field is either the number field itself or a quaternion division algebra over it. Then it is a well-known fact that $J = J_A$ for some $A = \operatorname{diag}(a_1, \ldots, a_n)$, with $a_i \in \mathbb{K}^*$.

The algebras are of type B or C, respectively.

Note that $\mathscr{H}(\mathscr{A}_{\mathbb{K}}, J) \otimes \mathbb{R}$ is a real compact Jordan algebra if and only if all a_i are positive.

2.4. We turn to the case of involution of the second kind. Then $(\mathscr{A}_{\mathbb{K}}, J)$ is isomorphic to $(\mathbb{D}_m, J_A)$, where $\mathbb{D}$ must be a central simple associative division algebra with positive definite involution of the second kind, and $A = \operatorname{diag}(a_1, \ldots, a_n)$, $a_i \in \mathbb{K}^*$ and a_i are positive. Such algebra $\mathbb{D}$ is a cyclic division algebra with positive definite involution j of the second kind and may be described as follows [17, §§ 4, 5]:

Let $\mathbb{K}(\zeta)$ be a real cyclic extension of $\mathbb{K}$, and $\mathbb{K}(\theta)$ be a complex quadratic extension of $\mathbb{K}$. Then $\mathbb{D}$ is an $s(= [\mathbb{K}(\zeta): \mathbb{K}])$-dimensional left vector space over $\mathbb{K}(\zeta, \theta)$ with a basis $\{1, \eta, \eta^2, \ldots, \eta^{s-1}\}$ such that $\eta c = {}^{\tau}c\eta$, for all $c \in \mathbb{K}(\zeta, \theta)$, where τ is a generator of the Galois group

$$\operatorname{Gal}(\mathbb{K}(\zeta, \theta)/\mathbb{K}(\theta)) = \operatorname{Gal}(\mathbb{K}(\zeta)/\mathbb{K}),$$

and such that $\eta^s = b \in \mathbb{K}(\theta)$. It is required that there exist $a \in \mathbb{K}(\zeta)$ such that $N_{\mathbb{K}(\zeta)/\mathbb{K}}(a) = N_{\mathbb{K}(\theta)/\mathbb{K}}(b)$, and that, for any k, $0 < k < s$, there exist no $c \in \mathbb{K}(\zeta, \theta)$ such that $N_{\mathbb{K}(\zeta,\theta)/\mathbb{K}(\theta)}(c) = b$. Then the positive definite involution j of the second kind is defined in the following way: j fixes elements of $\mathbb{K}(\zeta)$, and takes θ to $-\theta$, and η to $a\eta^{-1}$. We have $n = ms$.

This disposes the cases of types A, B and C. We shall devote the next two articles to type D Jordan algebras.

2.5. Let V be a vector space of dimension $\geqq 1$ over a field Φ of characteristic $\neq 2$, provided with a non-degenerate symmetric bilinear form f. Let $\mathscr{J} = \mathscr{J}(V, f) = \Phi \oplus V$ and define the multiplication of any two elements of $\mathscr{J}$ by $(\alpha_1 + v_1)(\alpha_2 + v_2) = (\alpha_1\alpha_2 + f(v_1, v_2)) + (\alpha_1 v_2 + \alpha_2 v_1)$, for $\alpha_1, \alpha_2 \in \Phi$, $v_1, v_2 \in V$. Then $\mathscr{J}$ is the central simple Jordan algebra of type D. We also note that $\alpha + v \to \alpha - v$ is an involution in $\mathscr{J}$.

THEOREM 2.5: *$\mathscr{H}(\mathscr{J}(V, f)_2, J_I)$ is a central simple Jordan algebra of type* D.

PROOF: Let $W = \Phi \oplus \Phi \oplus V$, and define a non-degenerate symmetric bilinear form g on W by $g(\alpha + \beta + v) = \alpha^2 + \beta^2 - f(v)$. Then

$$\begin{pmatrix} a & \alpha + v \\ \alpha - v & b \end{pmatrix} \to \frac{a+b}{2} + \left(\frac{a-b}{2} + \alpha + v\right)$$

defines a Jordan algebra isomorphism of $\mathscr{H}(\mathscr{J}(V, f)_2, J_I)$ onto $\mathscr{J}(W, g)$, a Jordan algebra of type D.

2.6. For the tube domain of type D, $n_0 \geqq 3$, we may take $G' = \mathrm{SO}(h)/\text{center}$, where h is a quadratic form in more than 6 variables, of $\mathbb{R}$-rank 2, with coefficients in $\mathbb{K}$. We may assume [14, p. 74] that $h = \alpha_1\alpha_2 + \alpha_3^2 - g$ such that g is the positive definite symmetric bilinear form of the Jordan algebra $\mathscr{J}(W, g)$ corresponding to the tube domain (cf. §§ 2.1, 2.5). The R-rank of $\alpha_3^2 - g$ is 1, and $\alpha_3^2 - g$ is in more than 4 variables, and hence is of $\mathbb{K}_\mathfrak{p}$-rank $\geqq 1$ for each prime ideal $\mathfrak{p}$ of $\mathbb{K}$. Thus, by Hasse principal for the quadratic forms, $\alpha_3^2 - g$ is of $\mathbb{K}$-rank $= 1$. Then we may write $g = x_1^2 + g_0$ in a suitable basis. If we write $g_0 = a(x_2^2 - f)$, then $\mathscr{J}(W, g)$ is isomorphic to $\mathscr{H}(\mathscr{J}(V, f)_2, J_A)$, with $A = \operatorname{diag}(a, 1)$. The isomorphism sends

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & \alpha + v \\ \alpha - v & y \end{pmatrix}$$

of $\mathscr{H}(\mathscr{J}(V, f)_2, J_A)$ to

$$\left(\frac{ax+y}{2}, \frac{ax-y}{2} + \alpha + v \right)$$

of $\mathscr{J}(W, g)$.

2.7. It is known [2] that, over an algebraic number field $\mathbb{K}$, a finite dimensional central simple exceptional Jordan algebra is reduced and isomorphic to $\mathscr{H}(\mathbb{O}_3, J_A)$, where $\mathbb{O}$ is an octonion algebra over $\mathbb{K}$, and $A = \operatorname{diag}(a_1, a_2, a_3)$, with $a_i \in \mathbb{K}^*$.

For a totally real number field $\mathbb{K}$, $\mathscr{H}(\mathbb{O}_3, J_A) \otimes \mathbb{R}$ is isomorphic to a compact real Jordan algebra if and only if $\mathbb{O}$ is a division algebra and a_i are positive.

2.8. We summarize the preceding results as

THEOREM 2.8: *The $\mathbb{K}$-structure $U'_{\mathbb{K}}$ of the Jordan algebra U' of* § 1.2 *for tube domains of types* B, C, D *or* E *is of the form $\mathscr{H}(\mathscr{A}_n, J_A)$, where $\mathscr{A}$ is $\mathbb{K}$, a division quaternion algebra over $\mathbb{K}$, $\mathscr{J}(V, f)$ of* § 2.6 *or an octonion division algebra over $\mathbb{K}$, respectively, and $A = \operatorname{diag}(a_1, \ldots, a_n)$, with all a_i totally positive in $\mathbb{K}$. (The last statement follows from the fact that the real extensions of the conjugates of the Jordan algebra $U'_{\mathbb{K}}$ are all compact (cf.* § 2.10).)

We know that $\mathscr{H}(\mathscr{A}_n, J_A) \otimes \mathbb{R}$ is isomorphic to $\mathscr{H}((\mathscr{A}_{\mathbb{R}})_n, J_I) = \mathscr{C}_n^{no}$ of § 2.1. The isomorphism may be described as follows: Any element of $\mathscr{H}(\mathscr{A}_n, J_A) \otimes \mathbb{R}$ is of the form AX, for $X \in \mathscr{H}(\mathscr{A}_n, J_I) \otimes \mathbb{R}$. The isomorphism sends AX to $X' \in \mathscr{H}(\mathscr{A}_{\mathbb{R}})_n, J_I)$, where if $X = (x_{ij})$, $X' = (x'_{ij})$, then $x'_{ij} = \sqrt{a_i a_j} x_{ij}$.

2.9. As for type A, the Jordan algebra is $\mathscr{H}(\mathbb{D}_m, J_A)$, where $\mathbb{D}$ is a cyclic division algebra of $2s^2$ dimensions over $\mathbb{K}$ with positive definite involution of the second kind, and $A = \operatorname{diag}(a_1, \ldots, a_m)$, with a_i totally positive in $\mathbb{K}$. We have $n = ms$.

Now we describe the isomorphism of $\mathscr{H}(\mathbb{D}_m, J_A) \otimes \mathbb{R}$ with $\mathscr{C}_n^2$, the $n \times n$ hermitian symmetric matrices.

As a first step, we have

LEMMA 2.9: *$\mathbb{D}_{\mathbb{R}}$ is isomorphic to $M(s, \mathbb{C})$.*

PROOF: [1] We use the notations of § 2.4. Define an isomorphism R_0 from $\mathbb{K}(\zeta, \theta) \otimes_{\mathbb{K}} \mathbb{R}$ into $M(s, \mathbb{C})$ by $R_0(\zeta)$ = right regular representation of ζ on $\mathbb{K}(\zeta, \theta)$ over $\mathbb{K}(\theta)$ with respect to the basis $\{\zeta_0, \zeta_1, \ldots, \zeta_{s-1}\}$, where $\zeta_i = {}^{\tau^i}\zeta$. If $\alpha = \Sigma c_i \zeta_i \in \mathbb{K}(\zeta, \theta) \otimes_{\mathbb{K}} \mathbb{R}$, then extend R_0 to α by defining $R_0(\alpha) = \Sigma c_i R_0(\zeta_i) \in M(s, \mathbb{C})$.

Let β be one of the s-th roots of $b \in \mathbb{K}(\theta) \subset \mathbb{C}$ in $\mathbb{C}$. Then $\beta \in \mathbb{C} \subset \mathbb{K}(\zeta, \theta) \otimes_{\mathbb{K}} \mathbb{R}$ is such that $\det R_0(\beta) = |R_0(\beta)| = b$. Define $R_0(\eta)$ = constant matrix β times the matrix of the linear transformation τ of $\mathbb{K}(\zeta, \theta)$ over $\mathbb{K}(\theta)$ with respect to $\{\zeta_0, \zeta_1, \ldots, \zeta_{s-1}\}$, and then R_0 can be extended to be an isomorphism from $\mathbb{D}_{\mathbb{R}}$ onto $M(s, \mathbb{C})$.

If no confusion is likely to arise, we shall use X for $R_0(X)$, $X \in \mathbb{D}_{\mathbb{R}}$, and τ for the matrix that represents it.

Now, the involution j on $\mathbb{D}_{\mathbb{R}} \cong M(s, \mathbb{C})$ is induced by a hermitian symmetric matrix $B_0 = B_0^*$ of $M(s, \mathbb{C})$ with $|B_0| = 1$ in the sense that $j(X) = B_0 X^* B_0^{-1}$ for all $X \in \mathbb{D}_{\mathbb{R}} \cong M(s, \mathbb{C})$. Put $B = \operatorname{diag}(B_0, B_0, \ldots, B_0)$, m factors. Then it is obvious that $(\mathbb{D}_m, J_A) \otimes \mathbb{R} \cong (\mathbb{C}_n, J_{AB})$, and the latter is in turn isomorphic to $(\mathbb{C}_n, J_I)$ by the correspondence $ABY \to CYC^*$, for any $Y \in (\mathbb{C}_n, J_I)$, where C is such that $CC^* = AB$.

The isomorphisms of the central simple associative algebras give the desired isomorphism for the Jordan algebras $\mathscr{H}(\mathbb{D}_m, J_A) \otimes \mathbb{R}$ and $\mathscr{C}_n^2 = \mathscr{H}(\mathbb{C}_n, J_I)$:

$$\begin{aligned} \mathscr{H}(\mathbb{D}_m, J_A) \otimes \mathbb{R} &\to \mathscr{H}(\mathbb{C}_n, J_{AB}) \to \mathscr{H}(\mathbb{C}_n, J_I) \\ AS &\to AB(B^{-1}S) \to C(B^{-1}S)C^*, \end{aligned}$$

for each $S \in \mathscr{H}(\mathbb{D}_m, J_I) \otimes \mathbb{R}$.

2.10. Let $G = R_{\mathbb{K}/\mathbb{Q}} G'$ be as in § 1.2. Write $\mathscr{J}$ (resp. $\mathscr{J}'$) for the abelian group U (resp. U') with Jordan algebra structure. Then $\mathscr{J}_{\mathbb{R}} = \prod_{\sigma \in \Sigma} {}^{\sigma}\mathscr{J}'_{\mathbb{R}}$, $\mathscr{J}_{\mathbb{Q}} = R_{\mathbb{K}/\mathbb{Q}} \mathscr{J}'_{\mathbb{K}}$ and ${}^{\sigma}\mathscr{J}'_{\mathbb{R}} = \mathscr{J}'_{\sigma_{\mathbb{K}}} \otimes \mathbb{R}$, where Σ is the set of all isomorphisms of $\mathbb{K}$ into $\mathbb{R}$.

In the sequel, we identify $\mathscr{J}$ (as a vector space) with the Lie algebra (a vector space) of the abelian group U [§ 7.2].

It is easy to see that ${}^{\sigma}\mathscr{J}'_{\mathbb{R}}$ are isomorphic to one another for all $\sigma \in \Sigma$. If $\mathscr{J}'$ is of type A, B, C, D or E, we say that the corresponding tube domain $\mathfrak{T}$ is, respectively, of type A, B, C, D or E.

3. Determinant and rank

In this section, we define the determinant and the rank of an element of a finite-dimensional central simple Jordan matrix algebra and develop some theorems that will be very useful for the calculation of exponential sums in § 10.

3.1. Let Φ be any field of characteristic $\neq 2$, Ω be its algebraic closure. Consider the finite-dimensional central simple Jordan algebras of the following types:

A. $\mathscr{H}(\mathbb{D}_n, J_I)$: $n \times n$ hermitian symmetric matrices over a quadratic extension of Φ.

B. $\mathscr{H}(\mathbb{D}_n, J_I)$: $n \times n$ symmetric matrices over Φ.

C. $\mathscr{H}(\mathbb{D}_n, J_I)$: $n \times n$ quaternion symmetric matrices over a quaternion algebra over Φ.

D. $\mathscr{H}(\mathscr{J}(V, f)_2, J_I)$: 2×2 ('hermitian') symmetric matrices over a Jordan algebra $\mathscr{J}(V, f)$ of a non-degenerate quadratic form f on a vector space V of dimension $\geqq 1$ over Φ.

E. $\mathscr{H}(\mathbb{O}_3, J_I)$: 3×3 octonion symmetric matrices over an octonion algebra over Φ.

We use the notation $\mathscr{H}(\mathscr{A}_n, J_I)$ to denote any one of the Jordan algebras listed above, n being the size of the matrix. In the following discussion, we also include the classes $n < 2$ of type D and $n < 3$ of type E.

3.2. In $\mathscr{A}$, we have the usual involution $a \to a^*$ such that

$$\operatorname{tr}(a) = a + a^* \in \Phi, \qquad n(a) = aa^* = a^*a \in \Phi,$$

and that $n(a)$ is a non-degenerate quadratic form and

$$(a, b) = \operatorname{tr}(ab^*) = \operatorname{tr}(a^*b)$$

defines a non-degenerate symmetric bilinear form.

THEOREM 3.2: *The generic minimum polynomial $M_X(\lambda)$ of $X \in \mathscr{H}(\mathscr{A}_n, J_I)$ is of degree n. The generic trace of X is the sum of diagonal entries.*

PROOF: The generic minimum polynomial (and hence the generic trace) is unchanged under the extension of base field; thus we may assume that $X \in (\mathscr{A}_n, J_I)_\Omega$. By [12, Chapt. VI, § 4], the theorem is obviously true for Jordan algebras of type A, B or E. (For $n = 2$ of type E, the Jordan algebra is isomorphic to a Jordan algebra of type D, for which the theorem will be proved later; for $n = 1$, it is trivial.)

For type C, we notice that $\mathscr{H}(\mathscr{A}_n, J_I)_\Omega \cong \mathscr{H}(\Omega_{2n}, J_S)$, where

$$S = \operatorname{diag}\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}.$$

The generic minimum polynomial calculated in $\mathscr{H}(\Omega_{2n}, J_S)$ is of degree n, hence it is also of degree n when calculated in $\mathscr{H}(\mathscr{A}_n, J_I)_\Omega$. The generic trace calculated in $\mathscr{H}(\Omega_{2n}, J_S)$ is one half of the sum of all the diagonal entries [12, Chapt. VI, § 4]; but a diagonal element a of a matrix in $\mathscr{H}(\mathscr{A}_n, J_I)_\Omega$ corresponds to the diagonal form

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

of the corresponding matrix in $\mathscr{H}(\Omega_{2n}, J_S)$. Therefore, the generic trace calculated in $\mathscr{H}(\mathscr{A}_n, J_I)_\Omega$ is the sum of all the diagonal elements.

For type D, we notice that (§ 2.5) $\mathscr{H}(\mathscr{J}(V, f)_2, J_I) \cong \mathscr{J}(W, g)$, where $\dim W = \dim V + 2$, and $g = x_1^2 + x_2^2 - f$. The isomorphism between them is

$$X = \begin{pmatrix} a & \alpha + v \\ \alpha - v & b \end{pmatrix} \to Y = \frac{a+b}{2} + \left(\frac{a-b}{2} + \alpha + v\right).$$

The generic minimum polynomial of Y is [12, Chapt. VI, § 4]

$$\text{(1)} \qquad M_Y(\lambda) = \lambda^2 - 2\left(\frac{a+b}{2}\right)\lambda + \left(\left(\frac{a+b}{2}\right)^2 - g\left(\frac{a-b}{2} + \alpha + v\right)\right)$$
$$= \lambda^2 - (a+b)\lambda + (ab - (\alpha^2 - f(v))).$$

Thus the generic minimum polynomial of X is of degree 2, and the generic trace of X is $a+b$, the sum of all the diagonal elements.

3.3. If $M_X(\lambda) = \lambda^n - \sigma_1(X)\lambda^{n-1} + \dots + (-1)^n\sigma_n(X)$, then the generic trace $\operatorname{tr}(X)$ of X is $\sigma_1(X)$, the generic norm $N(X)$ of X is $\sigma_n(X)$. It is known that $M_X(X) = 0$, from which we have

$$N(X)I = X[\sigma_{n-1}(X)I - \sigma_{n-2}(X)X + \dots + (-1)^{n-1}X^{n-1}].$$

If we put

$$\tilde{X} = \sigma_{n-1}(X)I - \sigma_{n-2}(X)X + \dots + (-1)^{n-1}X^{n-1},$$

then $X\tilde{X} = \tilde{X}X = N(X)I$. If $N(X) \neq 0$, then $X^{-1} = N(X)^{-1}\tilde{X}$.

We know that $N(X)$ is the determinant of X when X belongs to a Jordan algebra of type A or B. For type D, from (1), we have $N(X) = ab-(\alpha^2-f(v))$ if

$$X = \begin{pmatrix} a & \alpha+v \\ \alpha-v & b \end{pmatrix}.$$

And for type E, if

$$X = \begin{pmatrix} \xi_1 & x_{12} & x_{13} \\ x_{12}^* & \xi_2 & x_{23} \\ x_{13}^* & x_{23}^* & \xi_3 \end{pmatrix},$$

then

$$N(X) = \xi_1\xi_2\xi_3 - \xi_1 n(x_{23}) - \xi_2 n(x_{13}) - \xi_3 n(x_{12}) + \mathrm{tr}\,((x_{12}x_{23})x_{13}^*).$$

This, and the theorems of the next article, imply that the generic norm is a generalization of the usual determinant. We shall use the terms 'generic norm' and 'determinant' interchangeably, and denote by $|X|$ the determinant of X.

The notions of generic polynormial, generic norm, etc., also apply to $\mathscr{H}(\mathscr{A}_n, J_A)$, since $\mathscr{H}(\mathscr{A}_n, J_A)_\Omega$ is isomorphic to $\mathscr{H}((\mathscr{A}_\Omega)_n, J_I)$.

3.4. Define a unipotent transformation (cf., [5, § 2.1]) $(a)_{ij}$, $i \neq j$, $a \in \mathscr{A}$, of $\mathscr{H}(\mathscr{A}_n, J_I)$ by $(a)_{ij} \cdot X = (I + a^* e_{ji})X(I + ae_{ij})$. Let M be the group generated by $(a)_{ij}$ for all $a \in \mathscr{A}$ and $i \neq j$. If $\mu = (a)_{ij} \ldots (b)_{kl} \in M$, define μ^* by $\mu^* = (b^*)_{lk} \ldots (a^*)_{ji} \in M$. By abuse of language, we shall occasionally write $\mu^* X \mu$ instead of $\mu \cdot X$. Note that M contains $w_{ij} = (1)_{ij}(-1)_{ji}(1)_{ij}$ which permutes the i-th and j-th rows and the i-th and j-th columns. Note also that

$$(\mu \cdot X, Y) = (X, \mu^* \cdot Y),$$

where $(X, Y) = \mathrm{tr}\,(X \circ Y)$ is a non-degenerate symmetric bilinear form on the Jordan algebra $\mathscr{H}(\mathscr{A}_n, J_I)$.

THEOREM 3.4.1: *For any $X \in \mathscr{H}(\mathscr{A}_n, J_I)$, there is a $\mu \in M$ such that $\mu \cdot X$ is of diagonal form.*

PROOF: We may assume that $X \neq 0$. If $x_{ii} \neq 0$ for some i, then by applying w_{i1}, we may assume from the beginning that $x_{11} \neq 0$. If $x_{ii} = 0$

for all i, then $x_{ij} \neq 0$ for some $i \neq j$. Again we may assume that $i = 1$. If $y \in \mathscr{A}$ is such that $(y, x_{1j}) \neq 0$, then the $(1, 1)$-entry of $(y^*)_{j1} \cdot X$ is $(y, x_{1j}) \neq 0$. Therefore, again, we may assume that $x_{11} \neq 0$ to begin with. Then let $\mu = (-x_{11}^{-1}x_{12})_{12} \dots (-x_{11}^{-1}x_{1n})_{1n} \in M$, we have

$$\mu \cdot X = \begin{pmatrix} x_{11} & 0 \dots 0 \\ 0 & \\ \vdots & * \\ 0 & \end{pmatrix},$$

and hence, by induction, we may prove the theorem.

THEOREM 3.4.2: $|\mu \cdot X| = |X|$ *for any* $\mu \in M$.

PROOF: We may check this theorem directly for the generators of M for Jordan algebras of type D or E.

For special Jordan algebras of type A, B or C, we may use the result of [12, Chapt. VI, § 8]. It is known that if $Y \in M(n, \mathscr{A})$, then $X \to Y^*XY$ is a norm similarity of the Jordan algebra $\mathscr{H}(\mathscr{A}_n, J_I)$, and is a norm preserving if and only if $N(Y^*Y) = 1$. If μ is a generator of M, it is trivial to check that $N(\mu^*\mu) = 1$ for Jordan algebras of type A or B. For type C, we use the isomorphisms $\mathscr{A}_\Omega \cong M(2n, \Omega)$ and $\mathscr{H}(\mathscr{A}_n, J_I)_\Omega \cong \mathscr{H}(\Omega_{2n}, J_S)$, then it is a routine work to check our theorem.

Thus, if $\mu \cdot X = A$ is of diagonal form, then $N(X) = |X| = \prod_{i=1}^n a_i$, if $A = \operatorname{diag}(a_1, \dots, a_n)$.

If $Y \in \mathscr{H}(\mathscr{A}_n, J_I)$ and ξ is an $n \times 1$ column vector with entries in $\mathscr{A}$ (for type D, we assume $n < 2$, and type E, $n < 3$), we define $Y[\xi] = \xi^* Y \xi$ except for type E, $n = 2$, in which case we define

$$Y[\xi] = y_1 n(\xi_1) + y_2 n(\xi_2) + (y\xi_2, \xi_1)$$

if

$$Y = \begin{pmatrix} y_1 & y \\ y^* & y_2 \end{pmatrix}, \qquad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

THEOREM 3.4.3: (a) *If*

$$X = \begin{pmatrix} Y & \xi \\ \xi^* & x \end{pmatrix},$$

where Y is $(n-1)\times(n-1)$, ξ is $(n-1)\times 1$, and $x\in\Phi$, then $|X| = x|Y| - \tilde{Y}[\xi]$;
(b) $(\mu\cdot X)^{\sim} = (\mu^*)^{-1}\cdot\tilde{X}$.

PROOF: For Jordan algebras of type D or E, we may check the theorem by direct calculation. Now we treat the cases of types A, B and C.

(a) Suppose that $|Y| \neq 0$, then

$$\begin{pmatrix} I_{n-1} & 0 \\ -\xi^* Y^{-1} & 1\end{pmatrix}\begin{pmatrix} Y & \xi \\ \xi^* & x\end{pmatrix}\begin{pmatrix} I_{n-1} & -Y^{-1}\xi \\ 0 & 1\end{pmatrix} = \begin{pmatrix} Y & 0 \\ 0 & -Y^{-1}[\xi]+x\end{pmatrix},$$

and hence $|X| = |Y|(-Y^{-1}[\xi]+x) = x|Y| - \tilde{Y}[\xi]$.

Since $|X|$ and the entries of $\tilde{Y}$ are polynormial functions of the entries of X, and since $\{X \mid |Y| \neq 0\}$ is a Zariski-open dense subset, we conclude that the polynormial identity $|X| = x|Y| - \tilde{Y}[\xi]$ is true for all X.

(b) Form the $(n+1)\times(n+1)$ matrix

$$X_1 = \begin{pmatrix} X & \zeta \\ \zeta^* & 0\end{pmatrix},$$

where ζ is an arbitrary $(n+1)\times 1$ matrix. Since

$$\begin{pmatrix} \mu & 0 \\ 0 & 1\end{pmatrix}$$

is in the group M for the $(n+1)\times(n+1)$ matrices, we have

$$\begin{aligned}\tilde{X}[\zeta] &= -|X_1| = -\left|\begin{pmatrix} \mu^* & 0 \\ 0 & 1\end{pmatrix} X_1 \begin{pmatrix} \mu & 0 \\ 0 & 1\end{pmatrix}\right| = -\left|\begin{pmatrix} \mu\cdot X & \mu^*\zeta \\ \zeta^*\mu & 0\end{pmatrix}\right| \\ &= (\mu\cdot X)^{\sim}[\mu^*\zeta] = (\mu^*\cdot(\mu\cdot X)^{\sim})[\zeta].\end{aligned}$$

Since this holds true for any ζ, we have $\tilde{X} = \mu^*\cdot(\mu\cdot X)^{\sim}$ and hence $(\mu\cdot X)^{\sim} = (\mu^*)^{-1}\cdot\tilde{X}$.

We note that if $A = \operatorname{diag}(a_1,\ldots,a_n)$, then
$$\tilde{A} = \operatorname{diag}(a_2a_3\ldots a_n, a_1a_3\ldots a_n,\ldots,a_1a_2\ldots a_{n-1}).$$

3.5. Let $X\in\mathscr{J} = \mathscr{H}(\mathscr{A}_n, J_I)$, define $L_X(Y) = X\circ Y$ for $Y\in\mathscr{J}$, then $L_X\in\operatorname{Hom}_\Phi(\mathscr{J},\mathscr{J})$. Define $P_X = 2L_X^2 - L_{X^2}$ and

$$\mathscr{N}(X) = \{Y\in\mathscr{J} \mid P_X(Y) = 0\}.$$

Then $\mathcal{N}(X)$ is a linear subspace of $\mathcal{J}$. From the easily checked fact that $P_{\mu\cdot X} = \mu P_X \mu^*$ for $\mu \in M$, it follows that dim $\mathcal{N}(X) =$ dim $\mathcal{N}(\mu\cdot X)$ for any $\mu \in M$. If $\mu\cdot X$ is of diagonal form having r non-zero diagonal entries, then it is easy to see that

$$\dim \mathcal{N}(X) = \dim \mathcal{N}(\mu\cdot X) = \frac{n(n-1)}{2} n_0 + n - \left(\frac{r(r-1)}{2} n_0 + r\right).$$

Thus r depends only on X and is called the rank of X. Henceforth, we denote the rank of X by $R(X)$. Note that for any $X \in \mathcal{J}$, we can find $\mu \in M$, such that $\mu\cdot X$ is of diagonal form, and then $R(X) =$ number of non-zero diagonal entries of $\mu\cdot X$; it is independent of the way we diagonalize X.

LEMMA 3.5.1: *If there exists an $m \times m$ principal minor of $X \in \mathcal{J}$ with non-zero subdeterminant, then $R(X) \geqq m$.*

PROOF: We may assume that the principal minor consists of the first m rows and m columns:

$$X = \begin{pmatrix} Y & \xi \\ \xi^* & Z \end{pmatrix}, \qquad Y \text{ is } m\times m \text{ and } |Y| \neq 0.$$

Then there exists $\mu \in M(m)$ (= the group M for the $m \times m$ matrices), such that $\mu\cdot Y = A$ is of diagonal form. Then

$$\begin{pmatrix} \mu^* & 0 \\ 0 & I_{n-m} \end{pmatrix}\begin{pmatrix} Y & \xi \\ \xi^* & Z \end{pmatrix}\begin{pmatrix} \mu & 0 \\ 0 & I_{n-m} \end{pmatrix} = \begin{pmatrix} A & \mu^*\xi \\ \xi^*\mu & Z \end{pmatrix},$$

and

$$\begin{pmatrix} I_m & 0 \\ -(\xi^*\mu)A^{-1} & I_{n-m} \end{pmatrix}\begin{pmatrix} A & \mu^*\xi \\ \xi^*\mu & Z \end{pmatrix}\begin{pmatrix} I_m & -A^{-1}(\mu^*\xi) \\ 0 & I_{n-m} \end{pmatrix}$$
$$= \begin{pmatrix} A & 0 \\ 0 & -(\xi^*\mu)A^{-1}(\mu^*\xi)+Z \end{pmatrix},$$

and hence $R(X) \geqq R(A) = m$.

LEMMA 3.5.2: *If*

$$R\begin{pmatrix} 0 & \eta \\ \eta^* & 0 \end{pmatrix} = m > 0,$$

where η is of $(n-1)\times 1$, then

$$R\begin{pmatrix} 0 & \eta \\ \eta^* & x \end{pmatrix} = m$$

for any $x \in \Phi$.

PROOF: Let

$$\eta = \begin{pmatrix} \eta_1 \\ \cdot \\ \cdot \\ \cdot \\ \eta_{n-1} \end{pmatrix}.$$

Since

$$R\begin{pmatrix} 0 & \eta \\ \eta^* & 0 \end{pmatrix} = m > 0,$$

at least one of the η_i's, say η_1, is non-zero. Then there exists $\zeta_1 \in \mathscr{A}$, such that $(\zeta_1, \eta_1) = x$. And then we have

$$\begin{pmatrix} I_{n-1} & 0 \\ \zeta_1^* 0 \ldots 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \eta \\ \eta^* & 0 \end{pmatrix}\begin{pmatrix} I_{n-1} & \begin{matrix} \zeta_1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{matrix} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \eta \\ \eta^* & (\zeta_1, \eta_1) \end{pmatrix} = \begin{pmatrix} 0 & \eta \\ \eta^* & x \end{pmatrix}.$$

Thus

$$R\begin{pmatrix} 0 & \eta \\ \eta^* & x \end{pmatrix} = R\begin{pmatrix} 0 & \eta \\ \eta^* & 0 \end{pmatrix} = m$$

for any $x \in \Phi$.

LEMMA 3.5.3:

$$R\begin{pmatrix} 0 & \eta \\ \eta^* & x \end{pmatrix} = 0$$

if and only if $\eta = 0$, $x = 0$.

PROOF: Assume the rank is 0. If $x \neq 0$, then

$$\begin{pmatrix} I_{n-1} & -x^{-1}\eta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \eta \\ \eta^* & x \end{pmatrix}\begin{pmatrix} I_{n-1} & 0 \\ -x^{-1}\eta^* & 1 \end{pmatrix} = \begin{pmatrix} -x^{-1}\eta\eta^* & 0 \\ 0 & x \end{pmatrix},$$

which implies that the rank is non-zero, a contradiction. If $\eta \neq 0$, then by the proof of Lemma 3.5.2,

$$R\begin{pmatrix} 0 & \eta \\ \eta^* & 0 \end{pmatrix} = R\begin{pmatrix} 0 & \eta \\ \eta^* & 1 \end{pmatrix},$$

and the latter is $\geqq 1$ by the preceding argument. Thus both x and η are zero. The converse is trivial.

LEMMA 3.5.4:

$$R\begin{pmatrix} 0 & \xi \\ \xi^* & 0 \end{pmatrix} \leqq 2,$$

where ξ is of $(n-1)\times 1$.

PROOF: This is obviously true for type D or E. Thus one may assume that $n_0 \leqq 4$, and hence it is sufficient to show that

$$R\begin{pmatrix} 0 & \xi \\ \xi^* & 0 \end{pmatrix} \leqq 2 \qquad \text{for} \quad \xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_4 \end{pmatrix}.$$

By the method employed in the preceding two lemmas, it suffices to show that $R(\xi\xi^*) \leqq 1$. If for some i, say $i = 1$, $\xi_1\xi_1^* \neq 0$, then

$$(I - \sum_{i=2}^{n} (\xi_1\xi_1^*)^{-1}\xi_i\xi_1^* e_{i1})\xi\xi^*(I - \sum_{i=2}^{n} (\xi_1\xi_1^*)^{-1}\xi_1\xi_i^*)e_{1i} = \xi_1\xi_1^* e_{11},$$

and hence $R(\xi\xi^*) = 1$. If all $\xi_i\xi_i^* = 0$, and for some $i \neq j$, say $i = 1$, $j = 2$, $(\xi_1, \xi_2) \neq 0$, then $(\xi_1 + \xi_2)(\xi_1 + \xi_2)^* = (\xi_1, \xi_2) \neq 0$. Thus we may replace ξ_1 by $\xi_1 + \xi_2$ and again we have $R(\xi\xi^*) = 1$. Finally, assume that all $(\xi_i, \xi_j) = 0$. Then the subspace generated by ξ_i's is isotropic, and hence has dimension

$$\leqq \frac{n_0}{2} \leqq 2.$$

Thus we may assume that $\xi_3 = \xi_4 = 0$, and then the lemma is trivial.

COROLLARY: *Notation as in Theorem* 3.4.3, *we have*

$$R(X) \geqq R(Y) \geqq R(X)-2.$$

PROOF: Assume $R(Y) = k$, then there exists $\mu \in M(n-1)$ such that

$$\mu \cdot Y = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

where A is a $k \times k$ non-singular diagonal form. Then

$$\begin{pmatrix} \mu^* & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} Y & \xi \\ \xi^* & x \end{pmatrix}\begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 & \\ 0 & 0 & \mu^*\xi \\ \xi^* & \mu & x \end{pmatrix} = \begin{pmatrix} A & 0 & \eta_1 \\ 0 & 0 & \eta_2 \\ \eta_1^* & \eta_2^* & x \end{pmatrix},$$

where

$$\mu^*\xi = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},$$

η_1 and η_2 being of $k \times 1$ and $(n-1-k) \times 1$, respectively. We have further that

$$\begin{pmatrix} I_k & 0 & 0 \\ 0 & I_{n-k-1} & 0 \\ -\eta_1^* A^{-1} & 0 & 1 \end{pmatrix}\begin{pmatrix} A & 0 & \eta_1 \\ 0 & 0 & \eta_2 \\ \eta_1^* & \eta_2^* & x \end{pmatrix}\begin{pmatrix} I_k & 0 & -A^{-1}\eta_1 \\ 0 & I_{n-k-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & \eta_2 \\ 0 & \eta_2^* & x' \end{pmatrix},$$

with $x' = -A^{-1}[\eta_1]+x$. Thus

$$R(X) = R(A)+R\begin{pmatrix} 0 & \eta_2 \\ \eta_2^* & x' \end{pmatrix},$$

from which the corollary follows.

3.6. Let $\mathscr{J} = \prod_{i=1}^d \mathscr{J}_i$ be a direct product of the Jordan algebras $\mathscr{J}_i$ of § 3.1. Let $M = \prod_{i=1}^d M_i$, with M_i the group of § 3.4 for $\mathscr{J}_i$, then obviously we have

THEOREM 3.6: *Any matrix $X \in \mathcal{J}$ can be transformed by an element of M to the diagonal form (i.e., diagonal form in each factor).*

If $X = \prod_{i=1}^{d} X_i$, define the determinant $|X|$ of X by $|X| = \prod_{i=1}^{d} |X_i|$, and the trace $\mathrm{tr}(X)$ of X by $\mathrm{tr}(X) = \sum_{i=1}^{d} \mathrm{tr}(X_i)$. Then $|X|$ and $\mathrm{tr}(X)$ are the generic norm and generic trace, respectively, of X. The bilinear form on $\mathcal{J}$ induced by the trace is non-degenerate.

4. Parabolic subgroups and Bruhat decompositions

4.1. Let G be as in § 1.1. In particular, the $\mathbb{R}$-root system of G is of type C_n. Let $S = {}_{\mathbb{R}}T$ be a maximal $\mathbb{R}$-split torus of G and σ_j, $1 \leqq j \leqq n$, be the simple roots. Define S_j, $0 \leqq j \leqq n-1$, as the identity component of the group $\{s \in S | \sigma_i(s) = 1, i \neq n-j\}$. Let $Z(S_j)$ be the centralizer of S_j in G. Then $Z(S_j) = L_j \cdot S_j \cdot L'_j$, an almost direct product, where L_j and L'_j are almost simple and their $\mathbb{R}$-root systems are of types C_j and A_{n-j-1}, respectively. Let P_* be the minimal $\mathbb{R}$-parabolic subgroup corresponding to the positive $\mathbb{R}$-roots, and U be its unipotent radical [7, § 1].

Denote the Weyl group of G with respect to S by $W = {}_{\mathbb{R}}W(S, G)$, then, by assumption, W is of type C_n, which consists of all permutations of coordinates with respect to a maximal strongly orthogonal set of positive non-compact roots, together with all possible combinations of sign changes of these coordinates. Let W' be the subgroup of W consisting of all the permutations. Then

$$W = \bigcup_{j=0}^{n} W' \iota_{(j)} W', \tag{1}$$

where $\iota_{(j)} = \iota_{n-j+1}\iota_{n-j+2} \cdots \iota_n$, and ι_i is such that $\mathrm{Ad}\,\iota_i$ is the reflection with respect to the plane orthogonal to the i-th positive non-compact root (i.e., corresponding to the sign change of the i-th coordinate). (We let $\iota_{(0)}$ = the identity of G.)

Since the $\mathbb{R}$-root system is of type C_n, if $S = \{s = \mathrm{diag}(s_1, \ldots, s_n)\}$, we may order the roots in such a way that

i) the roots φ^i_j: $\varphi^i_j(s) = s_i s_j^{-1}$, $i \neq j$, are compact; positive if $i < j$, negative if $i > j$;
ii) the roots φ^{ij}: $\varphi^{ij}(s) = s_i s_j$, are positive non-compact, and
iii) the roots φ_{ij}: $\varphi_{ij}(s) = s_i^{-1} s_j^{-1}$, are negative non-compact.

We then take $\sigma_j = \varphi^j_{j+1}$, for $1 \leqq j \leqq n-1$, and $\sigma_n = \varphi^{nn}$ and form the simple $\mathbb{R}$-root system ${}_{\mathbb{R}}\Delta = \{\sigma_1, \ldots, \sigma_n\}$.

4.2. Let P_j be the maximal $\mathbb{R}$-parabolic subgroup of G generated by $Z(S_j)$ and U. By convention, let $S_n = \{\text{identity}\}$, $P_n = G$.

If $J \subset \{1, 2, \ldots, n\}$, let $\iota_J = \prod_{j \in J} \iota_j$ (if J is empty, then let $\iota_J = \text{identity}$); this is independent of the order of the product. In particular, we have $(k) = \{n-k+1, \ldots, n\}$.

LEMMA 4.2.1: $\iota_{(j)} \in Z(S_j) \subset P_j$.

PROOF: If $S = \{\operatorname{diag}(s_1, \ldots, s_n)\}$, then $S_j = \{\operatorname{diag}(s, \ldots, s, 1, \ldots, 1)\}$ (there are j 1's). Since $\iota_{(j)}$ only changes signs of the last j coordinates, it is obvious that $\iota_{(i)} \in Z(S_j) \subset P_j$.

LEMMA 4.2.2: *If J is non-empty, then $\iota_J \notin P_0$.*

PROOF: P_0 contains compact and positive non-compact root groups only. If $\iota_J \in P_0$, then P_0 would contain negative non-compact root groups, which is absurd.

4.3. Let I be the subgroup of W generated by all ι_k, $1 \leqq k \leqq n$, then I is an abelian normal subgroup of W.

LEMMA 4.3: *Let $w' \in W'$, $\iota \in I$, then*

$$\iota P_{*\mathbb{R}} w' P_{*\mathbb{R}} \subset P_{*\mathbb{R}} \iota w' P_{*\mathbb{R}} \quad \textit{and} \quad P_{*\mathbb{R}} w' P_{*\mathbb{R}} \iota \subset P_{*\mathbb{R}} w' \iota P_{*\mathbb{R}}.$$

PROOF: May assume $\iota = \iota_J$, for some non-empty J. We take $j \in J$ and write $\iota_J = \iota_j \iota_{J'}$, where $J' = J - \{j\}$. Assume that the lemma is true for $\iota_{J'}: \iota_{J'} P_{*\mathbb{R}} w' P_{*\mathbb{R}} \subset P_{*\mathbb{R}} \iota_{J'} w' P_{*\mathbb{R}}$. Since ι_j is a fundamental reflection with respect to a suitable ordering of W, and since, for any $w \in W$, if we let $l(w)$ denote the least number of fundamental reflections (with respect to this ordering) such that the product of them is w, then $l(\iota_j \iota_{J'} w') > l(\iota_{J'} w')$ for $w' \in W'$, therefore we have [8]

$$\iota_J P_{*\mathbb{R}} w' P_{*\mathbb{R}} = \iota_j \iota_{J'} P_{*\mathbb{R}} w' P_{*\mathbb{R}} \subset \iota_j P_{*\mathbb{R}} \iota_{J'} w' P_{*\mathbb{R}} \subset P_{*\mathbb{R}} \iota_j \iota_{J'} w' P_{*\mathbb{R}}$$
$$= P_{*\mathbb{R}} \iota_J w' P_{*\mathbb{R}}.$$

Thus, by induction on the size of J, we have our lemma. (Second formula can be proved in a similar way.)

4.4. Let us resume the assumption of § 1.2. By our assumption, both the simple $\mathbb{R}$- and $\mathbb{K}$-root systems ${}_{\mathbb{R}}\Delta'$ and ${}_{\mathbb{K}}\Delta'$ of G' are of type C. Let $r: {}_{\mathbb{R}}\Delta' \to {}_{\mathbb{K}}\Delta'$ be the restriction of the roots. If on ${}_{\mathbb{R}}\Delta' = \{\sigma_1, \ldots, \sigma_n\}$ we use the canonical numbering, and for each $\tau \in {}_{\mathbb{K}}\Delta'$, let $m(\tau)$ be the greatest index i such that $r(\sigma_i) = \tau$, and number the elements $\tau_1, \ldots, \tau_{n'}$ of ${}_{\mathbb{K}}\Delta'$ in such a way that $i < j$ if and only if $m(\tau_i) < m(\tau_j)$, and then write $m(j)$ for $m(\tau_j)$, then [7, Prop. 2.9] the numbering of ${}_{\mathbb{K}}\Delta'$ is the canonical one, and each $\tau \in {}_{\mathbb{K}}\Delta'$ is the restriction of one and only one simple $\mathbb{R}$-root.

Let ${}_{\mathbb{R}}T' = \{\operatorname{diag}(t_1, \ldots, t_n)\}$, then, by the numbering of the root systems and the above remarks, it is easy to see that

$$ {}_{\mathbb{K}}T' = \{\operatorname{diag}(t_1, \ldots, t_n) | t_i = t_j \quad \text{if} \quad m(k-1) < i, j \leqq m(k)\} $$

(by assumption $m(0) = 0$).

Thus the k-th maximal $\mathbb{K}$-parabolic subgroup ${}_{\mathbb{K}}P'_k$ of G' with respect to $({}_{\mathbb{K}}T', {}_{\mathbb{K}}\Delta')$ is just the $m(k)$-th maximal $\mathbb{R}$-parabolic subgroup ${}_{\mathbb{R}}P'_{m(k)}$ of G' with respect to $({}_{\mathbb{R}}T', {}_{\mathbb{R}}\Delta')$, and hence the k-th maximal $\mathbb{Q}$-parabolic subgroup ${}_{\mathbb{Q}}P_k = P_k$ of G with respect to $({}_{\mathbb{Q}}T, {}_{\mathbb{Q}}\Delta)$ is of the form $P_k = R_{\mathbb{K}/\mathbb{Q}}({}_{\mathbb{K}}P'_k) = R_{\mathbb{K}/\mathbb{Q}}({}_{\mathbb{R}}P'_{m(k)})$.

Since the $\mathbb{Q}$-root system of G is also of type C, we may define $\iota_k, \iota_{(k)}$, etc., as in §§ 4.1–4.3, and note that $\iota_k = \prod_\sigma {}^\sigma\iota'_{m(k)}$, $\iota_{(k)} = \prod_\sigma {}^\sigma\iota'_{(m(k))}$. Then what we did in §§ 4.1–4.3 is true if we replace $\mathbb{R}$ by $\mathbb{Q}$ and n by the $\mathbb{Q}$-rank of G.

Though the $\mathbb{Q}$-rank of G may be strictly less than the $\mathbb{R}$-rank of G', we still use n for the $\mathbb{Q}$-rank of G in the following discussions when no confusion is likely to arise.

4.5. Define $P_{ij} = P_i \cap P_j$, $P_{ijk} = P_i \cap P_j \cap P_k$, $P^*_{j\mathbb{Q}} = P_{0\mathbb{Q}}\iota_{(j)}P_{0\mathbb{Q}} \cap P_j$, $P^*_{ij\mathbb{Q}} = P_{0ij\mathbb{Q}}\iota_{(j)}P_{0ij\mathbb{Q}}$. Note that $P_0 = P$ is the $\mathbb{Q}$-maximal parabolic subgroup of § 1.2.

LEMMA 4.5:

$$ G_{\mathbb{Q}} = \bigcup_{j=0}^{n} P_{\mathbb{Q}}\iota_{(j)}P_{\mathbb{Q}}, $$

and the union is disjoint.

PROOF: By (1), and the Bruhat decompositions $G_{\mathbb{Q}} = P_{*\mathbb{Q}}WP_{*\mathbb{Q}}$, $P_{\mathbb{Q}} = P_{*\mathbb{Q}}W'P_{*\mathbb{Q}}$ [8, § 5.15], it follows that

$$ G_{\mathbb{Q}} = \bigcup_{j=0}^{n} P_{\mathbb{Q}}\iota_{(j)}P_{\mathbb{Q}}. $$

Now, if $P_{\mathbb{Q}}\iota_{(i)}P_{\mathbb{Q}} \cap P_{\mathbb{Q}}\iota_{(j)}P_{\mathbb{Q}} \neq \Phi$, then $\rho\iota_{(i)} = \iota_{(j)}\rho'$, for some $\rho, \rho' \in P_{\mathbb{Q}}$. Then, by Lemma 4.3,

$$ \iota_{(j)}\rho' \in \iota_{(j)}P_{\mathbb{Q}} = \iota_{(j)}P_{*\mathbb{Q}}W'P_{*\mathbb{Q}} \subset \bigcup_{w' \in W'} P_{*\mathbb{Q}}\iota_{(j)}w'P_{*\mathbb{Q}}; $$

but also

$$\rho\iota_{(i)} \in \bigcup_{w' \in W'} P_{*\mathbb{Q}} w' \iota_{(j)} P_{*\mathbb{Q}}.$$

Therefore,

$$P_{*\mathbb{Q}} \iota_{(j)} w'_1 P_{*\mathbb{Q}} \cap P_{*\mathbb{Q}} w'_2 \iota_{(i)} P_{*\mathbb{Q}} \neq \Phi$$

for some $w'_1, w'_2 \in W'$, which implies $\iota_{(j)} w'_1 = w'_2 \iota_{(i)}$ and hence $i = j$.

LEMMA 4.6:

$$P_{j\mathbb{Q}} = \bigcup_{i \leqq j} P_{0j\mathbb{Q}} \iota_{(i)} P_{0j\mathbb{Q}},$$

and the union is disjoint.

PROOF: Let W_j be the Weyl group of P_j with respect to the maximal $\mathbb{Q}$-split torus S, then W_j is generated by permutations of the first $n-j$ coordinates (i.e., of type A_{n-j-1}) and permutations and sign changes of the last j coordinates (i.e., of type C_j).

Since

$$W_j = \bigcup_{i \leqq j} W'_j \iota_{(i)} W'_j,$$

where W'_j denotes the group generated by the permutations of the first $n-j$ coordinates and the permutations of the last j coordinates, we have

$$\begin{aligned} P_{j\mathbb{Q}} &= P_{*\mathbb{Q}} W_j P_{*\mathbb{Q}} \qquad \text{(Bruhat decomposition)} \\ &= \bigcup_{i \leqq j} P_{*\mathbb{Q}} W'_j \iota_{(i)} W'_j P_{*\mathbb{Q}} = \bigcup_{i \leqq j} P_{0j\mathbb{Q}} \iota_{(i)} P_{0j\mathbb{Q}}. \end{aligned}$$

Since each $P_{0j\mathbb{Q}} \iota_{(i)} P_{0j\mathbb{Q}}$ is contained in $P_{\mathbb{Q}} \iota_{(i)} P_{\mathbb{Q}}$, the union is disjoint. As an easy consequence, we have

COROLLARY 4.7: $P^*_{j\mathbb{Q}} = P_{0j\mathbb{Q}} \iota_{(j)} P_{0j\mathbb{Q}}$ *and* $P^*_{j\mathbb{Q}} P_{0j\mathbb{Q}} = P^*_{j\mathbb{Q}}$.

LEMMA 4.8: *If* $i > j$, *then*

$$P_i \cap (P_{\mathbb{Q}} \iota_{(j)} P_{\mathbb{Q}}) = P_{0i\mathbb{Q}} \iota_{(j)} P_{0i\mathbb{Q}} = P_{0i\mathbb{Q}} P^*_{ij\mathbb{Q}} P_{0i\mathbb{Q}}.$$

PROOF: From

$$P_{i\mathbb{Q}} = \bigcup_{k \leqq i} P_{0i\mathbb{Q}} \iota_{(k)} P_{0i\mathbb{Q}},$$

we have

$$P_i \cap (P_{\mathbb{Q}} \iota_{(j)} P_{\mathbb{Q}}) = P_{0i\mathbb{Q}} \iota_{(j)} P_{0i\mathbb{Q}};$$

on the other hand,

$$P_{0i\mathbb{Q}} P^*_{ij\mathbb{Q}} P_{0i\mathbb{Q}} = P_{0i\mathbb{Q}} (P_{0ij\mathbb{Q}} \iota_{(j)} P_{0ij\mathbb{Q}}) P_{0i\mathbb{Q}} = P_{0i\mathbb{Q}} \iota_{(j)} P_{0i\mathbb{Q}}.$$

4.9. Let Φ be any field containing $\mathbb{Q}$, U be the unipotent radical of P corresponding to all positive non-compact roots. If $J \subset \{1, 2, \ldots, n\}$, denote U_J the subgroup of U generated by the subgroups corresponding to the positive non-compact roots $\varphi^{ij}, i, j \in J$. If α is any positive non-compact root not among the above, and if U_α is the subgroup of U_Φ corresponding to α, then $\iota_J^{-1} U_\alpha \iota_J \subset P$. Therefore, if $\rho \in U_\Phi$, we have $\rho \iota_J = \rho' \iota_J \rho''$, with $\rho' \in U_J$ and $\rho'' \in P$.

5. Group actions and functional determinant

We shall study the group actions on the tube domains and calculate the corresponding functional determinants, which will be very useful in the studying of Eisenstein series.

5.1. In this article we retain the assumption of § 1.1. Let G_h be the group of holomorphic automorphisms of $\mathfrak{T}$. Then we have $[G_h: G^0_{\mathbb{R}}] = 1$ or 2. In every case where $[G_h: G^0_{\mathbb{R}}] = 2$, G_h contains an element τ of order 2, not in $G^0_{\mathbb{R}}$, such that τ operates on $\mathfrak{T}$ by a linear transformation of $\mathbb{C}^{\mathbb{M}}$ [4, § 2].

It is known that the group $G^0_{\mathbb{R}}$ is generated by $\iota = \iota_{(n)}$, which acts on $\mathfrak{T}$ by $Z \cdot \iota = -Z^{-1}$ for $Z \in \mathfrak{T}$, and the group $U_{\mathbb{R}}$ of real translations $t_X: Z \to Z + X, X \in \mathscr{J}_{\mathbb{R}} = \mathbb{R}^{\mathbb{M}}$, and that G_h is generated by ι and $\mathrm{Aut}(\mathfrak{K}) \cdot U_{\mathbb{R}}$, where $\mathrm{Aut}(\mathfrak{K})$ is the group of all linear automorphisms of the cone $\mathfrak{K}$. In the notation of § 1.1, we also have

$$P \cap G^0_{\mathbb{R}} = N(F_0) = \mathrm{Aut}(\mathfrak{K})^0 \cdot U_{\mathbb{R}}.$$

5.2. Let $S = \{s = (s_1, \ldots, s_n)\}$ be an n-dimensional torus. Let $\mathscr{J}$ be a Jordan algebra of § 2.1, and let $\mathscr{J}_{\mathbb{C}}$ be its complexification. Let S act on $\mathscr{J}_{\mathbb{C}}$ from the left by $s \cdot Z = s \cdot (z_{ij}) = (s_i s_j z_{ij})$, with $s \in S$, $Z = (z_{ij}) \in \mathscr{J}_{\mathbb{C}}$. Let M be the group of unipotent transformations of $\mathscr{J}_{\mathbb{C}}$ constructed in § 3.4. Now we let $S_{\mathbb{R}}$ act on the tube domain $\mathfrak{T} \subset \mathscr{J}_{\mathbb{C}}$ from the right by $Z \cdot s = s^{-1} \cdot Z$, $Z \in \mathfrak{T}$, $s \in S_{\mathbb{R}}$, and let $M_{\mathbb{R}}$ act on the tube domain $\mathfrak{T}$ from the right by $Z \cdot \mu = \mu^* \cdot Z$, $Z \in \mathfrak{T}$, $\mu \in M_{\mathbb{R}}$. Note that

$$(s \cdot X, Y) = (X, s \cdot Y) \quad \text{and} \quad (\mu \cdot X, Y) = (X, \mu^* \cdot Y)$$

for any $s \in S_{\mathbb{R}}$, $\mu \in M_{\mathbb{R}}$, $X, Y \in \mathscr{J}_{\mathbb{R}}$, and that $\mathfrak{K}$ is self-adjoint with respect to (,), we know that $S_{\mathbb{R}}$ and $M_{\mathbb{R}}$ actually map $\mathfrak{T}$ onto itself. Therefore $G^0_{\mathbb{R}}$ (as a transformation group on $\mathfrak{T}$) contains $S_{\mathbb{R}}$ and $M_{\mathbb{R}}$ (as transformation groups on $\mathfrak{T}$) as subgroups. And, moreover, $S_{\mathbb{R}}$ is a maximal $\mathbb{R}$-split torus of $G^0_{\mathbb{R}}$. Since $s(y)_{ij}s^{-1} = (s_i s_j^{-1} y)_{ij}$, $(y)_{ij}$ belongs to the root subgroup of G corresponding to the compact root φ^i_j. $G^0_{\mathbb{R}}$ also contains $U_{\mathbb{R}}$; and since $st_{xe_{ij}}s^{-1} = t_{(s_i s_j x)e_{ij}}$, $t_{xe_{ij}}$ belongs to the root subgroup of G corresponding to the positive non-compact root φ^{ij}.

LEMMA 5.2: *If* $\mu \in M_{\mathbb{R}}$, $X \in \mathscr{J}_{\mathbb{R}}$, *then* $\mu t_X \mu^{-1} = t_{(\mu^*)^{-1} \cdot X}$.

PROOF: This follows by a direct calculation.

In general, when a tube domain is not necessarily irreducible, we may define the group actions componentwise.

5.3. In this article we consider G in general setting, and retain the notations of § 1.2 and § 2.10.

For each $\mathscr{C}^{n_0}_n$ of § 2.1, we define $\mathbb{N} = \mathbb{N}(n, n_0) = (n-1)n_0 + 2$. In the rest of this paper we fix the notation ι for the element $\iota_{(n)}$.

THEOREM 5.3: $j(Z, \iota) = \pm |Z|^{-\mathbb{N}}$, $Z \in \mathfrak{T}$.

PROOF: Note that

$$j(Z, \iota) = \prod_\sigma j({}^\sigma Z', {}^\sigma \iota) \quad \text{and} \quad |Z|^{-\mathbb{N}} = \prod_\sigma |{}^\sigma Z'|^{-\mathbb{N}},$$

so it suffices to prove the theorem for the irreducible tube domain corresponding to the simple compact real Jordan algebra $\mathscr{C}^{n_0}_n$.

Now let $\mathfrak{T}$ be irreducible. If $j(Z, \iota) = |Z|^{-\mathbb{N}}$ for $Z = iY \in i\mathfrak{K}$, then, since $j(Z, \iota)$ and $|Z|^{-\mathbb{N}}$ are analytic functions on $\mathfrak{T}$, $j(Z, \iota) = |Z|^{-\mathbb{N}}$ will be true for all $Z \in \mathfrak{T}$.

For any given $Z = iY$, there is $\mu \in M_{\mathbb{R}}$ such that $Z \cdot \mu$ is of diagonal form. Since $\iota = \mu\iota(\mu^*)^{-1}$ (Theorem 3.4.3(b)), we have

$$j(Z, \iota) = j(Z, \mu\iota(\mu^*)^{-1}) = j(Z \cdot \mu, \iota),$$

and since $|Z \cdot \mu| = |Z|$, thus if we can prove that $j(Z_0, \iota) = |Z_0|^{-\mathbb{N}}$ for all diagonal elements $Z_0 \in \mathfrak{T}$, then $j(Z, \iota) = |Z|^{-\mathbb{N}}$ is true for all $Z \in i\mathfrak{K}$, and hence for all $Z \in \mathfrak{T}$.

Now let Z be a variable element and Z_0 be a fixed diagonal element in $\mathfrak{T}$. Write $Z = (z_{ij})$ and

$$z_{ij} = \sum_{k=1}^{n_0} z_{ij}^{(k)} c_k$$

for $i \neq j$, where $\{c_1, \ldots, c_{n_0}\}$ is a basis of $\mathscr{C}_{\mathbb{C}}$ over $\mathbb{C}$, and $z_{ij}^{(k)} \in \mathbb{C}$. Also write $e_{ji}^{(k)} + e_{ij}^{(k)} = c_k^* e_{ji} + c_k e_{ij}$. From $Z\tilde{Z} = |Z|I$, we have

$$\left(\frac{\partial Z}{\partial z_{ij}^{(k)}}\right)\tilde{Z} + Z\left(\frac{\partial \tilde{Z}}{\partial z_{ij}^{(k)}}\right) = \frac{\partial |Z|}{\partial z_{ij}^{(k)}} I_n. \tag{1}$$

Using (1), we then have

$$\begin{aligned}\frac{\partial(Z^{-1})}{\partial z_{ij}^{(k)}} = \frac{\partial(|Z|^{-1}\tilde{Z})}{\partial z_{ij}^{(k)}} &= -|Z|^{-2}\frac{\partial |Z|}{\partial z_{ij}^{(k)}}\tilde{Z} + |Z|^{-1}\frac{\partial \tilde{Z}}{\partial z_{ij}^{(k)}} \\ &= -|Z|^{-2}\frac{\partial |Z|}{\partial z_{ij}^{(k)}}\tilde{Z} + |Z|^{-1}Z^{-1}\left(\frac{\partial |Z|}{\partial z_{ij}^{(k)}} I_n - \frac{\partial Z}{\partial z_{ij}^{(k)}}\tilde{Z}\right) \\ &= -|Z|^{-1}Z^{-1}\frac{\partial Z}{\partial z_{ij}^{(k)}}\tilde{Z} = -Z^{-1}(e_{ji}^{(k)} + e_{ij}^{(k)})Z^{-1}.\end{aligned}$$

Thus,

$$\left.\frac{\partial Z^{-1}}{\partial z_{ij}^{(k)}}\right|_{Z_0} = -Z_0^{-1}(e_{ji}^{(k)} + e_{ij}^{(k)})Z_0^{-1} = -z_i^{-1}z_j^{-1}(e_{ji}^{(k)} + e_{ij}^{(k)}),$$

where $Z_0 = \operatorname{diag}(z_1, \ldots, z_n)$.

By counting the dimensions of the entries, it follows easily that $j(Z_0, \iota) = \pm |Z_0|^{-\mathbb{N}}$. This completes the proof.

Observe that $j(Z, t_X) = 1$ for any $t_X \in U_{\mathbb{R}}$, and also that $j(Z, \tau) = \pm 1$, thus essentially we have calculated the functional determinant for any $g \in G_h$.

6. Boundary components and functional determinant

In this section we describe the boundary components of the tube domain $\mathfrak{T}$ and relate the functional determinant on $\mathfrak{T}$ with those on the boundary components. The result is one of the key points that we are able to reduce the treatment of the Fourier coefficients to those for the biggest cell of the Bruhat decomposition.

6.1. (cf., [5, § 6]). Let $\mathscr{C}$ be the real algebra of § 2.1. Let $M(j, \mathscr{C}_{\mathbb{C}})$ denote the set of all $j \times j$ matrices with entries in $\mathscr{C}_{\mathbb{C}}$.

For $0 \leqq j \leqq n$, let

$$\mathfrak{T}_j = \left\{ Z_{(j)} \in M(j, \mathscr{C}_{\mathbb{C}}) \middle| \begin{pmatrix} iI_{n-j} & 0 \\ 0 & Z_{(j)} \end{pmatrix} \in \mathfrak{T} \right\};$$

in particular, $\mathfrak{T}_n = \mathfrak{T}$ and $\mathfrak{T}_0 = \{0\}$.

If $Z_{(j)} \in \mathfrak{T}_j$, let

$$Z^{\lambda}_{(j)} = \begin{pmatrix} i\lambda I_{n-j} & 0 \\ 0 & Z_{(j)} \end{pmatrix}, \ \lambda > 0.$$

Define $\mathfrak{T}^{\infty}_j$ (resp. $\mathfrak{T}^0_j$) as the set of the limits of sequences of the form $\{Z^{\lambda_n}_{(j)}\}$ for $Z_{(j)} \in \mathfrak{T}_j$ and $\lambda_n \to \infty$ (resp. $\lambda_n \to 0$). We remark that we embed $\mathfrak{T}$ in its compact dual $\mathfrak{T}^*$ and then take limits in $\mathfrak{T}^*$.

From what we know about the actions of the groups $S_{\mathbb{R}}$, $M_{\mathbb{R}}$, $U_{\mathbb{R}}$ and the element $\imath$ on the tube domain $\mathfrak{T}$, we conclude the following

(1) $\mathfrak{T}^{\infty}_j \cdot \imath = \mathfrak{T}^0_j$ and $\mathfrak{T}^0_j \cdot \imath = \mathfrak{T}^{\infty}_j$;

(2) $Z(S_j)$ stabilizes both $\mathfrak{T}^{\infty}_j$ and $\mathfrak{T}^0_j$, $0 \leqq j \leqq n-1$;

(3) the normalizer $N(\mathfrak{T}^{\infty}_j)_{\mathbb{R}}$ of $\mathfrak{T}^{\infty}_j$ is $P_{j\mathbb{R}}$;

(4) the group $L_{j\mathbb{R}}$, $1 \leqq j \leqq n-1$, acts non-trivially on $\mathfrak{T}^{\infty}_j$ and on $\mathfrak{T}^0_j$; and

(5) $\mathfrak{T}^{\infty}_j$ and $\mathfrak{T}^0_j$ are proper boundary components of $\mathfrak{T}$, $0 \leqq j \leqq n-1$.

6.2. Let's retain the assumption of § 1.2. Then by § 4.4, it is clear that the rational boundary components $\mathfrak{T}^{\infty}_k$ and $\mathfrak{T}^0_k$ associated with P_k are direct products of boundary components of ${}^{\sigma}\mathfrak{T}'$, i.e.,

$$\mathfrak{T}^{\infty}_k = \prod_{\sigma} {}^{\sigma}\mathfrak{T}'^{\infty}_{m(k)}, \qquad \mathfrak{T}^0_k = \prod_{\sigma} {}^{\sigma}\mathfrak{T}'^{0}_{m(k)}.$$

6.3. Let $J \subset \{1, 2, \ldots, n\}$, then denote by Z_J the submatrix of $Z \in \mathfrak{T}$ consisting of (i, j) entries, $i, j \in J$, of each factor of Z.

THEOREM 6.3: *$j(Z, \imath_J)^2 = \pm |Z_J|^{-2\mathbb{N}}$, where $\mathbb{N}$ is as in* § 5.3.

PROOF: By suitable permutations of rows and columns, we may assume that $J = \{k\} = \{n-k+1, \ldots, n\}$. We may also assume that $\mathfrak{T}$ is irreducible. By Lemma 4.2.1, $\imath_{(k)} \in P_k \cap G^0_{\mathbb{R}}$ which implies [7, § 7.9] that $j(Z, \imath_{(k)})$ is constant along the fibers of the canonical projection π_{kn} from $\mathfrak{T}^{\infty}_n$ onto $\mathfrak{T}^{\infty}_k$. Thus we may assume that

$$Z = \begin{pmatrix} iI_{n-k} & 0 \\ 0 & Z_{(k)} \end{pmatrix}.$$

If the theorem is true for all

$$Z = \begin{pmatrix} iI_{n-k} & 0 \\ 0 & Z_{(k)} \end{pmatrix} \in i\mathfrak{R},$$

then it is also true for all

$$Z = \begin{pmatrix} iI_{n-k} & 0 \\ 0 & Z_{(k)} \end{pmatrix}.$$

Now, let

$$Z = \begin{pmatrix} iI_{n-k} & 0 \\ 0 & iY \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} I_{n-k} & 0 \\ 0 & Y \end{pmatrix} \in \mathfrak{R},$$

then there exists $\mu \in M_{(k)}$ (= the group generated by all $(y)_{ij}$, $y \in \mathscr{C}$, $i, j \in (k)$) such that $Z \cdot \mu$ is of diagonal form. Let $(k)' = \{1, 2, \ldots, n-k\}$, then, from $\iota = \mu\iota(\mu^*)^{-1}$, we have

$$\iota_{(k)'}\iota_{(k)} = \mu\iota_{(k)'}\iota_{(k)}(\mu^*)^{-1} = \iota_{(k)'}\mu\iota_{(k)}(\mu^*)^{-1},$$

from which we have $\iota_{(k)} = \mu\iota_{(k)}(\mu^*)^{-1}$ and

$$j(Z, \iota_{(k)}) = j(Z, \mu\iota_{(k)}(\mu^*)^{-1}) = j(Z \cdot \mu, \iota_{(k)}).$$

Therefore, from the fact that

$$|Z_{(k)}|^{-2\mathbb{N}} = \pm|Z|^{-2\mathbb{N}} = \pm|Z \cdot \mu|^{-2\mathbb{N}} = \pm|(Z \cdot \mu)_{(k)}|^{-2\mathbb{N}},$$

we may assume that Z is of diagonal form. Write

$$Z = \operatorname{diag}(z_1, \ldots, z_n) = \operatorname{diag}(z, z_n),$$

with $z = \operatorname{diag}(z_1, \ldots, z_{n-1})$, and write $z^{-1} = \operatorname{diag}(z_1^{-1}, \ldots, z_{n-1}^{-1})$. Then, by the fact $\iota_n = t_{e_{nn}}\iota^{-1}t_{e_{nn}}\iota t_{e_{nn}}$ and Theorem 5.3, we have

$$\begin{aligned}
&j(Z, \iota_n)^2 = j(Z, t_{e_{nn}}\iota^{-1}t_{e_{nn}}\iota t_{e_{nn}})^2 = j((z, z_n+1), \iota^{-1}t_{e_{nn}}\iota t_{e_{nn}})^2 \\
&= |z|^{-2\mathbb{N}}|z_n+1|^{-2\mathbb{N}}j((-z^{-1}, -(z_n+1)^{-1}), t_{e_{nn}}\iota t_{e_{nn}})^2 \\
&\quad = |z|^{-2\mathbb{N}}|z_n+1|^{-2\mathbb{N}}j((-z^{-1}, 1-(z_n+1)^{-1}), \iota t_{e_{nn}})^2 \\
&\quad = |z_n+1|^{-2\mathbb{N}}|1-(z_n+1)^{-1}|^{-2\mathbb{N}}j((z, -(1-(z_n+1)^{-1})^{-1}), t_{e_{nn}})^2 \\
&\quad = |z_n|^{-2\mathbb{N}}.
\end{aligned}$$

Since the action of ι_n has no effect on the coordinates of Z except z_n, and this is also true for any i instead of n, and since $\iota_{(k)} = \iota_{n-k+1} \cdots \iota_n$, it is easy to see that $j(Z, \iota_{(k)})^2 = \pm|Z_{(k)}|^{-2\mathbb{N}}$ for diagonal elements Z of $\mathfrak{T}$. This completes our proof.

6.4. Define $N_i = P_i \cap G^0_{\mathbb{R}}$, $N_{ij} = P_{ij} \cap G^0_{\mathbb{R}}$. We know that if $g \in N_{k\mathbb{Q}}$, then [7, § 7.9] $j(Z, g)$ is constant along the fibers of $\pi_{kn}: \mathfrak{T}^\infty_n \to \mathfrak{T}^\infty_k$; thus we may regard $j_n(Z_{(k)}, g) \equiv j(Z, g)$ as a function on the tube domain $\mathfrak{T}_k$ of degree k; where n refers to the degree of the original tube domain $\mathfrak{T} = \mathfrak{T}_n$. On the other hand, if we identify $\mathfrak{T}^\infty_k$ with $\mathfrak{T}_k$, then g acts on $\mathfrak{T}_k$ as a holomorphic automorphism; thus we may consider the functional determinant of g at $Z_{(k)} \in \mathfrak{T}_k$, which we denote by $j_k(Z_{(k)}, g)$. We are going to study the relation between $j_n(Z_{(k)}, g)$ and $j_k(Z_{(k)}, g)$ for

$$g \in L^*_{k\mathbb{Q}} = L_{k\mathbb{Q}} \cap L^0_{k\mathbb{R}} \subset N_{k\mathbb{Q}}.$$

We remark that L_k and $P^\#_k = L_k \cap P$ are the groups for $\mathfrak{T}_k$ as G and P for $\mathfrak{T}_n$.

The number $\mathbb{N}$ depends on n; thus write $\mathbb{N}(n)$ for $\mathbb{N}$ whenever it is necessary.

For $\iota_{(k)} \in L^*_{k\mathbb{Q}}$, we have seen that $j_n(Z_{(k)}, \iota_{(k)})^2 = \pm|Z_{(k)}|^{-2\mathbb{N}(n)}$; on the other hand, since $\iota_{(k)}$ plays the same role for L_k and $\mathfrak{T}_k$ as ι does for G and $\mathfrak{T}$, thus we have $j_k(Z_{(k)}, \iota_{(k)})^2 = |Z_{(k)}|^{-2\mathbb{N}(k)}$ (Theorem 5.3). Therefore $j_n(Z_{(k)}, \iota_{(k)})^{2/\mathbb{N}(n)} = \pm j_k(Z_{(k)}, \iota_{(k)})^{2/\mathbb{N}(k)}$.

THEOREM 6.4: $j_n(Z_{(k)}, g)^{2/\mathbb{N}(n)} = \pm j_k(Z_{(k)}, g)^{2/\mathbb{N}(k)}$, *for all* $g \in L^*_{k\mathbb{Q}}$.

PROOF: By Lemma 4.5, we have

$$L_{k\mathbb{Q}} = \bigcup_{j \leqq k} P^\#_{k\mathbb{Q}} \iota_{(j)} P^\#_{k\mathbb{Q}}.$$

By Theorems 5.3 and 6.3, the theorem is true for all $\iota_{(j)}, 0 \leqq j \leqq k$; therefore we have only to prove the case when $g \in P^\#_{k\mathbb{Q}}$.

Let $S_{(k)} = \{s \in S | s = (1, \ldots, 1, s_{n-k+1}, \ldots, s_n)\}$ then $P^\#_k$ is generated by $Z_k(S_{(k)})$, the centralizer of $S_{(k)}$ in L_k, and elements which act on $\mathfrak{T}_k$ by real transformations. Since both $j_n(*, \rho)$ and $j_k(*, \rho)$ are constant functions on $\mathfrak{T}_k$ for fixed $\rho \in P^\#_{k\mathbb{R}}$, they are real continuous characters on $P^\#_{k\mathbb{R}}$. Since $Z_k(S_{(k)})$ is a direct product of $S_{(k)}$ and a compact subgroup of L_k and since a real continuous character can take only ± 1 on a compact group, we only have to check our theorem for elements of $S_{(k)}$, which can be completed by a routine work.

Part II. EISENSTEIN SERIES AND THEIR FOURIER COEFFICIENTS

7. Automorphic forms and Eisenstein series

We shall define certain Eisenstein series on a tube domain, then apply the knowledge from the preceding three sections to obtain the first step of the reduction of the treatment of the Fourier coefficients.

7.1. Let G be in general. (cf. [6, § 1].) For the sake of technical simplification, we assume G to be simply-connected (in any case, we could replace G by its simply-connected covering). Let P be the maximal parabolic subgroup of § 1.2, and let Γ be an arithmetic subgroup of G contained in $G^0_{\mathbb{R}}$ subject to the following restriction: For each prime p in $\mathbb{Q}$ and a prime ideal $\mathfrak{p}$ in $\mathbb{K}$ dividing p, we can choose a special maximal compact subgroup $\mathscr{K}_{\mathfrak{p}}$ of $G'_{\mathfrak{p}} = G'_{\mathbb{K}_{\mathfrak{p}}}$, such that $\mathscr{K}_p = \prod_{\mathfrak{p}|p} \mathscr{K}_{\mathfrak{p}}$ is a special maximal compact subgroup of $G_p = G_{\mathbb{Q}_p}$, and that $\Gamma \subset \Gamma_{Z_p} \subset \mathscr{K}_p$, where Γ_{Z_p} is the closure of Γ in G_p, and that we have $G'_{\mathfrak{p}} = \mathscr{K}_{\mathfrak{p}} P'_{\mathfrak{p}}$ and $G_p = \mathscr{K}_p P_p$. A maximal arithmetic subgroup with this property is called special. Thus Γ is an arithmetic subgroup of a special arithmetic subgroup, an assumption we shall assume henceforth.

It is known that $G_{\mathbb{Q}}$ is the disjoint union of a finite number of double cosets $\Gamma a P_{\mathbb{Q}}$, $a \in A \subset G_{\mathbb{Q}} \cap G^0_{\mathbb{R}}$. Let $l = l_1/\mathbb{N}$ be a large, positive fraction with $\mathbb{N} = (n-1)n_0 + 2$ [§ 5.3] and l_1 a multiple of $4l_0$ for a certain fixed l_0 (see the remark of § 7.3). Then, by § 5.3 and [4, § 5], the series

$$E_{l,a}(Z) = \sum_{\gamma \in \Gamma/\Gamma_{0,a}} j(Z, \gamma a)^l, \quad \text{where} \quad \Gamma_{0,a} = \Gamma \cap aPa^{-1},$$

converges normally on $\mathfrak{T}$ and represents there an automorphic form with respect to Γ.

Our main interest, in this paper, is to pick, in a suitable way, a number $c(a)$ for each a, and then show that the Fourier coefficients of the Eisenstein series $E_{l,a} = \sum_{a \in A} E_{l,a} c(a)^l$ are all rational numbers for all the tube domains considered.

7.2. The mapping $X \to t_X$ from the Jordan algebra $\mathscr{J}$ onto the abelian group U (cf. § 5.1) gives an isomorphism of $\mathscr{J}$ onto U. Hence $\mathscr{J}$ can be viewed as the Lie algebra of U.

If $g \in G_{\mathbb{Q}} \subset G_p = \mathscr{K}_p P_p$, we write $g = \gamma \rho$, with $\gamma \in \mathscr{K}_p$, $\rho \in P_p$, then define $c_p(g) = |\det(\mathrm{Ad}_{\mathscr{J}}\, \rho)|_p^{-1}$ (see § 0.4 for the notation $|\;|_p$), and $c(g) = \prod_p c_p(g)$.

LEMMA 7.2: *$c_p(g)$ is independent of the representation of g as $\gamma\rho$.*

PROOF: Assume $g = \gamma\rho = \gamma_1\rho_1$, then $\rho\rho_1^{-1} = \gamma^{-1}\gamma_1 \in \mathscr{K}_p \cap P_p$. Since the continuous character $|\det(\mathrm{Ad}\,\mathscr{J}^*)|_p$ is trivial on the compact subgroup $\mathscr{K}_p \cap P_p$, we have $|\det(\mathrm{Ad}_{\mathscr{J}}\,\rho)|_p = |\det(\mathrm{Ad}_{\mathscr{J}}\,\rho_1)|_p$, from which the lemma follows.

LEMMA 7.3:

$$E_l(Z) = \sum_{a\in A}\ \sum_{\gamma\in\Gamma/\Gamma_{0,a}} j(Z,\gamma a)^l c(a)^l = \sum_{g\in G_{\mathbb{Q}}/P_{\mathbb{Q}}} j(Z,g)^l c(g)^l.$$

PROOF: First we contend that there is a $1-1$ onto correspondence φ between $G_{\mathbb{Q}}/P_{\mathbb{Q}}$ and $\bigcup_{a\in A}\Gamma/\Gamma_{0,a}$. We know that $G_{\mathbb{Q}} = \bigcup_{a\in A}\Gamma a P_{\mathbb{Q}}$ is a disjoint union. If $g = \gamma a\rho \in G_{\mathbb{Q}}$, then define $\varphi(gP_{\mathbb{Q}}) = \gamma\Gamma_{0,a}$. If $g = \gamma a\rho = \gamma_1 a\rho_1$, then $\gamma_1^{-1}\gamma = a(\rho_1\rho^{-1})a^{-1} \in \Gamma \cap aP_{\mathbb{Q}}a^{-1} = \Gamma_{0,a}$. Thus φ is well defined. It is also easy to check that φ is $1-1$ and onto.

If $g = \gamma a\rho$, then

$$\begin{aligned} j(Z,g)^l c(g)^l = j(Z,\gamma a\rho)^l c(a\rho)^l &= j(Z,\gamma a)^l j(*,\rho)^l c(a)^l (\det(\mathrm{Ad}_{\mathscr{J}}\,\rho))^l \\ &= j(Z,\gamma a)^l c(a)^l; \end{aligned}$$

here we use the product formula (§ 0.4) and the fact that

$$j(*,\rho) = (\det(\mathrm{Ad}_{\mathscr{J}}\,\rho))^{-1}$$

[7, § 1.9].

REMARK: l_0 of § 7.1 is chosen so that $j(Z,\gamma a)^l c(a)^l$ is independent of the choice of the coset representation of γ in $\Gamma/\Gamma_{0,a}$ [4, § 5].

7.4. Let F be an automorphic form on $\mathfrak{T}$ of weight l with respect to an arithmetic group Γ of § 7.1. Then, by definition, we have

$$\text{(1)} \qquad F(Z\cdot\gamma)j(Z,\gamma)^l = F(Z), \qquad \text{for all} \quad \gamma\in\Gamma;$$

and then

$$\text{(2)} \qquad F(Z\cdot\gamma) = F(Z), \qquad \text{for all} \quad \gamma\in\Gamma_0 = \Gamma\cap P.$$

Therefore,

$$\text{(3)} \qquad F(Z+\mathbb{S}) = F(Z), \qquad \text{for all} \quad \mathbb{S}\in\Lambda' = \Gamma\cap U,$$

and

$$F(Z \cdot \mu) = F(Z), \qquad \text{for all} \quad \mu \in M_{\mathbb{R}} \cap \Gamma. \tag{4}$$

(3) implies that F has a Fourier expansion

$$F(Z) = \sum_{\mathbb{T} \in \Lambda} a(\mathbb{T}) \varepsilon((\mathbb{T}, Z)), \tag{5}$$

where $\varepsilon(\ ) = e^{2\pi i(\ )}$, and $(\mathbb{T}, Z) = \operatorname{tr}(\mathbb{T} \circ Z)$ is the generic trace of the Jordan product $\mathbb{T} \circ Z$ of $\mathbb{T}$ and Z, and Λ is the dual lattice of Λ' with respect to the nondegenerate symmetric bilinear form $(\ ,\ )$. Actually, the sum is over $\mathbb{T} \in \Lambda^+ = \Lambda \cap \bar{\mathfrak{R}}$ (provided $\mathfrak{T}$ has no simple 1-dimensional factors); i.e.,

$$F(Z) = \sum_{\mathbb{T} \in \Lambda^+} a(\mathbb{T}) \varepsilon((\mathbb{T}, Z)). \tag{6}$$

7.5. $E_l(Z)$ is an automorphic form on $\mathfrak{T}$; thus we have

$$E_l(Z) = \sum_{\mathbb{T} \in \Omega^+} a_l(\mathbb{T}) \varepsilon((\mathbb{T}, Z)). \tag{7}$$

By Lemma 4.5, we have

$$G_{\mathbb{Q}} = \bigcup_{k=0}^{n} P_{\mathbb{Q}} \iota_{(k)} P_{\mathbb{Q}}.$$

But now we should pay attention to the fact that, if n denotes the real rank of G, then only those k corresponding to rational boundary components should appear in this decomposition. We refer to such k as 'rational' k, and denote by $\sum_{k=0}^{n_*}$ the summation over those 'rational' k's. In the remainder of Section 7, we assume that k is 'rational'.

By the decomposition of $G_{\mathbb{Q}}$, we may write

$$E_l(Z) = \sum_{k=0}^{n_*} E_l^{(k)}(Z), \tag{8}$$

with

$$E_l^{(k)}(Z) = \sum_{g \in P_{\mathbb{Q}} \iota_{(k)} P_{\mathbb{Q}} / P_{\mathbb{Q}}} j(Z, g)^l c(g)^l. \tag{9}$$

It is obvious that $E_l^{(k)}(Z + \mathbb{S}) = E_l^{(k)}(Z)$ for all $\mathbb{S} \in \Lambda'$. Thus $E_l^{(k)}(Z)$ also has a Fourier expansion

$$E_l^{(k)}(Z) = \sum_{\mathbb{T} \in \Lambda^+} a_l^{(k)}(\mathbb{T}) \varepsilon((\mathbb{T}, Z)). \tag{10}$$

Hence, we have

$$a_l(\mathbb{T}) = \sum_{k=0}^{n_*} a_l^{(k)}(\mathbb{T}). \tag{11}$$

7.6. Let $\mu \in M_{\mathbb{Q}}$ be fixed. We are going to consider the Eisenstein series ${}^{\mu}E_l$ with respect to the arithmetic subgroup ${}^{\mu}\Gamma = \mu\Gamma\mu^{-1}$ of $G^0_{\mathbb{R}}$.

For any $g \in G_{\mathbb{Q}}$, $g = \mu g_1 = \mu\gamma a\rho = (\mu\gamma\mu^{-1})\mu a\rho$, thus

$$G_{\mathbb{Q}} = \bigcup_{\mu a \in \mu A} {}^{\mu}\Gamma \mu a P_{\mathbb{Q}}$$

is a double set decomposition of $G_{\mathbb{Q}}$ with respect to ${}^{\mu}\Gamma$. Let ${}^{\mu}c(\mu a)$ be the number chosen for μa with respect to this decomposition, then it is easy to see that ${}^{\mu}c(\mu a) = c(a)$. Then

$$\begin{aligned} {}^{\mu}E_l(Z) &= \sum_{a \in A} \sum_{\mu\gamma\mu^{-1} \in {}^{\mu}\Gamma/{}^{\mu}\Gamma_{0,a}} j(Z, \mu\gamma a)^l c(a)^l \\ &= \sum_{a \in A} \sum_{\gamma \in \Gamma/\Gamma_{0,a}} j(Z \cdot \mu, \gamma a)^l c(a)^l = E_l(Z \cdot \mu) \\ &= \sum_{\mathbb{T} \in \Lambda^+} a_l(\mathbb{T}) \varepsilon((\mathbb{T}, Z \cdot \mu)) = \sum_{\mathbb{T} \in \Lambda^+} a_l(\mathbb{T}) \varepsilon((\mu \cdot \mathbb{T}, Z)). \end{aligned}$$

Since ${}^{\mu}\Gamma \cap U = {}^{\mu}(\Gamma \cap U) = {}^{\mu}\Lambda' = (\mu^*)^{-1} \cdot \Lambda'$, by Lemma 5.2, the dual lattice of ${}^{\mu}\Gamma \cap U$ is $\mu \cdot \Lambda$. Thus $a_l(\mathbb{T})$ is the Fourier coefficient of ${}^{\mu}E_l(Z)$ at $\mu \cdot \mathbb{T} \in \mu \cdot \Lambda$.

If we write $\mathbb{T} = ({}^{\sigma}\mathbb{T}')$, then all the factors have the same rank, which will be called the rank $R(\mathbb{T})$ of $\mathbb{T}$. We shall see in § 8.8 that, if $R(\mathbb{T}) = m$ is not 'rational', then $a_l(\mathbb{T}) = 0$. On the other hand, if $R(\mathbb{T}) = k$ is 'rational', then there is $\mu \in M_{\mathbb{Q}}$ such that $\mu \cdot \mathbb{T}$ has the form

$$\begin{pmatrix} 0 & 0 \\ 0 & {}^{\sigma}\mathbb{T}'_1 \end{pmatrix}$$

for each factor, where ${}^{\sigma}\mathbb{T}'_1$ is $k \times k$. Thus we have

LEMMA 7.6: *In order to prove the rationality of the Fourier coefficient* $a_l(\mathbb{T})$, *we may assume that* $\mathbb{T}$ *is of the form*

$$\begin{pmatrix} 0 & 0 \\ 0 & {}^{\sigma}\mathbb{T}'_1 \end{pmatrix}$$

for each factor, where ${}^{\sigma}\mathbb{T}'_1$ *is* $k \times k$, *and* $k = R(\mathbb{T})$.

7.7. It is known [7, § 10.14] that if F is an automorphic form on $\mathfrak{T}$, then the restriction of F to a fundamental set $\mathscr{F}$ for Γ in $\mathfrak{T}$ has a continuous extension to $\bar{\mathscr{F}}$ such that the restriction to $\bar{\mathscr{F}} \cap \mathfrak{T}_k^\infty$ of that extension coincides with an automorphic form $\Phi_k F$ on $\bar{\mathscr{F}} \cap \mathfrak{T}_k^\infty$. If the Fourier expansion of F is given by (6), then it follows from § 6.1 that the Fourier expansion of $\Phi_k F$, as an automorphic form on $\mathfrak{T}_k^\infty$, is (by identifying $\mathfrak{T}_k^\infty \cong \mathfrak{T}_k^0$) (cf., [5, § 7.2])

$$\Phi_k F(Z) = \sum_{\mathbb{T} \in \Lambda_k^+} a_l(\mathbb{T})\varepsilon((\mathbb{T}, Z)),$$

where

$$\Lambda_k^+ = \left\{ \mathbb{T} \in \Lambda^+ \mid \text{each factor of } \mathbb{T} \text{ is of the form} \begin{pmatrix} 0 & 0 \\ 0 & {}^\sigma\mathbb{T}_1' \end{pmatrix}, \right.$$
$$\left. \text{and } {}^\sigma\mathbb{T}_1' \text{ is } k \times k \right\}.$$

Denote the mapping dual to π_{kn}, the canonical mapping from $\mathfrak{T}$ onto $\mathfrak{T}_k^\infty$, by π_{kn}^*. If $g \in P_k P \cap G_{\mathbb{R}}^0$, then [7, § 7.9] $j(Z, g)^l$ is constant along the fibers of π_{kn}, and if $g \notin P_k P \cap G_{\mathbb{R}}^0$, then the limit of $j(Z, g)^l$ as $Z \to \mathfrak{T}_k^\infty$ is zero. Since $(P_k P)_{\mathbb{Q}} = P_{k\mathbb{Q}} P_{\mathbb{Q}}$ and since $P_{k\mathbb{Q}} P_{\mathbb{Q}}/P_{\mathbb{Q}}$ may be identified with $P_{k\mathbb{Q}}/P_{0k\mathbb{Q}}$, we have

$$(\pi_{kn}^* \Phi_k E_l)(Z) = \sum_{g \in P_{k\mathbb{Q}}/P_{0k\mathbb{Q}}} j(Z, g)^l c(g)^l.$$

It is also clear that $\varepsilon((\mathbb{T}, Z))$ is constant along the fibers of π_{kn} if $\mathbb{T} \in \Lambda_k^+$; thus we have

$$\sum_{g \in P_{k\mathbb{Q}}/P_{0k\mathbb{Q}}} j(Z, g)^l c(g)^l = \sum_{\mathbb{T} \in \Lambda_k^+} a_l(\mathbb{T})\varepsilon((\mathbb{T}, Z)). \tag{12}$$

Now note that $P_{k\mathbb{Q}}/P_{0k\mathbb{Q}}$ may be identified with $L_{k\mathbb{Q}}/L_k \cap P_{\mathbb{Q}}$, and by Theorem 6.4, that $j(Z, g)^l = j_k(Z_{(k)}, g)^{l\mathbb{N}(n)/\mathbb{N}(k)}$ and $c(g)^l = c_k(g)^{l\mathbb{N}(n)/\mathbb{N}(k)}$ for $g \in L_{k\mathbb{Q}}^*$, we conclude that the left hand side of (12) is the Eisenstein series of weight $l\mathbb{N}(n)/\mathbb{N}(k)$ with respect to L_k and its arithmetic subgroup $\Gamma \cap L_k$, and the right hand side of (12) is its Fourier expansion. Combining Lemma 7.6, we have

LEMMA 7.7: *In order to prove the rationality of the Fourier coefficient $a_l(\mathbb{T})$, we may assume that $\mathbb{T}$ is of maximal rank.*

LEMMA 7.8.1: *Let $\mu_1, \mu_2 \in M_{\mathbb{Q}}$, $t_{X_1}, t_{X_2} \in U_{(k)\mathbb{Q}}$, then*

$$\mu_1 t_{X_1} \iota_{(k)} P_{\mathbb{Q}}/P_{\mathbb{Q}} = \mu_2 t_{X_2} \iota_{(k)} P_{\mathbb{Q}}/P_{\mathbb{Q}}$$

implies that $\mu_1 \iota_{(k)} P_{\mathbb{Q}}/P_{\mathbb{Q}} = \mu_2 \iota_{(k)} P_{\mathbb{Q}}/P_{\mathbb{Q}}$.

PROOF: It may be assumed that μ_2 = identity. Then there exists $\rho \in P_{\mathbb{Q}}$ such that $\mu_1 t_{X_1} \iota_{(k)} = t_{X_2} \iota_{(k)} \rho$. Let $w_1, w_2 \in W'$ be such that $\mu_1 \in P_{*\mathbb{Q}} w_1 P_{*\mathbb{Q}}$, $\rho \in P_{*\mathbb{Q}} w_2 P_{*\mathbb{Q}}$. (See § 4 for notations.) Then

$$P_{*\mathbb{Q}} w_1 P_{*\mathbb{Q}} \iota_{(k)} \cap \iota_{(k)} P_{*\mathbb{Q}} w_2 P_{*\mathbb{Q}} \neq \Phi.$$

By Lemma 4.3, we have

$$P_{*\mathbb{Q}} w_1 \iota_{(k)} P_{*\mathbb{Q}} \cap P_{*\mathbb{Q}} \iota_{(k)} w_2 P_{*\mathbb{Q}} \neq \Phi,$$

from which we conclude that $w_1 \iota_{(k)} = \iota_{(k)} w_2$. It follows that $w_1 = w_2 \in P_{k\mathbb{Q}}$ and that $\mu_1 \in P_{k\mathbb{Q}}$, from which $\iota_{(k)}^{-1} \mu_1 \iota_{(k)} \in P_{\mathbb{Q}}$ and hence our lemma.

By § 4.9, every element of $P_{\mathbb{Q}} \iota_{(k)} P_{\mathbb{Q}}/P_{\mathbb{Q}}$ can be represented by $\mu t_X \iota_{(k)} P_{\mathbb{Q}}/P_{\mathbb{Q}}$, $\mu \in M_{\mathbb{Q}}$, $t_X \in U_{(k)\mathbb{Q}}$. Thus, by the preceding lemma, we have

$$E_l^{(k)}(Z) = \sum_{\mu \iota_{(k)} \in M_{\mathbb{Q}} \iota_{(k)} P_{\mathbb{Q}}/P_{\mathbb{Q}}} {}^{\mu}E_l^{(k)}(Z),$$

with

$${}^{\mu}E_l^{(k)}(Z) = \sum_{g = \mu t_X \iota_{(k)} \in \mu U_{(k)\mathbb{Q}} \iota_{(k)} P_{\mathbb{Q}}/P_{\mathbb{Q}}} j(Z, g)^l c(g)^l.$$

Put $g_1 = t_{\mathbb{S}} g$, then $c(g_1) = c(g)$ and

$$g_1 P_{\mathbb{Q}} = t_{\mathbb{S}} g P_{\mathbb{Q}} = t_{\mathbb{S}} \mu t_X \iota_{(k)} P_{\mathbb{Q}} = \mu t_{\mu^* \cdot \mathbb{S} + X_{\iota(k)}} P_{\mathbb{Q}} = \mu t_{((\mu^* \cdot \mathbb{S})_{(k)} + X)} \iota_{(k)} P_{\mathbb{Q}}.$$

Thus we have ${}^{\mu}E_l^{(k)}(Z + \mathbb{S}) = {}^{\mu}E_l^{(k)}(Z)$, and hence the Fourier expansion

$${}^{\mu}E_l^{(k)}(Z) = \sum_{\mathbb{T} \in \Lambda^+} {}^{\mu}a_l^{(k)}(\mathbb{T}) \varepsilon((\mathbb{T}, Z)),$$

with

$${}^{\mu}a_l^{(k)}(\mathbb{T}) = \int_{\Lambda'_{\mathscr{F}}} {}^{\mu}E_l^{(k)}(Z) e^{-2\pi i(\mathbb{T}, Z)} dX_{\Lambda'}$$

where $\Lambda'_{\mathscr{F}}$ is a fundamental period parallelogram of the lattice Λ', and $dX_{\Lambda'}$ is the measure with respect to a basis of Λ'.

Let $\mathscr{J}_{(k)}$ denote the subspace of the real Jordan algebra $\mathscr{J}$ corresponding to the subgroup $U_{(k)}$ of $U_{\mathbb{R}}$, and $\mathscr{J}_{(k)*}$ denote the orthogonal complement of $\mathscr{J}_{(k)}$ in $\mathscr{J}$. Write $\mathbb{T} = \mathbb{T}_1 + \mathbb{T}_2 \in \mu \cdot \mathscr{J}_{(k)} + \mu \cdot \mathscr{J}_{(k)*}$.

If $\mathbb{T}_2 \neq 0$, then ${}^{\mu}\mathrm{a}_l^{(k)}(\mathbb{T}) = 0$, because ${}^{\mu}E_l^{(k)}(Z)$ is constant when X varies in $\Lambda'_{\mathscr{F}} \cap (\mu^*)^{-1} \cdot \mathscr{J}_{(k)*}$, in which $\varepsilon((\mathbb{T}, Z))$ is non-trivial. Thus

$$ {}^{\mu}E_l^{(k)}(Z) = \sum_{\mathbb{T} \in \Lambda^+ \cap \mu \cdot \mathscr{J}_{(k)}} {}^{\mu}a_l^{(k)}(\mathbb{T})\varepsilon((\mathbb{T}, Z)). $$

Since the rank of $\mathbb{T} \in \Lambda^+ \cap \mu \cdot \mathscr{J}_{(k)}$ is $\leqq k$, combining the results of the preceding lemmas, we have

LEMMA 7.8.2: *In order to prove the rationality of the Fourier coefficients, it is sufficient to prove this for the Fourier coefficients of* $E_l^{(n)}(Z)$.

7.9. Now we look at the biggest cell $P_{\mathbb{Q}}\iota P_{\mathbb{Q}}$ and the corresponding series $E_l^{(n)}$. Every element of $P_{\mathbb{Q}}\iota P_{\mathbb{Q}}/P_{\mathbb{Q}}$ can be written uniquely in the form $t_X\iota$.

If we let $\chi(X) = c(t_X\iota)$, then we have

$$ (13) \qquad E_l^{(n)}(Z) = \sum_{X \in \mathscr{J}_{\mathbb{Q}}} j(Z, t_X\iota)^l \chi(X)^l = \sum_{X \in \mathscr{J}_{\mathbb{Q}}} |Z+X|^{-\mathbb{N}l}\chi(X)^l $$

$$ = \sum_{X \in \mathscr{J}_{\mathbb{Q}}/\Lambda'} \Big(\sum_{\mathbb{S} \in \Lambda'} |Z+X+\mathbb{S}|^{-\mathbb{N}l}\Big)\chi(X)^l, $$

which is ready for applying the Poisson summation formula.

8. Gamma integral and Poisson summation formula

The purpose of this section is to apply some generalized gamma integral and Poisson summation formula and obtain a formula for the Fourier coefficients of (13) of § 7.9.

LEMMA 8.1: *Let* $\mathfrak{R}$ *be an irreducible homogeneous self-adjoint cone of* § 2.1, *and* dX *be the ordinary euclidean measure of the ambient space of* $\mathfrak{R}$. *Then*

$$ (1) \qquad \int_{\mathfrak{R}} |X|^{\rho - \mathbb{N}/2} e^{-tr\, X} dX = \pi^{n(\mathbb{N}-2)/4} \prod_{j=0}^{n-1} \gamma(\rho - jn_0/2), $$

where $\rho > \mathbb{N}/2 - 1$, $\mathbb{N} = (n-1)n_0 + 2$ *as before, and* $\gamma(\ )$ *is the ordinary gamma integral.*

PROOF: This can be done by changing variable $X \to T^*T$ (cf., § 2.1) and then reducing to the ordinary gamma integral [10, § 24.6].

LEMMA 8.2: *Let Z be in the irreducible tube domain $\mathfrak{T}$ corresponding to $\mathfrak{R}$. Then, for $\rho > \mathbb{N}/2-1$, we have*

$$(2)\qquad \int_{\mathfrak{R}} |X|^{\rho-\mathbb{N}/2}\varepsilon((X, Z))dX = (2\pi i)^{-n\rho}\pi^{n(\mathbb{N}-2)/4}\prod_{j=0}^{n-1}\gamma(\rho-jn_0/2)|Z|^{-\rho}.$$

PROOF: This follows from Lemma 8.1 by appropriate changes of variables.

8.3. Let Λ, Λ' be mutually dual lattices with respect to an inner product $(\ ,\ )$ in $\mathbb{R}^{\mathbb{M}}$. Let f be a continuous function with continuous partial derivatives up to order v in $\mathbb{R}^{\mathbb{M}}$. Let $g(X) = \sum_{\lambda\in\Lambda} f(X+\lambda)$, and assume that this series and the corresponding series with f replaced by any of its partial derivatives of order $\leqq v$ converge normally in $\mathbb{R}^{\mathbb{M}}$. If v is sufficiently large, then Poisson summation formula says

$$(3)\qquad \sum_{\lambda\in\Lambda} f(\lambda) = \sum_{\lambda'\in\Lambda'} \hat{f}(\lambda'),$$

where

$$\hat{f}(\lambda') = \int_{\mathbb{R}^{\mathbb{M}}} f(X)e^{-2\pi i(X,\lambda')}dX_{\Lambda},$$

and the measure dX_{Λ} is with respect to (a basis of) Λ.

LEMMA 8.4: *Let Λ, Λ' be as in § 7, then for sufficiently large ρ, we have*

$$(4)\qquad \sum_{\mathbb{T}\in\Lambda\cap\mathfrak{R}} |\mathbb{T}|^{\rho-\mathbb{N}/2}\varepsilon((\mathbb{T}, Z)) = v(\Lambda)^{-1}(2\pi i)^{-n\rho}\pi^{n(\mathbb{N}-2)/4} \cdot \prod_{j=0}^{n-1}\gamma(\rho-jn_0/2)\sum_{\mathbb{S}\in\Lambda'}|Z+\mathbb{S}|^{-\rho},$$

where $v(\Lambda)$ is the volume (of a fundamental period parallelogram) of Λ with respect to dX.

PROOF: Apply Poisson summation formula (§ 8.2) to the function f_Z defined on the ambient space of $\mathfrak{R}$ by

$$f_Z(X) = \begin{cases} |X|^{\rho-\mathbb{N}/2}\varepsilon((X, Z)) & \text{if } X\in\mathfrak{R},\\ = 0 & \text{otherwise,}\end{cases}$$

we obtain

$$\sum_{\mathbb{T}\in\Lambda\cap\mathfrak{R}} |\mathbb{T}|^{\rho-\mathbb{N}/2}\varepsilon((\mathbb{T},Z)) = \sum_{\mathbb{S}\in\Lambda'}\int_{\mathfrak{R}} |X|^{\rho-\mathbb{N}/2}\varepsilon((X,Z))e^{-2\pi i(X,\mathbb{S})}dX_{\Lambda}$$

$$= v(\Lambda)^{-1}\sum_{\mathbb{S}\in\Lambda'}\int_{\mathfrak{R}} |X|^{\rho-\mathbb{N}/2}\varepsilon((X,Z-\mathbb{S}))dX$$

$$= v(\Lambda)^{-1}(2\pi i)^{-n\rho}\pi^{n(\mathbb{N}-2)/4}\prod_{j=0}^{n-1}\gamma(\rho-jn_0/2)\sum_{\mathbb{S}\in\Lambda'}|Z+\mathbb{S}|^{-\rho}.$$

LEMMA 8.5: *For* $R(\mathbb{T}) = n, \mathbb{T}$ *irreducible, the Fourier coefficient* $a_l(\mathbb{T}) = a_l^{(n)}(\mathbb{T})$ *of* (13) *in* § 7.9 *has the following expression*

$$(5)\quad a_l(\mathbb{T}) = v(\Lambda)(2\pi i)^{n\mathbb{N}l}\pi^{-n(\mathbb{N}-2)/4}\prod_{j=0}^{n-1}\gamma(\mathbb{N}l-jn_0/2)^{-1} \cdot |\mathbb{T}|^{\mathbb{N}l-\mathbb{N}/2}\sum_{X\in\mathscr{I}_{\mathbb{Q}}/\Lambda'}\varepsilon((\mathbb{T},X))\chi(X)^l.$$

PROOF: From (4), we have

$$\sum_{\mathbb{S}\in\Lambda'}|Z+\mathbb{S}|^{-\rho} = v(\Lambda)(2\pi i)^{n\rho}\pi^{-n(\mathbb{N}-2)/4}\prod_{j=0}^{n-1}\gamma(\rho-jn_0/2)^{-1} \sum_{\mathbb{T}\in\Lambda\cap\mathfrak{R}}|\mathbb{T}|^{\rho-\mathbb{N}/2}\varepsilon((\mathbb{T},Z)).$$

Let $\rho = \mathbb{N}l$, and apply this formula, with Z replaced by $Z+X$, to the inner sum of (13) of § 7.9, we have

$$(6)\quad E_l^{(n)}(Z) = \sum_{X\in\mathscr{I}_{\mathbb{Q}}/\Lambda'}\Big\{v(\Lambda)(2\pi i)^{n\mathbb{N}l}\pi^{-n(\mathbb{N}-2)/4}\prod_{j=0}^{n-1}\gamma(\mathbb{N}l-jn_0/2)^{-1} \times\sum_{\mathbb{T}\in\Lambda\cap\mathfrak{R}}|\mathbb{T}|^{\mathbb{N}l-\mathbb{N}/2}\varepsilon((\mathbb{T},Z+X))\Big\}\chi(X)^l,$$

$$= \sum_{\mathbb{T}\in\Lambda\cap\mathfrak{R}}\Big\{v(\Lambda)(2\pi i)^{n\mathbb{N}l}\pi^{-n(\mathbb{N}-2)/4}\prod_{j=0}^{n-1}\gamma(\mathbb{N}l-jn_0/2)^{-1} \cdot|\mathbb{T}|^{\mathbb{N}l-\mathbb{N}/2}\sum_{X\in\mathscr{I}_{\mathbb{Q}}/\Lambda'}\varepsilon((\mathbb{T},X))\chi(X)^l\Big\}\varepsilon((\mathbb{T},Z)),$$

from which (5) follows.

8.6. We lift the irreducibility assumption of $\mathfrak{R}$, and let $d = [\mathbb{K}:\mathbb{Q}]$, then we have

LEMMA 8.6: *Let* $\rho > \mathbb{N}/2$, *and* dX *be the ordinary euclidean measure on the ambient space of* $\mathfrak{R}$, *then*

$$(7)\qquad \int_{\mathfrak{K}} |X|^{\rho-\mathbb{N}/2} e^{-tr\,X} dX = \pi^{dn(\mathbb{N}-2)/4} \prod_{j=0}^{n-1} \gamma(\rho - jn_0/2)^d.$$

PROOF: This follows from the fact that the left hand side of this equation can be written as a product of the gamma integrals of Lemma 8.1 for the irreducible components of the cone $\mathfrak{K}$.

Proceeding as in §§ 8.2–8.5, we have

LEMMA 8.7: *The Fourier coefficient* $a_l(\mathbb{T})$, $R(\mathbb{T}) = n$, *is given by*

$$(8)\quad a_l(\mathbb{T}) = v(\Lambda)(2\pi i)^{dn\mathbb{N}l} \pi^{-dn(\mathbb{N}-2)/4} \prod_{j=0}^{n-1} \gamma(\mathbb{N}l - jn_0/2)^{-d} \cdot |\mathbb{T}|^{\mathbb{N}l-\mathbb{N}/2} \sum_{X \in \mathscr{J}_{\mathbb{Q}}/\Lambda'} \varepsilon((\mathbb{T}, Z))\chi(X)^l.$$

8.8. From (6), we see that $a_l^{(n)}(\mathbb{T}) = 0$ if $R(\mathbb{T}) \neq n$. By the result of § 7.7, we also see that $a_l^{(k)}(\mathbb{T}) = 0$ if $R(\mathbb{T}) \neq k$. That is, $E_l^{(k)}$ only gives the Fourier coefficients $a_l(\mathbb{T})$ with $R(\mathbb{T}) = k$. In particular, this implies that if $R(\mathbb{T}) = m$ is not 'rational', then $a_l(\mathbb{T}) = 0$; that is needed in § 7.6.

9. Content and Hensel's lemma

We know from Section 8 that the formula for $a_l(\mathbb{T})$ involves an infinite series

$$(1)\qquad S = \sum_{X \in \mathscr{J}_{\mathbb{Q}}/\Lambda'} \varepsilon((\mathbb{T}, X))\chi(X)^l.$$

In this section, we shall calculate $\chi(X)$ in terms of a new concept, the content of X. And then apply Hensel's lemma [3] and express S as a product of finite series.

9.1. Following the same idea as in [16, § 7; 5, ; 9], we may write $S = \prod_p S_p$, where S_p is given by

$$(2)\qquad S_p = \sum_{X \in \mathscr{J}_p/\Lambda'_p} \varepsilon_p((\mathbb{T}, X))\chi_p(X)^l,$$

where $\Lambda'_p = \Lambda' \otimes \mathbb{Z}_p$, $\mathscr{J}_p = \mathscr{J}_{\mathbb{Q}_p}$, $\chi_p(X) = c_p(t_{X^l})$ and ε_p is the character of $\mathbb{Q}_p/\mathbb{Z}_p$ such that if $q \in \mathbb{Q}_p$, $q \equiv \xi = \sum_{\nu \geqq 0} a_\nu p^{-\nu} \pmod{\mathbb{Z}_p}$, with $a_\nu \in \mathbb{Z}$, then $\varepsilon_p(q) = \varepsilon(\xi) = e^{2\pi i \xi}$. We add the convention that if $a \in \mathbb{Z}$ and if $\bar{a}$ is the image of a in $\mathbb{F}_p$, the finite field of p elements, then $\varepsilon_p(p^{-1}\bar{a}) = \varepsilon_p(p^{-1}a)$.

9.2. Since $\mathscr{J} = R_{\mathbb{K}/\mathbb{Q}}\mathscr{J}'$ implies that $\mathscr{J}_p = \prod_{\mathfrak{p}|p}(R_{\mathbb{K}_\mathfrak{p}/\mathbb{Q}_p}\mathscr{J}')_{\mathbb{Q}_p}$ [20, p. 9],

$X \in \mathscr{J}_p$ can be written as $X = \prod_{\mathfrak{p}|p}\prod_{\sigma:{}^\sigma\mathfrak{p}=\mathfrak{p}} {}^\sigma X'_{\mathfrak{p}}$. Then

$$(\mathbb{T}, X) = \sum_{\mathfrak{p}|p} \mathrm{Tr}_{\mathbb{K}_{\mathfrak{p}}|\mathbb{Q}_p}(\mathbb{T}'_{\mathfrak{p}}, X'_{\mathfrak{p}}).$$

On the other hand, if $t_X\iota = \gamma\rho \in \mathscr{K}_p P_p$, then $\rho = \prod_{\mathfrak{p}|p}\prod_{\sigma:{}^\sigma\mathfrak{p}=\mathfrak{p}} {}^\sigma\rho'_{\mathfrak{p}}$, and hence

$$\begin{aligned}\chi_p(X) = |\det(Ad_{\mathscr{J}}\rho)|_p^{-1} &= |\prod_{\mathfrak{p}|p}\prod_{\sigma:{}^\sigma\mathfrak{p}=\mathfrak{p}} \det(Ad_{\mathscr{J}'}{}^\sigma\rho'_{\mathfrak{p}})|_p^{-1}\\ &= \prod_{\mathfrak{p}|p} |N_{\mathbb{K}_{\mathfrak{p}}/\mathbb{Q}_p}(\det(Ad_{\mathscr{J}'}\rho'_{\mathfrak{p}}))|_p^{-1}.\end{aligned}$$

If we define $c_{\mathfrak{p}}(g)$, $\chi_{\mathfrak{p}}(X'_{\mathfrak{p}})$ in the same way as we did for $c_p(g)$ and $\chi_p(X)$, then $\chi_{\mathfrak{p}}(X'_{\mathfrak{p}}) = c_{\mathfrak{p}}(t_{X'_{\mathfrak{p}}}\iota') = |\det(Ad_{\mathscr{J}'}\rho')|_{\mathfrak{p}}^{-1}$. Thus, from (2), we have

$$S_p = \prod_{\mathfrak{p}|p} \sum_{X'_{\mathfrak{p}}\in\mathscr{J}'_{\mathfrak{p}}/\Lambda'_{\mathfrak{p}}} \varepsilon_p(\mathrm{Tr}_{\mathbb{K}_{\mathfrak{p}}/\mathbb{Q}_p}(\mathbb{T}'_{\mathfrak{p}}, X'_{\mathfrak{p}}))|N_{\mathbb{K}_{\mathfrak{p}}/\mathbb{Q}_p}(\chi_{\mathfrak{p}}(X'_{\mathfrak{p}}))|_p^{-l},$$

where $\Lambda'_{\mathfrak{p}} = \Lambda'_{\mathfrak{p}} \otimes \mathbb{Z}_{\mathfrak{p}}$. If we write

$$S_{\mathfrak{p}} = \sum_{X'_{\mathfrak{p}}\in\mathscr{J}'_{\mathfrak{p}}/\Lambda'_{\mathfrak{p}}} \varepsilon_p(\mathrm{Tr}_{\mathbb{K}_{\mathfrak{p}}/\mathbb{Q}_p}(\mathbb{T}'_{\mathfrak{p}}, X'_{\mathfrak{p}}))|N_{\mathbb{K}_{\mathfrak{p}}/\mathbb{Q}_p}(\chi_{\mathfrak{p}}(X'_{\mathfrak{p}}))|_p^{-l}, \tag{3}$$

then we have $S_p = \prod_{\mathfrak{p}|p} S_{\mathfrak{p}}$.

In the subsequent discussions, for simplicity, if there is no confusion, we shall drop the subscripts and superscripts of an element.

LEMMA 9.3: *If* $X \in \mathscr{J}'_{\mathfrak{p}}$, $u \in \mathbb{Z}^*_{\mathfrak{p}}$, *then* $\chi_{\mathfrak{p}}(uX) = \chi_{\mathfrak{p}}(X)$.

PROOF: Let $\sqrt{u}$ be a square root of u in the algebraic closure of $\mathbb{K}_{\mathfrak{p}}$. Let $\sqrt{u}$ also denote the element $\mathrm{diag}(\sqrt{u}, \ldots, \sqrt{u})$ of the maximal split torus S' of G'. Let $t_X\iota = \gamma\rho \in \mathscr{K}_{\mathfrak{p}}P'_{\mathfrak{p}}$, then

$$t_{uX}\iota = \sqrt{u}t_X\sqrt{u}^{-1}\iota = \sqrt{u}t_X\iota\sqrt{u} = \sqrt{u}\gamma\rho\sqrt{u} = (\sqrt{u}\gamma\sqrt{u})(\sqrt{u}^{-1}\rho\sqrt{u}).$$

Since $\sqrt{u}^{-1}t_Y\sqrt{u} = t_{u^{-1}Y}$ for any $Y \in \mathscr{J}'_{\mathfrak{p}}$, and $\sqrt{u}$ commutes with any element of $Z(S'_0)$, it follows that $\sqrt{u}^{-1}\rho\sqrt{u} \in P'_{\mathfrak{p}}$, the group generated by t_Y and $Z(S'_0)$. Observing that we may take $\mathscr{K}_{\mathfrak{p}}$ to be the group denoted by M in [9, p. 869] and that $\varphi(\sqrt{u})$ is in $\mathbb{Z}^*_{\mathfrak{p}}$ for any $\mathbb{K}$-root φ of G', it follows that $\sqrt{u}\gamma\sqrt{u}$ is in $\mathscr{K}_{\mathfrak{p}}$, and hence $\chi_{\mathfrak{p}}(uX) = \chi_{\mathfrak{p}}(X)$.

9.4. Notations as in § 1.1. The maximal parabolic subgroup P is the semi-direct product of $Z(S_0)$ and U. An element k of $Z(S_0)$, through the adjoint representation on U, is a norm similarity of the Jordan algebra $\mathscr{J}$. Let $\nu(k)$ be the number such that $N(\mathrm{Ad}(k)\cdot X) = \nu(k)N(X)$ for all $X \in J$, then $\nu(k)$ is a rational character of $Z(S_0)$. If $\rho = ku \in P = Z(S_0) \cdot U$, let $\nu(\rho) = \nu(k)$, then $\nu(\rho)$ is a rational character of P. On the other hand, $\det(\mathrm{Ad}\,\rho)$ is also a rational character of P. By restricting to S_0, it is easy to see that $\det(\mathrm{Ad}\,\rho) = \nu(\rho)^{\mathbb{N}/2}$, where, again, $\mathbb{N} = (n-1)n_0+2$. If $X_p \in \mathscr{J}'_{\mathfrak{p}}$, then $t_{X_{\mathfrak{p}}}\iota = \gamma\rho \in G'_{\mathfrak{p}} = \mathscr{K}_{\mathfrak{p}}P'_{\mathfrak{p}}$. Define the $\mathfrak{p}$-adic content $\kappa_{\mathfrak{p}}(X_{\mathfrak{p}})$ to be $|\nu(\rho)|_{\mathfrak{p}}^{\frac{1}{2}}$. If $X = (X_{\mathfrak{p}}) \in \mathscr{J}$, define the content $\kappa(X)$ of X to be $\prod_{\mathfrak{p}}\kappa_{\mathfrak{p}}(X_{\mathfrak{p}})$. $\kappa_{\mathfrak{p}}$ is well defined and we have $\chi_{\mathfrak{p}}(X_{\mathfrak{p}}) = \kappa_{\mathfrak{p}}(X_{\mathfrak{p}})^{-\mathbb{N}}$ and $\chi(X) = \kappa(X)^{-\mathbb{N}}$.

In order to calculate $\kappa_{\mathfrak{p}}(X)$, we have to study the $\mathbb{K}_{\mathfrak{p}}$-structure of $\mathscr{J}'$. For $\mathscr{J}' = \mathscr{H}(\mathbb{D}_m, J_A)$, the Jordan algebra of § 2.9, it is known [6, § 3] that, $\forall'$ (= for all but a finite number of) prime ideals $\mathfrak{p}$ of $\mathbb{K}$, $\mathscr{J}'_{\mathfrak{p}}$ is isomorphic either to $\mathscr{H}((\mathbb{K}(\theta)_{\mathfrak{p}})_n, J_A)$, when $\mathfrak{p}$ is a prime in $\mathbb{K}(\theta)$, or to

$$\mathscr{H}((\mathbb{K}_{\mathfrak{p}} \oplus \mathbb{K}_{\mathfrak{p}})_n, J_A),$$

when $\mathfrak{p}$ splits in $\mathbb{K}(\theta)$. Furthermore, the elements a_i of A are units of $\mathbb{Z}_{\mathfrak{p}}$, $\forall'_{\mathfrak{p}}$. For other types of Jordan algebras, the situation is even simpler and similar, and we summarize the result as

THEOREM 9.4.1: *$\forall'\mathfrak{p}$, $\mathscr{J}'_{\mathfrak{p}} = \mathscr{H}(\mathscr{A}_n, J_A)$, where $\mathscr{A}$ is an algebra of* § 3.1 *with $\Phi = \mathbb{K}_{\mathfrak{p}}$, $A = \operatorname{diag}(a_1, \ldots, a_n)$ and $a_i \in \mathbb{Z}_{\mathfrak{p}}^*$. Moreover, we may assume that $\mathscr{J}'_{\mathbb{Z}_{\mathfrak{p}}} = \mathscr{H}((\mathscr{A}_{\mathbb{Z}_{\mathfrak{p}}})_n, J_A)$, which is isomorphic to a component of $\Gamma_{\mathbb{Z}_{\mathfrak{p}}} \cap U$.*

Assumption as in the theorem, any element $X \in \mathscr{J}'_{\mathfrak{p}}$ is of the form $X = AY$ with $Y \in \mathscr{H}(\mathscr{A}_n, J_I)$. An element of M' (the subgroup M of § 5.2 for G') acts on $\mathscr{J}'_{\mathfrak{p}}$ by $\mu \cdot X = A(\sqrt{A}^{-1}(\mu \cdot (\sqrt{A}Y\sqrt{A}))\sqrt{A}^{-1})$, and $M'_{\mathbb{Z}_{\mathfrak{p}}}$ is generated by $(\sqrt{a_j/a_i}y)_{ij}$, $y \in \mathscr{A}_{\mathbb{Z}_{\mathfrak{p}}}$ (cf., § 2.8).

LEMMA 9.4.2: *$\forall'\mathfrak{p}$, if $X = AY \in \mathscr{J}'_{\mathfrak{p}}$, then there exists $\mu \in M'_{\mathbb{Z}_{\mathfrak{p}}}$ such that $\mu \cdot X$ is of diagonal form, with diagonal entries $\eta_i = v_i\pi^{u_i}$, where π is a prime element in $\mathbb{K}_{\mathfrak{p}}$* (§ 0.4), *$v_i \in \mathbb{Z}_{\mathfrak{p}}^*$, and μ_i are rational integers or $+\infty$ (by which we mean $\eta_i = 0$) such that $\mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_n$.*

PROOF: Choose a basis $\{e_1, \ldots, e_{n_0}\}$ of $\mathscr{A}_{\mathbb{K}}$, then $\forall'\mathfrak{p}$, $\mathscr{A}_{\mathbb{Z}_{\mathfrak{p}}} = \sum_i \mathbb{Z}_{\mathfrak{p}}e_i$, and the norm form on $\mathscr{A}_{\mathfrak{p}}$ has coefficients in $\mathbb{Z}_{\mathfrak{p}}^*$. Using these facts, the proof then follows the same idea of the proof for a special case in [5, § 3.4].

Assumptions as in Theorem 9.4.1 and Lemma 9.4.2, let $\bar{\kappa}_{\mathfrak{p}}(X) = \prod_{\mu_i<0}|\pi|_{\mathfrak{p}}^{\mu_i}$. We shall show that $\bar{\kappa}_{\mathfrak{p}}(X) = \kappa_{\mathfrak{p}}(X)$, and hence $\kappa_{\mathfrak{p}}(X)$ is unchanged by an element of $M'_{\mathbb{Z}_{\mathfrak{p}}}$, and is computable when X is diagonalized by an element of $M'_{\mathbb{Z}_{\mathfrak{p}}}$.

9.5. As in [5, § 7.7], for fixed j, $1 \leqq j \leqq n$, define an injection i_j of the group SL$(2,\mathbb{K}_{\mathfrak{p}})$ into $G'_{\mathfrak{p}}$ as follows. If $a \in \mathbb{K}_{\mathfrak{p}}$, let

$$u_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}; \quad \textit{also let } \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then SL $(2, \mathbb{K}_{\mathfrak{p}})$ is generated by u_a, $a \in \mathbb{K}_{\mathfrak{p}}$ and σ. We define $i_j(\sigma) = \iota_j$ and $i_j(u_a) = t_{ae_{jj}}$. If we restrict ourselves to the subdomain

$$\{\operatorname{diag}(i, \ldots, i, z, i, \ldots, i) | z \text{ is the } j\text{-th diagonal entry}\},$$

which is isomorphic to the upper half plane, then ι_j and $t_{ae_{jj}}$ play the same roles as σ and u_a do, respectively, on the upper half plane. Thus, i_j is clearly an isomorphism onto its image. Note that $\forall' \mathfrak{p}, \gamma \in \mathrm{SL}(2, \mathbb{Z}_{\mathfrak{p}})$ if and only if $i_j(\gamma) \in G'_{\mathbb{Z}_{\mathfrak{p}}}$.

Lemma 9.5.1: *Assumptions as in* § 9.4. *If* $t_{ae_{jj}} \iota_j \rho \in i_j(\mathrm{SL}\,(2, \mathbb{Z}_{\mathfrak{p}}))$, $a \in \mathbb{K}_{\mathfrak{p}}$, $\rho \in P'_{\mathfrak{p}}$, *then* $|\det(\mathrm{Ad}_{\mathscr{g}'}\rho)|_{\mathfrak{p}} = \bar{\kappa}_{\mathfrak{p}}(ae_{jj})^{-\mathbb{N}}$.

Proof: Let

$$\gamma = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

be mapped by i_j to

$$t_{ae_{jj}} i_j \rho, \; \rho = i_j \left(\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \right) t_{be_{jj}}.$$

Note that

$$i_j \left(\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \right) = \operatorname{diag}(1, \ldots, 1, s, 1, \ldots, 1) \in S,$$

with s in the j-th entry. Then

$$|\det(\mathrm{Ad}_{\mathscr{g}'}\rho)|_{\mathfrak{p}} = \left| \det\left(\mathrm{Ad}_{\mathscr{g}'} i_j \left(\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \right)\right) \right|_{\mathfrak{p}}$$

$$= \left| j\left(*, i_j \left(\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \right)\right) \right|_{\mathfrak{p}}^{-1} = |s^{\mathbb{N}}|_{\mathfrak{p}} = \bar{\kappa}_{\mathfrak{p}}(ae_{jj})^{-\mathbb{N}}.$$

The last equality follows from a direct calculation (cf., [5, § 7.7]).

$\mathrm{SL}(2, \mathbb{K}_{\mathfrak{p}})$ has two types of elements, corresponding to the Bruhat decomposition, one of which is the type of elements γ mentioned above, and the other one is the type of elements in upper triangular form. For the second type of elements, the lemma is trivial.

LEMMA 9.5.2: *$i_j(\mathrm{SL}(2, \mathbb{K}_{\mathfrak{p}}))$ are mutually commutative for all j, $1 \leqq j \leqq n$.*

PROOF: We may check the generators for commutativity.

THEOREM 9.5.3: *Assumptions as in § 9.4. If $t_X \iota \rho \in G'_{\mathbb{Z}_{\mathfrak{p}}}$, $X \in \mathscr{J}'_{\mathfrak{p}}$, $\rho \in P'_{\mathfrak{p}}$, then $|\det(\mathrm{Ad}_{\mathscr{J}'}\rho)|_{\mathfrak{p}} = \bar{\kappa}_{\mathfrak{p}}(X)^{-\mathbb{N}}$.*

PROOF: There exists $\mu \in M'_{\mathbb{Z}_{\mathfrak{p}}}$ such that $\mu \cdot X = \sum c_j e_{jj}$. We have $t_{\mu \cdot X} \iota \mu^* \rho = (\mu^*)^{-1} t_X \iota \rho \in G'_{\mathbb{Z}_{\mathfrak{p}}}$. Since $|\det(\mathrm{Ad}_{\mathscr{J}'}\mu^*)|_{\mathfrak{p}} = 1$, we may assume that X is of diagonal form $\sum c_j e_{jj}$. Choose $\rho_j \in i_j(\mathrm{SL}(2, \mathbb{K}_{\mathfrak{p}}))$ such that $t_{c_j e_j} \iota_j \rho_j \in G'_{\mathbb{Z}_{\mathfrak{p}}}$, then $\prod_j t_{c_j e_{jj}} \iota_j \rho_j \in G'_{\mathbb{Z}_{\mathfrak{p}}}$, and by Lemma 9.5.2, this element is $t_X \iota \prod_j \rho_j$. Note that if both $t_X \iota \rho$ and $t_X \iota \rho'$ are in $G'_{\mathbb{Z}_{\mathfrak{p}}}$, then $\rho^{-1}\rho' \in G'_{\mathbb{Z}_{\mathfrak{p}}}$, and hence $|\det(\mathrm{Ad}_{\mathscr{J}'}\rho)|_{\mathfrak{p}} = |\det(\mathrm{Ad}_{\mathscr{J}'}\rho')|_{\mathfrak{p}}$. Therefore, we may assume that $\rho = \prod_j \rho_j$, and then

$$|\det(\mathrm{Ad}_{\mathscr{J}'}\rho)|_{\mathfrak{p}} = \prod_j |\det(\mathrm{Ad}_{\mathscr{J}'}\rho_j)|_{\mathfrak{p}} = \prod_j \bar{\kappa}_{\mathfrak{p}}(c_j e_{jj})^{-\mathbb{N}} = \bar{\kappa}_{\mathfrak{p}}(X)^{-\mathbb{N}}.$$

COROLLARY: *Assumptions as in § 9.4. If $X \in \mathscr{J}'_{\mathfrak{p}}$, then $\bar{\kappa}_{\mathfrak{p}}(X) = \kappa_{\mathfrak{p}}(X)$.*

PROOF: Let $t_X \iota = \gamma \rho$, then $t_X \iota \rho^{-1} = \gamma \in G'_{\mathbb{Z}_{\mathfrak{p}}}$ $(\forall' \mathfrak{p}, G'_{\mathbb{Z}_{\mathfrak{p}}} = K'_{\mathfrak{p}})$. Thus, by the theorem $\bar{\kappa}_{\mathfrak{p}}(X)^{-\mathbb{N}} = |\det(\mathrm{Ad}_{\mathscr{J}'}\rho^{-1})|_{\mathfrak{p}} = \chi_{\mathfrak{p}}(X) = \kappa_{\mathfrak{p}}(X)^{-\mathbb{N}}$.

9.6. Identify $H_{\mathfrak{p}} = Z(S'_0)_{\mathfrak{p}}$ as a subgroup of $\mathrm{End}(\mathscr{J}'_{\mathfrak{p}})$ by adjoint representation. Let $H' = H_{\mathfrak{p}} \cap \Gamma_{\mathbb{Z}_{\mathfrak{p}}}$, then H' is an open subgroup of $H_{\mathbb{Z}_{\mathfrak{p}}} = \{g \in H_{\mathfrak{p}} | g \cdot \Lambda'_{\mathfrak{p}} \subset \Lambda'_{\mathfrak{p}}\}$. Let $H(m) = \{g \in H_{\mathfrak{p}} | g = e \bmod \pi^m\}$, where e is the identity transformation of $\mathscr{J}'_{\mathfrak{p}}$. Then for m large, $H(m) \subset H'$. Let $\mathscr{J}'_{\mathfrak{p},m}$ denote the set $\pi^{-m}\Lambda'_{\mathfrak{p}}/\Lambda'_{\mathfrak{p}} \backslash \pi^{-(m-1)}\Lambda'_{\mathfrak{p}}/\Lambda'_{\mathfrak{p}}$. Then $\mathscr{J}'_{\mathfrak{p},m}$ is a finite set and is stable under $H'/H(m)$, and hence is a finite union of the orbits of $H'/H(m)$. Moreover, each point of the orbit occurs the same number of times.

LEMMA 9.6: *$\sum_{g \in H'/H(m)} \varepsilon_p(\mathrm{Tr}(\mathbb{T}, g \cdot X))$ is zero for $m > v_{\mathfrak{p}}$, where $v_{\mathfrak{p}}$ is a fixed positive integer depending on $\mathfrak{p}$, and is $1 \forall' \mathfrak{p}$.*

PROOF: It is sufficient to prove that the linear functional $\tau: \mathscr{J}'_{\mathfrak{p}} \to \mathbb{K}_{\mathfrak{p}}$ defined by $\tau(X) = (\mathbb{T}, X)$ satisfies the conditions of [3, Prop. 2] (here we replace $H_{\mathbb{Z}_{\mathfrak{p}}}$ of Prop. 2 by H', a modification which is justifiable). If $X \in \Lambda'_{\mathfrak{p}}$, define $h_X: \mathrm{End}(\mathscr{J}'_{\mathfrak{p}}) \to \mathbb{K}_{\mathfrak{p}}$ by $h_X(g) = \tau(g \cdot X) = (\mathbb{T}, g \cdot X)$ for

$g \in \mathrm{End}(\mathscr{J}'_{\mathfrak{p}})$, then if X is non-zero, the conditions of Prop. 2 are (i) $h_X \not\equiv 0$, (ii) h_X^0 is nowhere tangent to H', where h_X^0 denotes the hyperplane annihilated by h_X, (iii) $\forall' \mathfrak{p}$, if X is a primitive vector of $\Lambda'_{\mathfrak{p}}$, then h_X is primitive relative to H'.

As done in [3, Part 2] and [5, § 10.2] we intend to verify the above three conditions by showing the following: If X is a primitive vector in $\mathscr{J}'_{\mathfrak{p}}$, then we can find some $g \in H(m)$ such that $(\mathbb{T}, g \cdot X - X) \not\equiv 0(\pi^{m+\nu_{\mathfrak{p}}})$ for some fixed positive integer $\nu_{\mathfrak{p}}$, which is $1 \forall' \mathfrak{p}$.

Given $AX_1, AX_2 \in \mathscr{H}(\mathscr{A}_n, J_A)$, we have $(AX_1, AX_2) = (AX_1 A, X_2)$. Thus we may assume that both $\mathbb{T}$ and X of $h_X = (\mathbb{T}, g \cdot X)$ are in $\mathscr{H}(\mathscr{A}_n, J_I)$. Furthermore, for any

$$\mu \in M'_{\mathbb{K}}, h_X(g) = (\mathbb{T}, g \cdot X) = (\mu \cdot T, (\mu^*)^{-1} g \mu^* (\mu^*)^{-1} \cdot X)$$

and $(\mu^*)^{-1} g \mu^* \in H(m)$. Thus we may choose some special μ and assume, from the beginning, that $\mathbb{T}$ is of diagonal form (Theorem 3.4.1, Theorem 2.8 and § 2.4).

Now consider the following two types of elements of $H(m)$:

(i) elements g of $H(m)$, which act on X by

$$g \cdot X = (I + \pi^m y e_{ii}) X (I + \pi^m y e_{ii}),$$

with $y \in \mathbb{Z}_{\mathfrak{p}}^*$;

(ii) elements g of $H(m)$ of the form $(\pi^m y)_{ij}$, with $i \neq j$ and y primitive in $\mathscr{C}_{\mathbb{Z}_{\mathfrak{p}}}$.

Dividing the possible cases into the case when some diagonal element x_{ii} of X is primitive and the case when all diagonal elements of X are divisible by π and some x_{ij}, $i \neq j$, is primitive, it is easy to see that we can find g and $\nu_{\mathfrak{p}}$ which verify the requirement.

COROLLARY: *For any* $\mathfrak{p}$, $S_{\mathfrak{p}} \in \mathbb{Q}$.

PROOF: This follows from [3, Lemma 2].

9.7. From the result of § 9.6 and the fact that $\chi_{\mathfrak{p}}(X) = \chi_{\mathfrak{p}}(g \cdot X)$ for any $g \in H'$, by (3), we know that, $\forall' \mathfrak{p}$,

$$\begin{aligned} S_{\mathfrak{p}} &= \sum_{X \in \pi^{-1}\Lambda'_{\mathfrak{p}}/\Lambda'_{\mathfrak{p}}} \varepsilon_p(\mathrm{Tr}(T, X)) |N(\kappa_{\mathfrak{p}}(X))|_p^{\mathbb{N}l} \\ &= \sum_{Y \in \mathscr{J}'_{\mathbb{Z}_{\mathfrak{p}}}/\pi \mathscr{J}'_{\mathbb{Z}_{\mathfrak{p}}}} \varepsilon_p(\mathrm{Tr}(T, \pi^{-1} Y)) |N(\kappa_{\mathfrak{p}}(\pi^{-1} Y))|_p^{\mathbb{N}l} \end{aligned}$$

$$= \sum_{Y \in \mathscr{J}'_{\mathbb{Z}_\mathfrak{p}}/\pi \mathscr{J}'_{\mathbb{Z}_\mathfrak{p}}} \varepsilon_p(p^{-1}\operatorname{Tr}(T, Y))|N(\kappa_\mathfrak{p}(\pi^{-1}Y))|_p^{\mathbb{N}l}.$$

Observing that, $\forall'\mathfrak{p}$, $\mathscr{J}'_{\mathbb{Z}_\mathfrak{p}}/\mathscr{J}'_{\mathbb{Z}_\mathfrak{p}}$ may be identified with $\mathscr{H}(\mathscr{A}_n, J_A)$ as sets, where $\mathscr{A}$ is the algebra of § 3.1 with $\Phi = \mathbb{F}_{p^\alpha}$, the field of p^α elements, $\alpha = [\mathbb{K}_\mathfrak{p} : \mathbb{Q}_p]$, and that, $\forall'\mathfrak{p}$, $Y \in \mathscr{J}'_{\mathbb{Z}_\mathfrak{p}}/\pi\mathscr{J}'_{\mathbb{Z}_\mathfrak{p}}$ can be diagonalized by an element of $M'_{\mathbb{Z}_\mathfrak{p}}$, we have, by the corollary to Theorem 9.5.3, that, $\forall'\mathfrak{p}$, $|N(\kappa_\mathfrak{p}(\pi^{-1}Y))|_p^{-\mathbb{N}l} = p^{-\alpha R(Y)\mathbb{N}l}$, where $R(Y)$ is the rank of Y as an element of $\mathscr{H}(\mathscr{A}_n, J_A)$. On the other hand, $\mathbb{T}$ may be identified to an element of $\mathscr{H}(\mathscr{A}_n, J_A)$ with maximal rank $\forall'\mathfrak{p}$. By doing so, we may replace $\operatorname{Tr}_{\mathbb{K}_\mathfrak{p}/\mathbb{Q}_p}$ by $\operatorname{Tr}_{\mathbb{F}_{p^\alpha}/\mathbb{F}_p}$, and we have, $\forall'\mathfrak{p}$,

$$S_\mathfrak{p} = \sum_{Y \in \mathscr{H}(\mathscr{A}_n, J_A)} \varepsilon_p(p^{-1}\operatorname{Tr}_{\mathbb{F}_{p^\alpha}/\mathbb{F}_p}(\mathbb{T}, Y))p^{-\alpha R(Y)\mathbb{N}l}, \tag{4}$$

a finite exponential sum to be evaluated in the next section.

10. Exponential sums

10.1. Let $p \neq 2$ be a rational prime and α be a positive integer. Let $\mathbb{F}_{p^\alpha}$ be the field of p^α elements. Then $\mathbb{F}_{p^\alpha}$ consists of all the roots of $x^{p^\alpha} = x$. Let ζ be a primitive root of $x^{p^\alpha - 1} = 1$. Then any element of $\mathbb{F}_{p^\alpha}$ is a power of ζ and it is obvious that we have

LEMMA 10.1.1: *ζ^k is a square element of $\mathbb{F}_{p^\alpha}$ if and only if k is even.*

For a non-zero element $x \in \mathbb{F}_{p^\alpha}$, we define $(x/p)_\alpha = 1$ if x is a square element and $(x/p)_\alpha = -1$ if x is a non-square element. When $\alpha = 1$, then $(x/p) = (x/p)_1$ is the usual Legendre symbol.

Put

$$e_{p^\alpha}(x) = \exp\left(\frac{2\pi i}{p}\operatorname{Tr}_{\mathbb{F}_{p^\alpha}/\mathbb{F}_p}(x)\right) \qquad \text{for } x \in \mathbb{F}_{p^\alpha}.$$

Define

$$\mathfrak{G}_\alpha(a) = \sum_{x \in \mathbb{F}_{p^\alpha}} e_{p^\alpha}(ax^2) \qquad \text{for a fixed } a \in \mathbb{F}^*_{p^\alpha}.$$

Then $\mathfrak{G} = \mathfrak{G}_1(1)$ is the usual Gaussian sum, for which we have

$$\mathfrak{G}^2 = \left(\frac{-1}{p}\right)p.$$

LEMMA 10.1.2: $\mathfrak{G}_\alpha(a) = (a/p)_\alpha \mathfrak{G}_\alpha(1)$.

PROOF: Let

$$R_1 = \sum_{x \neq 0,\text{ square}} e_{p^\alpha}(x), \qquad R_{-1} = \sum_{x:\text{ non-square}} e_{p^\alpha}(x).$$

If a is a square b^2, then

$$\mathfrak{G}_\alpha(a) = \sum_{bx \in \mathbb{F}_{p^\alpha}} e_{p^\alpha}(b^2x^2) = \mathfrak{G}_\alpha(1).$$

If a is non-square, then

$$\mathfrak{G}_\alpha(a) = 1 + 2R_{-1} = 2(1 + R_{-1}) - 1 = -2R_1 - 1 = -\mathfrak{G}_\alpha(1),$$

because $1 + R_1 + R_1 = 0$ and Lemma 10.1.1.

The trace induces an inner product $(x, y) = \operatorname{Tr}(xy)$ on $\mathbb{F}_{p^\alpha}$ over $\mathbb{F}_p$. If $\{\zeta_1, \ldots, \zeta_\alpha\}$ is a basis of $\mathbb{F}_{p^\alpha}$ over $\mathbb{F}_p$, then $\delta_\alpha = \det((\zeta_i, \zeta_j))$ is called the discriminant of $\mathbb{F}_{p^\alpha}$; it is determined up to a square (in $\mathbb{F}_p$) by the choosing of the bases. Thus (δ_α/p) is uniquely determined for the field $\mathbb{F}_{p^\alpha}$. It can be shown that $(\delta_\alpha/p) = (-1)^{\alpha-1}$.

LEMMA 10.1.3: $\mathfrak{G}_\alpha(a) = (-1)^{\alpha-1}(a/p)_\alpha \mathfrak{G}^\alpha$.

PROOF: Choose an orthogonal basis $\{\zeta_1, \ldots, \zeta_\alpha\}$ of $\mathbb{F}_{p^\alpha}$ with respect to the inner product. Write $x = \sum_{i=1}^\alpha x_i\zeta_i$ for $x \in \mathbb{F}_{p^\alpha}$, where $x_i \in \mathbb{F}_p$. Then

$$\mathfrak{G}_\alpha(1) = \sum_{x \in \mathbb{F}_{p^\alpha}} e_{p^\alpha}(x^2) = \sum_{x \in \mathbb{F}_{p^\alpha}} e_p\Big(\sum_{i=1}^{\alpha} x_i^2 \operatorname{Tr}\zeta_i^2\Big) = \prod_{i=1}^{\alpha} \left(\frac{\operatorname{Tr}\zeta_i^2}{p}\right)\mathfrak{G}^\alpha$$

$$= \left(\frac{\delta_\alpha}{p}\right)\mathfrak{G}^\alpha = (-1)^{\alpha-1}\mathfrak{G}^\alpha.$$

The lemma then follows from Lemma 10.1.2.

REMARK: If $a \in \mathbb{F}_p \subset \mathbb{F}_{p^\alpha}$, then

$$\mathfrak{G}_\alpha(a) = \sum_{x \in \mathbb{F}_{p^\alpha}} e_{p^\alpha}(ax^2) = \sum_{x \in \mathbb{F}_{p^\alpha}} e_p\Big(a\sum_{i=1}^{\alpha} x_i^2 \operatorname{Tr}\zeta_i^2\Big) = \left(\frac{a}{p}\right)^\alpha \left(\frac{\delta_\alpha}{p}\right)\mathfrak{G}^\alpha.$$

Thus $(a/p)_\alpha = (a/p)^\alpha$, if $a \in \mathbb{F}_p$.

LEMMA 10.1.4: *If $a \neq 0$ in $\mathbb{F}_{p^\alpha}$, then*

$$\sum_{x \in \mathbb{F}^*_{p^\alpha}} \left(\frac{x}{p}\right)_\alpha e_{p^\alpha}(ax) = \mathfrak{G}_\alpha(a) = (-1)^{\alpha-1} \left(\frac{a}{p}\right)_\alpha \mathfrak{G}^\alpha.$$

PROOF:

$$\sum_{x \neq 0} \left(\frac{x}{p}\right)_\alpha e_{p^\alpha}(ax) = \left(\frac{a}{p}\right)_\alpha \sum_{y = ax} \left(\frac{y}{p}\right)_\alpha e_{p^\alpha}(y) = \left(\frac{a}{p}\right)_\alpha (R_1 - R_{-1})$$

$$= \left(\frac{a}{p}\right)_\alpha (1 + 2R_1) = \left(\frac{a}{p}\right)_\alpha \mathfrak{G}_\alpha(1) = (-1)^{\alpha-1} \left(\frac{a}{p}\right)_\alpha \mathfrak{G}^\alpha = \mathfrak{G}_\alpha(a),$$

where R_1, R_{-1} were defined in Lemma 10.1.2.

10.2. Let $\mathscr{J}_{p^\alpha}$ be a Jordan algebra of § 3.1, with $\Phi = \mathbb{F}_{p^\alpha}$. Define $(X, Y) = \operatorname{tr}(X \circ Y)$, for $X, Y \in \mathscr{J}_{p^\alpha}$. For any fixed $\mathbb{T} \in \mathscr{J}_{p^\alpha}$, we define, for m, n such that $0 \leqq m \leqq n$,

$$S_{p^\alpha}(n, m)(\mathbb{T}) = \sum_{R(X) = m} e_{p^\alpha}((\mathbb{T}, X)), \tag{1}$$

and, if w is an indeterminant, define

$$S_{p^\alpha}(n)(\mathbb{T})(w) = \sum_{m=0}^{n} S_{p^\alpha}(n, m)(\mathbb{T}) w^m. \tag{2}$$

If there is no confusion likely to arise, we write $S_{p^\alpha}(\mathbb{T})$ or $S_{p^\alpha}(\mathbb{T})(w)$ or $S_{p^\alpha}(w)$ or $S_{p^\alpha}(n)(w)$ for $S_{p^\alpha}(n)(\mathbb{T})(w)$, and $S_{p^\alpha}(n, m)$ for $S_{p^\alpha}(n, m)(\mathbb{T})$.

Let $\mathscr{A}$ be the algebra of § 3.1 for $\mathscr{J}_{p^\alpha}$, $\dim \mathscr{A} = n_0$. Let $|f|$ be the discriminant of the norm form f on $\mathscr{A}$, then $\sigma = (|f|/p)_\alpha$ is well defined (not depending on the choice of the basis).

LEMMA 10.2.1: *If $Y \in \mathscr{J}_{p^\alpha}$ is of full rank, then*

$$\sum_{\eta \in \mathscr{A}^n} e_{p^\alpha}(Y[\eta]) = (\sigma((-1)^{\alpha-1} \mathfrak{G}^\alpha)^{n_0})^n \left(\frac{|Y|}{p}\right)_\alpha^{n_0},$$

where we assume $n < 2$ for type D Jordan algebras and $n < 3$ for type E Jordan algebras.

PROOF: If $\mu = (y)_{ij} \in M$, define $\mu\eta = (I + ye_{ij})\eta$. Then it is clear that $(\mu \cdot Y)[\eta] = Y[\mu\eta]$ (cf., Theorem 3.4.3 for type E case). Now, if $\mu = (y_1)_{i_1 j_1} \cdots (y_k)_{i_k j_k} \in M$, we define inductively

$$\mu\eta = (y_1)_{i_1 j_1} \cdots (y_{k-1})_{i_{k-1} j_{k-1}} ((y_k)_{i_k j_k} \eta).$$

It is clear that $(\mu \cdot Y)[\eta] = Y[\mu\eta]$. $\eta \to \mu\eta$ is a non-singular linear transformation of $\mathscr{A}^n$ onto itself; thus if we assume that $\mu^{-1} \cdot Y = A$ is of diagonal form, then we have

$$\sum_{\eta\in\mathscr{A}^n} e_{p^\alpha}(Y[\eta]) = \sum_{\eta\in\mathscr{A}^n} e_{p^\alpha}((\mu \cdot A)[\eta]) = \sum_{\eta\in\mathscr{A}^n} e_{p^\alpha}(A[\mu\eta])$$

$$= \sum_{\xi\in\mathscr{A}^n} e_{p^\alpha}(A[\xi]) = (\sigma((-1)^{\alpha-1}\mathfrak{G}^\alpha)^{n_0})^n \left(\frac{|Y|}{p}\right)_\alpha^{n_0}.$$

LEMMA 10.2.2: *$S_{p^\alpha}(n, m)(\mu \cdot \mathbb{T}) = S_{p^\alpha}(n, m)(\mathbb{T})$ for any $\mu \in M$.*

PROOF: This is because

$$e_{p^\alpha}(\mu \cdot \mathbb{T}, X) = e_{p^\alpha}(\mathbb{T}, \mu^* \cdot X) \quad \text{and} \quad R(\mu^* \cdot X) = R(X)$$

for any $\mu \in M$.

For a given $\mathbb{T} \in \mathscr{J}_{p^\alpha}$, we can find a $\mu \in M$ such that $\mu \cdot \mathbb{T}$ is of diagonal form. Therefore, in calculating exponential sums, it suffices to treat the case when $\mathbb{T}$ is of diagonal form, which we shall assume henceforth.

LEMMA 10.3: *If n_0 is even, then $S_{p^\alpha}(n, n) = (-1)^n(\sigma\mathfrak{G}^{\alpha n_0})^{n(n-1)/2}$.*

PROOF: Let

$$X = \begin{pmatrix} Y & \xi \\ \xi^* & x \end{pmatrix},$$

where Y is of $(n-1)\times(n-1)$. If $R(Y) < n-1$, then $|Y| = 0$ and hence $|X| = -\tilde{Y}[\xi]$. Thus for such Y and a fixed ξ, x can be arbitrary (such that $R(X) = n$). Therefore,

$$\sum_{R(X)=n,\, R(Y)\neq n-1} e_{p^\alpha}((\mathbb{T}, X)) = 0.$$

Thus we may assume that $R(Y) = n-1$ (i.e., $|Y| \neq 0$). Then

$$x|Y| - \tilde{Y}[\xi] = |X| = \tau \neq 0, \qquad x = \tau|Y|^{-1} + Y^{-1}[\xi],$$

and then

$$S_{p^\alpha}(n, n)(\mathbb{T}) = \sum_{R(Y)=n-1,\ \tau\in\mathbb{F}^*{}_{p^\alpha},\ \xi\in\mathscr{A}^{n-1}} e_{p^\alpha}((\mathbb{T}_{n-1}, Y))e_{p^\alpha}((t_n, \tau|Y|^{-1} + Y^{-1}[\xi]))$$

$$= -\sum_{R(Y)=n-1,\, \xi\in\mathscr{A}^{n-1}} e_{p^\alpha}((\mathbb{T}_{n-1}, Y))e_{p^\alpha}((t_n, Y^{-1}[\xi])$$

$$= -(\sigma\mathfrak{G}^{\alpha n_0})^{n-1} \sum_{R(Y)=n-1} e_{p^\alpha}((\mathbb{T}_{n-1}, Y))$$

$$= -(\sigma\mathfrak{G}^{\alpha n_0})^{n-1} S_{p^\alpha}(n-1, n-1)(\mathbb{T}_{n-1})$$

$$= \ldots = (-1)^n(\sigma\mathfrak{G}^{\alpha n_0})^{n(n-1)/2}.$$

Note that we let $\mathbb{T}_{n-1} = \mathrm{diag}\,(t_1, \ldots, t_{n-1})$ when $\mathbb{T}_n = \mathbb{T} = \mathrm{diag}\,(t_1, \ldots, t_n)$.

Write

$$X = \begin{pmatrix} Y & \xi \\ \xi^* & x \end{pmatrix}$$

as before. Assume that $R(X) = m$ and $R(Y) = k$. Then by the proof of the corollary to Lemma 3.5.4, X is M-equivalent to

$$\begin{pmatrix} A & 0 & 0 \\ 0 & 0 & \eta_2 \\ 0 & \eta_2^* & x' \end{pmatrix},$$

where $A = \mathrm{diag}\,(a_1, \ldots, a_k)$, η_2 is $(n-k-1) \times 1$, and $x' = -A^{-1}[\eta_1] + x$ for some η_1 of $k \times 1$. Thus we have

$$R\begin{pmatrix} 0 & \eta_2 \\ \eta_2^* & x' \end{pmatrix} = m-k,$$

which is $\leqq 2$ by Lemma 3.5.4.

LEMMA 10.4:

$$S_{p^\alpha}(n, m) = \sum_{R(Y)=m,\, \eta_1 \in \mathscr{A}^m} e_{p^\alpha}((\mathbb{T}_{n-1}, Y)) e_{p^\alpha}((t_n, A^{-1}[\eta_1])$$

$$+ \sum_{R(Y)=m-1,\, x' \in \mathbb{F}_{p^\alpha}^*,\, \eta_1 \in \mathscr{A}^{m-1}} e_{p^\alpha}((\mathbb{T}_{n-1}, Y)) e_{p^\alpha}((t_n, x' + A^{-1}[\eta_1])).$$

PROOF:

i) If $R(Y) = k = m$, then

$$R\begin{pmatrix} 0 & \eta_2 \\ \eta_2^* & x' \end{pmatrix} = 0$$

which implies that $\eta_2 = 0$, $x' = 0$ and

$$X = \begin{pmatrix} Y & \xi \\ \xi^* & A^{-1}[\eta_1] \end{pmatrix}.$$

Then

$$\sum_{R(X)=m,\, R(Y)=m} e_{p^\alpha}((\mathbb{T}, X)) = \sum_{R(Y)=m,\, \eta_1 \in \mathscr{A}^m} e_{p^\alpha}((\mathbb{T}_{n-1}, Y)) e_{p^\alpha}((t_n, A^{-1}[\eta_1]).$$

ii) If $R(Y) = k = m-1$, then

$$R\begin{pmatrix} 0 & \eta_2 \\ \eta_2^* & x' \end{pmatrix} = 1. \quad \text{If} \quad R\begin{pmatrix} 0 & \eta_2 \\ \eta_2^* & 0 \end{pmatrix} = 1 \quad \text{then} \quad R\begin{pmatrix} 0 & \eta_2 \\ \eta_2^* & z \end{pmatrix} = 1$$

for all $z \in \mathbb{F}_{p^\alpha}$ by Lemma 3.5.2. Thus, fixing such Y and ξ, the exponential sum over all x is zero, since x can be any element of $\mathbb{F}_{p^\alpha}$. If

$$R\begin{pmatrix} 0 & \eta_2 \\ \eta_2^* & 0 \end{pmatrix} = 0$$

then $\eta_2 = 0$ and $x' \neq 0$. Thus

$$\sum_{R(X)=m,\, R(Y)=m-1} e_{p^\alpha}((\mathbb{T}, X))$$

$$= \sum_{R(Y)=m-1,\, x' \in \mathbb{F}_{p^\alpha}^*,\, \eta_1 \in \mathscr{A}^{m-1}} e_{p^\alpha}((\mathbb{T}_{n-1}, Y)) e_{p^\alpha}((t_n, x' + A^{-1}[\eta_1])).$$

iii) If $R(Y) = k = m-2$, then

$$R\begin{pmatrix} 0 & \eta_2 \\ \eta_2^* & x' \end{pmatrix} = 2.$$

The argument for the case

$$R\begin{pmatrix} 0 & \eta_2 \\ \eta_2^* & 0 \end{pmatrix} > 0$$

is the same as that in ii) and the corresponding sum is zero. If

$$R\begin{pmatrix} 0 & \eta_2 \\ \eta_2^* & 0 \end{pmatrix} = 0,$$

then $\eta_2 = 0$ and

$$R\begin{pmatrix} 0 & \eta_2 \\ \eta_2^* & x' \end{pmatrix} \leqq 1,$$

contradicting the assumption.

Combining i), ii) and iii), the lemma follows.

THEOREM 10.5: *If n_0 is even, then*

$$S_{p^\alpha}(w) = \prod_{k=0}^{n-1} (1-(\sigma \mathfrak{G}^{\alpha n_0})^k w).$$

PROOF: By Lemma 10.4, using Lemma 10.2.1, we have

$$\begin{aligned}(3)\quad S_{p^\alpha}(n, m) = & \sum_{R(Y)=m,\, \eta_1 \in \mathscr{A}^m} e_{p^\alpha}((\mathbb{T}_{n-1}, Y)) e_{p^\alpha}((t_n, A^{-1}[\eta_1])) \\ & - \sum_{R(Y)=m-1,\, \eta_1 \in \mathscr{A}^{m-1}} e_{p^\alpha}((\mathbb{T}_{n-1}, Y)) e_{p^\alpha}((t_n, A^{-1}[\eta_1])) \\ = & (\sigma \mathfrak{G}^{\alpha n_0})^m S_{p^\alpha}(n-1, m) - (\sigma \mathfrak{G}^{\alpha n_0})^{m-1} S_{p^\alpha}(n-1, m-1).\end{aligned}$$

Note that, by convention, we define $S_{p^\alpha}(n, m)$ to be zero when $m > n$ or $m < 0$.

Substitute (3) in (2), we have

$$\begin{aligned} S_{p^\alpha}(n)(w) &= \sum_{m=0}^{n} (\sigma \mathfrak{G}^{\alpha n_0})^m S_{p^\alpha}(n-1, m) w^m - (\sigma \mathfrak{G}^{\alpha n_0})^{m-1} S_{p^\alpha}(n-1, m-1) w^m \\ &= \sum_{m=0}^{n-1} S_{p^\alpha}(n-1, m)(\sigma \mathfrak{G}^{\alpha n_0} w)^m - w \sum_{m=0}^{n-1} S_{p^\alpha}(n-1, m)(\sigma \mathfrak{G}^{\alpha n_0} w)^m \\ &= (1-w) S_{p^\alpha}(n-1)(\sigma \mathfrak{G}^{\alpha n_0} w). \end{aligned}$$

Thus, by induction, the theorem follows.

THEOREM 10.6: *For the case of type D, n_0 odd, we have*

$$S_{p^\alpha}(2)(\mathbb{T})(w) = \left(1 + \sigma \mathfrak{G}^{\alpha(n_0+1)} \left(\frac{|\mathbb{T}|}{p}\right)_\alpha w\right)(1-w);\ S_{p^\alpha}(1)(w) = -1.$$

PROOF: By Lemma 10.4,

$$S_{p^\alpha}(2, 2)(\mathbb{T}) = -\sum_{y_1 \neq 0,\, \xi \in \mathscr{A}} e_{p^\alpha}((t_1, y_1)) e_{p^\alpha}((t_2, y_1^{-1}[\xi]))$$

$$= -\sigma((-1)^{\alpha-1}\mathfrak{G}^{\alpha})^{n_0}\left(\frac{t_2}{p}\right)_\alpha \sum_{y_1 \neq 0} e_{p^\alpha}((t_1, y_1))\left(\frac{y_1}{p}\right)_\alpha = -\sigma\mathfrak{G}^{\alpha(n_0+1)}\left(\frac{|\mathbb{T}|}{p}\right)_\alpha.$$

Now,

$$S_{p^\alpha}(2,1)(\mathbb{T}) = -S_{p^\alpha}(2,0)(\mathbb{T}) - S_{p^\alpha}(2,2)(\mathbb{T}) = -1+\sigma\mathfrak{G}^{\alpha(n_0+1)}\left(\frac{|\mathbb{T}|}{p}\right)_\alpha.$$

Hence

$$S_{p^\alpha}(2)(\mathbb{T})(w) = 1 + \left(-1+\sigma\mathfrak{G}^{\alpha(n_0+1)}\left(\frac{|\mathbb{T}|}{p}\right)_\alpha\right)w - \sigma\mathfrak{G}^{\alpha(n_0+1)}\left(\frac{|\mathbb{T}|}{p}\right)_\alpha w^2$$

$$= \left(1+\sigma\mathfrak{G}^{\alpha(n_0+1)}\left(\frac{|\mathbb{T}|}{p}\right)_\alpha w\right)(1-w).$$

10.7. For the case of type $B(\sigma = 1)$, the computation is much more complicated. By Lemma 10.4,

$$S_{p^\alpha}(n,m) = \sum_{R(Y)=m,\ \xi\in\mathscr{A}^m} e_{p^\alpha}((\mathbb{T}_{n-1}, Y))e_{p^\alpha}((t_n, A^{-1}[\xi]))$$

$$- \sum_{R(Y)=m-1,\ \xi\in\mathscr{A}^{m-1}} e_{p^\alpha}((\mathbb{T}_{n-1}, Y))e_{p^\alpha}((t_n, A^{-1}[\xi]))$$

$$= ((-1)^{\alpha-1}\mathfrak{G}^{\alpha})^m\left(\frac{t_n}{p}\right)_\alpha^m \sum_{R(Y)=m}\left(\frac{|A|}{p}\right)_\alpha e_{p^\alpha}((\mathbb{T}_{n-1}, Y))$$

$$-((-1)^{\alpha-1}\mathfrak{G}^{\alpha})^{m-1}\left(\frac{t_n}{p}\right)_\alpha^{m-1} \sum_{R(Y)=m-1}\left(\frac{|A|}{p}\right)_\alpha e_{p^\alpha}((\mathbb{T}_{n-1}, Y)).$$

Now, apply the argument of Lemma 10.4 again, but noting that, when A and x' for X are replaced by B and y' for Y, then in i) of the proof of Lemma 10.4, we have $|A| = |B|$, and in ii) we have $y'|B| = |A|$. Thus

$$S_{p^\alpha}(n,m) = ((-1)^{\alpha-1}\mathfrak{G}^{\alpha})^m\left(\frac{t_n}{p}\right)_\alpha$$

$$\times\left\{\sum_{R(Z)=m,\ \eta\in\mathscr{A}^m}\left(\frac{|A|}{p}\right)_\alpha e_{p^\alpha}((\mathbb{T}_{n-2}, Z))e_{p^\alpha}((t_{n-1}, A^{-1}[\eta]))\right.$$

$$\left.+\sum_{R(Z)=m-1,\ \eta\in\mathscr{A}^{m-1}}\left(\frac{|A|}{p}\right)_\alpha e_{p^\alpha}((\mathbb{T}_{n-2}, Z))e_{p^\alpha}((t_{n-1}, |A||B|^{-1}+B^{-1}[\eta]))\right\}$$

$$-((-1)^{\alpha-1}\mathfrak{G}^{\alpha})^{m-1}\left(\frac{t_n}{p}\right)_{\alpha}^{m-1}$$

$$\times\left\{\sum_{R(Z)=m-1,\,\eta\in\mathscr{A}^{m-1}}\left(\frac{|A|}{p}\right)_{\alpha}e_{p^\alpha}((\mathbb{T}_{n-2},Z))e_{p^\alpha}((t_{n-1},A^{-1}[\eta]))\right.$$

$$\left.+\sum_{R(Z)=m-2,\,\eta\in\mathscr{A}^{m-2}}\left(\frac{|A|}{p}\right)_{\alpha}e_{p^\alpha}((\mathbb{T}_{n-2},Z))e_{p^\alpha}((t_{n-1},|A||B|^{-1}+B^{-1}[\eta]))\right\}$$

$$=((-1)^{\alpha-1}\mathfrak{G}^{\alpha})^m\left(\frac{t_n}{p}\right)_{\alpha}^{m}\left\{((-1)^{\alpha-1}\mathfrak{G}^{\alpha})^m\left(\frac{t_{n-1}}{p}\right)_{\alpha}^{m}S_{p^\alpha}(n-2,m)\right.$$

$$\left.+((-1)^{\alpha-1}\mathfrak{G}^{\alpha})^m\left(\frac{t_{n-1}}{p}\right)_{\alpha}^{m}S_{p^\alpha}(n-2,m-1)\right\}$$

$$-((-1)^{\alpha-1}\mathfrak{G}^{\alpha})^{m-1}\left(\frac{t_n}{p}\right)_{\alpha}^{m-1}\left\{((-1)^{\alpha-1}\mathfrak{G}^{\alpha})^{m-1}\left(\frac{t_{n-1}}{p}\right)_{\alpha}^{m-1}S_{p^\alpha}(n-2,m-1)\right.$$

$$\left.+((-1)^{\alpha-1}\mathfrak{G}^{\alpha})^{m-1}\left(\frac{t_{n-1}}{p}\right)_{\alpha}^{m-1}S_{p^\alpha}(n-2,m-2)\right\}.$$

Thus we have

(4) $S_{p^\alpha}(n,m)(\mathbb{T})$

$$=\mathfrak{G}^{2\alpha m}\left(\frac{t_nt_{n-1}}{p}\right)_{\alpha}^{m}\{S_{p^\alpha}(n-2,m)(\mathbb{T}_{n-2})+S_{p^\alpha}(n-2,m-1)(\mathbb{T}_{n-2})\}$$

$$-\mathfrak{G}^{2\alpha(m-1)}\left(\frac{t_nt_{n-1}}{p}\right)_{\alpha}^{m-1}\{S_{p^\alpha}(n-2,m-1)(\mathbb{T}_{n-2})+S_{p^\alpha}(n-2,m-2)(\mathbb{T}_{n-2})\}.$$

THEOREM 10.8: *For the case of type B, we have*

$$S_{p^\alpha}(2n+1)(\mathbb{T})(w)=(1-w)\prod_{k=1}^{n}(1-\mathfrak{G}^{4k\alpha}w^2).$$

PROOF: First, by induction, we want to prove that

$$S_{p^\alpha}(2n+1,2m)+S_{p^\alpha}(2n+1,2m+1)=0. \tag{5}$$

For $m=0$, $S_{p^\alpha}(2n+1,0)=1$; on the other hand, by (4),

$$S_{p^\alpha}(2n+1,1)=\mathfrak{G}^{2\alpha}\left(\frac{t_{2n+1}t_{2n}}{p}\right)_{\alpha}(S_{p^\alpha}(2n-1,1)+1)-1,$$

from which, by induction on n, we have $S_{p^\alpha}(2n+1, 1) = -1$. Thus (5) holds true for $m = 0$. Then by (4), the induction process goes, and (5) is true in general. Now also by (4), we have

$$(6)\quad S_{p^\alpha}(2n+1, 2m) = -S_{p^\alpha}(2n+1, 2m+1) \\ = \mathfrak{G}^{4\alpha m}\{S_{p^\alpha}(2n-1, 2m-1) + S_{p^\alpha}(2n-1, 2m)\}.$$

Substitute (6) in (2), we have

$$\begin{aligned} S_{p^\alpha}(2n+1)(w) &= \sum_{m=0}^{n} S_{p^\alpha}(2n+1, 2m)w^{2m} + \sum_{m=0}^{n} S_{p^\alpha}(2n+1, 2m+1)w^{2m+1} \\ &= (1-w)\sum_{m=0}^{n} S_{p^\alpha}(2n+1, 2m)w^{2m}. \end{aligned}$$

Let

$$A_n(w) = \sum_{m=0}^{n} S_{p^\alpha}(2n+1, 2m)w^{2m}.$$

Then

$$\begin{aligned} A_n(w) &= \sum_{m=0}^{n} \mathfrak{G}^{4\alpha m}(S_{p^\alpha}(2n-1, 2m) + S_{p^\alpha}(2n-1, 2m-1))w^{2m} \\ &= \sum_{m=0}^{n-1} \mathfrak{G}^{4\alpha m} S_{p^\alpha}(2n-1, 2m)w^{2m} - \sum_{m=1}^{n-1} \mathfrak{G}^{4\alpha m} S_{p^\alpha}(2n-1, 2m-2)w^{2m} \\ &= A_{n-1}(\mathfrak{G}^2 w) - \mathfrak{G}^{4\alpha} w^2 A_{n-1}(\mathfrak{G}^2 w) \\ &= (1-\mathfrak{G}^{4\alpha w^2})A_{n-1}(\mathfrak{G}^{2w}) = \ldots = \prod_{k=1}^{n}(1-\mathfrak{G}^{4\alpha k w^2}), \end{aligned}$$

and the theorem follows.

THEOREM 10.9: *For the case of type B, we have*

$$S_{p^\alpha}(2n)(\mathbb{T})(w) = (1-w)\left(1 + \mathfrak{G}^{2\alpha n}\left(\frac{|\mathbb{T}|}{p}\right)_\alpha w\right)\prod_{k=1}^{n-1}(1-\mathfrak{G}^{4\alpha k}w^2).$$

We need a lemma.

Let $N_k = \{1, 2, \ldots, k\}$ be the set of the integers from 1 to k. If M is a subset of N_k, let $|M|$ = member of elements in M, and

$$S(M) = \sum_{m\in M} m, \qquad \mathfrak{G}(k, m) = \sum_{M\subset N_k,\, |M|=m} \mathfrak{G}^{4\alpha S(M)}.$$

(The notations N_k, M are used, in the above meaning, in this article only.)

LEMMA: $\mathfrak{G}(k, m) = \mathfrak{G}^{4\alpha m}(\mathfrak{G}(k-1, m) + \mathfrak{G}(k-1, m-1))$.

PROOF:

$$\begin{aligned}\mathfrak{G}(k, m) &= \sum_{M\subset N_k,\, |M|=m} \mathfrak{G}^{4\alpha S(M)} \\ &= \sum_{M\subset N_k-N_1,\, |M|=m} \mathfrak{G}^{4\alpha S(M)} + \mathfrak{G}^{4\alpha} \sum_{M'\subset N_k-N_1,\, |M'|=m-1} \mathfrak{G}^{4\alpha S(M')} \\ &= \mathfrak{G}^{4\alpha m} \sum_{M_1\subset N_{k-1},\, |M_1|=m} \mathfrak{G}^{4\alpha S(M_1)} + \mathfrak{G}^{4\alpha m} \sum_{M'_1\subset N_{k-1},\, |M'_1|=m-1} \mathfrak{G}^{4\alpha S(M'_1)} \\ &= \mathfrak{G}^{4\alpha m}(\mathfrak{G}(k-1, m) + \mathfrak{G}(k-1, m-1)).\end{aligned}$$

PROOF OF THE THEOREM: Let

$$B_n(w) = (1-w)\left(1 + \mathfrak{G}^{2\alpha n}\left(\frac{|\mathbb{T}_{2n}|}{p}\right)_\alpha w\right)\prod_{k=1}^{n-1}(1 - \mathfrak{G}^{4\alpha k}w^2).$$

We contend that $S_{p^\alpha}(2n)(w) = B_n(w)$, hence our theorem.

For $n = 1$, $S_{p^\alpha}(2)$ is the $S_{p^\alpha}(2)$ in Theorem 10.6, with $\sigma = 1$, $n_0 = 1$. Thus $S_{p^\alpha}(2) = B_1$. Suppose $S_{p^\alpha}(2j) = B_j$ is true up to $j = n-1$, then the comparison of the coefficients of $S_{p^\alpha}(2n-2)$ and B_{n-1} gives

$$\begin{aligned}&S_{p^\alpha}(2n-2, 2m)(\mathbb{T}_{2n-2}) \\ &\qquad = (-1)^m\left(\frac{|\mathbb{T}_{2n-2}|}{p}\right)_\alpha \mathfrak{G}^{2\alpha(n-1)}\mathfrak{G}(n-2, m-1) + (-1)^m\mathfrak{G}(n-2, m)\end{aligned}$$

and

$$S_{p^\alpha}(2n-2, 2m+1)(\mathbb{T}_{2n-2}) = (-1)^m\left\{\left(\frac{|\mathbb{T}_{2n-2}|}{p}\right)_\alpha \mathfrak{G}^{2\alpha(n-1)} - 1\right\}\mathfrak{G}(n-2, m).$$

Hence

(7) $\quad S_{p^\alpha}(2n-2, 2m+1)(\mathbb{T}_{2n-2}) + S_{p^\alpha}(2n-2, 2m)(\mathbb{T}_{2n-2})$

$$= (-1)^m \left(\frac{|\mathbb{T}_{2n-2}|}{p}\right)_\alpha \mathfrak{G}^{2\alpha(n-1)}(\mathfrak{G}(n-2,m)+\mathfrak{G}(n-2,m-1))$$

$$= (-1)^m \left(\frac{|\mathbb{T}_{2n-2}|}{p}\right)_\alpha \mathfrak{G}^{2\alpha(n-1)-4\alpha m}\mathfrak{G}(n-1,m)$$

and

$$(8)\quad S_{p^\alpha}(2n-2,2m)+S_{p^\alpha}(2n-2,2m-1)$$
$$= (-1)^m(\mathfrak{G}(n-2,m)+\mathfrak{G}(n-2,m-1)) = (-1)^m\mathfrak{G}^{-4\alpha m}\mathfrak{G}(n-1,m).$$

From (4), we have

$$\begin{aligned}
&S_{p^\alpha}(2n,2m)(\mathbb{T}_{2n})+S_{p^\alpha}(2n,2m+1)(\mathbb{T}_{2n})\\
&\quad = \mathfrak{G}^{4\alpha m}(S_{p^\alpha}(2n-2,2m)(\mathbb{T}_{2n-2})+S_{p^\alpha}(2n-2,2m-1)(\mathbb{T}_{2n-2}))\\
&\qquad -\mathfrak{G}^{2\alpha(2m-1)}\left(\frac{t_{2n}t_{2n-1}}{p}\right)_\alpha (S_{p^\alpha}(2n-2,2m-1)(\mathbb{T}_{2n-2})\\
&\qquad +S_{p^\alpha}(2n-2,2m-2)(\mathbb{T}_{2n-2}))\\
&\qquad +\mathfrak{G}^{2\alpha(2m+1)}\left(\frac{t_{2n}t_{2n-1}}{p}\right)_\alpha (S_{p^\alpha}(2n-2,2m+1)(\mathbb{T}_{2n-2})\\
&\qquad +S_{p^\alpha}(2n-2,2m)(\mathbb{T}_{2n-2}))\\
&\qquad -\mathfrak{G}^{4\alpha m}(S_{p^\alpha}(2n-2,2m)(\mathbb{T}_{2n-2})+S_{p^\alpha}(2n-2,2m-1)(\mathbb{T}_{2n-2}))\\
&\quad = -\mathfrak{G}^{2\alpha(2m-1)}\left(\frac{t_{2n}t_{2n-1}}{p}\right)_\alpha (-1)^{m-1}\left(\frac{|\mathbb{T}_{2n-2}|}{p}\right)_\alpha\\
&\qquad\qquad \times\mathfrak{G}^{2\alpha(n-1)}\mathfrak{G}^{-4\alpha(m-1)}\mathfrak{G}(n-1,m-1)\\
&\qquad +\mathfrak{G}^{2\alpha(2m+1)}\left(\frac{t_{2n}t_{2n-1}}{p}\right)_\alpha (-1)^{m}\left(\frac{|\mathbb{T}_{2n-2}|}{p}\right)_\alpha\\
&\qquad\qquad \times\mathfrak{G}^{2\alpha(n-1)}\mathfrak{G}^{-4\alpha m}\mathfrak{G}(n-1,m)\\
&\quad = (-1)^m\left(\frac{|\mathbb{T}_{2n}|}{p}\right)_\alpha \mathfrak{G}^{2\alpha n}(\mathfrak{G}(n-1,m-1)+\mathfrak{G}(n-1,m)).
\end{aligned}$$

Thus (7) holds true for $2n$. Similarly, (8) holds true for $2n$.
Write

$$B_n(w) = \sum_{m=0}^{2n} b_m w^m, \qquad S_{p^\alpha}(2n)(w) = \sum_{m=0}^{2n} s_m w^m, \quad \text{where} \quad s_m = S_{p^\alpha}(n,m).$$

Then we have

$$b_{2m}+b_{2m+1} = s_{2m}+s_{2m+1} \quad \text{and} \quad b_{2m}+b_{2m-1} = s_{2m}+s_{2m-1}.$$

Since $b_0 = s_0 = 1$, we have, by induction, $b_k = s_k$ for all k. Thus $B_n = S_{p^\alpha}(2n)$, hence our theorem.

11. Volume of the Lattice

11.1. Because our purpose is to prove the rationality of the Fourier coefficients $a_l(\mathbb{T}) = a_l^{(n)}(\mathbb{T})$ of § 8.7, it is enough to determine $v(\Lambda)$ up to a multiplicative rational constant, an observation which simplifies the calculation.

First, since Λ and $\mathscr{J}_{\mathbb{Z}}(= U_{\mathbb{Z}})$ are commeasurable, we may take $\Lambda = \mathscr{J}_{\mathbb{Z}}$ and a $\mathbb{Q}$-basis of $\mathscr{J}_{\mathbb{Q}}$ as a basis of Λ.

11.2. Now, we restrict ourselves to the cases of § 2.8, where the $\mathbb{K}$-structure of the Jordan algebra $\mathscr{J}'_{\mathbb{K}}$ is of the form $\mathscr{H}(\mathscr{A}_n, J_A)$.

In terms of the 'standard' $\mathbb{R}$-basis of $(R_{\mathbb{K}/\mathbb{Q}}\mathbb{K}) \otimes \mathbb{R} \cong \prod_\sigma {}^\sigma\mathbb{K} \otimes \mathbb{R}$, a $\mathbb{Q}$-basis of $R_{\mathbb{K}/\mathbb{Q}}\mathbb{K}$ has a volume equal to $\Delta^{\frac{1}{2}}$, the square root of the discriminant Δ of $\mathbb{K}$ over $\mathbb{Q}$. By 'standard' we mean that based on which we define the euclidean measure dX of § 8.6.

Again, let $\mathbb{N} = (n-1)n_0+2$, then $\mathscr{H}(\mathscr{A}_n, J_A)$ has $n\mathbb{N}/2$ dimensions over $\mathbb{Q}$, which contributes a factor $\Delta^{n\mathbb{N}/4}$ to the volume $v(\Lambda)$ of Λ.

Next, turn to the isomorphism between

$$\mathscr{H}(\mathscr{A}_n, J_A) \otimes \mathbb{R} \quad \text{and} \quad \mathscr{H}(\mathscr{A}_n, J_I) \otimes \mathbb{R}$$

of § 2.8. It obviously contributes another factor $(N(|A|))^{\mathbb{N}/2}$ to $v(\Lambda)$, where $N(|A|)$ is the norm down to $\mathbb{Q}$ of the determinant $|A|$ of A.

Finally, let the norm form of $\mathscr{A}_{\mathbb{K}}$ be

$$f(x) = \sum_{i=1}^{n_0} \alpha_i x_i^2, \quad \text{where} \quad x = \sum_{i=1}^{n_0} x_i e_i$$

is with respect to a $\mathbb{K}$-basis $\{e_1, \ldots, e_{n_0}\}$ of $\mathscr{A}$, and α_i are totally positive numbers of $\mathbb{K}$. (Note: for type D, in the notation of § 2.8, the norm form would be $\alpha^2 - f$.) Then the isomorphism which maps $x = (x_i)$ to $(\sqrt{\alpha_i} x_i)$ of $\mathscr{C}$ of § 2.1, transfers a K-basis of $\mathscr{A}$ onto the standard R-basis of $\mathscr{C}_n^{n_0}$. This isomorphism contributes the third factor $(N(|f|))^{n(n-1)/4}$ to $v(\Lambda)$, where $|f| = \prod_{i=1}^{n_0} \alpha_i$. Thus in conclusion, we have

THEOREM 11.2: *Assumptions as in* § 2.8,

$$v(\Lambda) \equiv (N(|A|))^{\mathbb{N}/2}(N(|f|))^{n(n-1)/4}\Delta^{n\mathbb{N}/4}, \text{ mod } \mathbb{Q}^*.$$

We shall devote the rest of Section 11 to the cases of type A.

11.3. Notations as in §§ 2.4 and 2.9. If we use the notation

$$\mathbb{D} = (\mathbb{K}(\zeta, \theta), \tau, b) = (1, \eta, \ldots, \eta^{s-1})/\mathbb{K}(\zeta, \theta),$$

then

$$\mathbb{D}_{\sigma_{\mathbb{K}}} = ({}^\sigma\mathbb{K}({}^\sigma\zeta, {}^\sigma\theta), \tau, {}^\sigma b) = ({}^\sigma 1, {}^\sigma\eta, \ldots, {}^\sigma\eta^{s-1})/{}^\sigma\mathbb{K}({}^\sigma\zeta, {}^\sigma\theta).$$

Let $\{{}^\sigma w_i | 1 \leqq i \leqq s^2\}$ denote the set $\{R({}^\sigma\zeta_j{}^\sigma\eta^k), 0 \leqq j, k \leqq s-1\}$ and $\{{}^\sigma u_i | 1 \leqq i \leqq s^2\}$ the set of the standard $\mathbb{C}$-basis of $\mathbb{D}_{\sigma_{\mathbb{K}}} \otimes \mathbb{R}$, when the latter is identified with $M(s, \mathbb{C})$ by Lemma 2.9. Let $\{\xi_i, 1 \leqq i \leqq d\}$ be a basis of $\mathbb{K}$ over $\mathbb{Q}$, then $\{{}^\sigma\xi_i, 1 \leqq i \leqq d\}$ is a basis of ${}^\sigma\mathbb{K}$ over $\mathbb{Q}$. Then the $\mathbb{Q}$-basis of $(R_{\mathbb{K}/\mathbb{Q}}\mathbb{D})_{\mathbb{R}}$ is

$$\{\sum_\sigma {}^\sigma\xi_i{}^\sigma w_j, \sum_\sigma {}^\sigma\xi_i{}^\sigma\theta^\sigma w_j, 1 \leqq i \leqq d, 1 \leqq j \leqq s^2\}.$$

Let

$${}^\sigma w_j = \sum_k {}^\sigma c_{kj}{}^\sigma u_k, {}^\sigma c_{kj} \in \mathbb{C},$$

then

$$\sum_\sigma {}^\sigma\xi_i{}^\sigma w_j = \sum_{\sigma, k} {}^\sigma c_{kj}{}^\sigma\xi_i{}^\sigma u_k$$

and

$$\sum_\sigma {}^\sigma\xi_i{}^\sigma\theta^\sigma w_j = \sum_{\sigma, k} {}^\sigma c_{kj}{}^\sigma\xi_i{}^\sigma\theta^\sigma u_k.$$

Thus the lattice formed by the $\mathbb{Q}$-basis of $(R_{\mathbb{K}/\mathbb{Q}}\mathbb{D})_{\mathbb{R}}$ has a volume, with respect to $\{{}^\sigma u_k, \sqrt{-1}{}^\sigma u_k\}$,

$$v_1 = \left| \begin{matrix} {}^\sigma\xi_i\mathbb{R}({}^\sigma c_{kj}) & {}^\sigma\xi_i\mathbb{I}({}^\sigma c_{kj}) \\ {}^\sigma\xi_i\sqrt{-1}{}^\sigma\theta\mathbb{I}({}^\sigma c_{kj}) & {}^\sigma\xi_i(-\sqrt{-1}){}^\sigma\theta\mathbb{R}({}^\sigma c_{kj}) \end{matrix} \right|_{(i,j),(\sigma,k)}$$

$$= \left| \begin{matrix} {}^\sigma\xi_i\mathbb{R}({}^\sigma W) & {}^\sigma\xi_i\mathbb{I}({}^\sigma W) \\ {}^\sigma\xi_i(\sqrt{-1}{}^\sigma\theta\mathbb{I}({}^\sigma W) & {}^\sigma\xi_i(-\sqrt{-1}){}^\sigma\theta\mathbb{R}({}^\sigma W) \end{matrix} \right|,$$

where ${}^\sigma W = ({}^\sigma c_{kj})$, $1 \leqq k, j \leqq s^2$; $\mathbb{R}({}^\sigma c_{kj})$ = real part of ${}^\sigma c_{kj}$, $\mathbb{I}({}^\sigma c_{kj})$ = imaginary part of ${}^\sigma c_{kj}$; $\mathbb{R}({}^\sigma W) = (\mathbb{R}({}^\sigma c_{kj}))_{k,j}$, $\mathbb{I}({}^\sigma W) = (\mathbb{I}({}^\sigma c_{kj}))_{k,j}$.

By some manipulations on rows and columns of the determinant, we have

$$v_1 \equiv \Delta^{s^2} N_{\mathbb{K}/\mathbb{Q}}(-\theta^2)^{s^2/2} \prod_\sigma |{}^\sigma W||{}^\sigma \bar{W}|. \tag{1}$$

11.4. To calculate $|W|$, let $\{\zeta_j\eta^k, 0 \leqq j, k \leqq s-1\}$ be ordered in the way below: $\zeta_0, \ldots, \zeta_{s-1}, \zeta_0\eta, \ldots, \zeta_{s-1}\eta, \ldots, \zeta_{s-1}\eta^{s-1}$. Let e_{ij} be the standard basis of $M(s, \mathbb{C})$ ordered by $e_{11}, \ldots, e_{1s}, e_{21}, \ldots, e_{ss}$. If $(a_{ij}) \in M(s, \mathbb{C})$, define $L((a_{ij})) = (a_{11}, \ldots, a_{1s}, a_{21}, \ldots, a_{ss})$, a row vector.

LEMMA 11.4.1: *Let $E_i, F \in M(s, \mathbb{C})$, $1 \leqq i \leqq s$, then*

$$\begin{pmatrix} L(E_1F) \\ L(E_2F) \\ \vdots \\ L(E_sF) \end{pmatrix} = \begin{pmatrix} L(E_1) \\ L(E_2) \\ \vdots \\ L(E_s) \end{pmatrix} \begin{pmatrix} F & & & 0 \\ & F & & \\ & & \ddots & \\ 0 & & & F \end{pmatrix}.$$

PROOF: This is an easy consequence from the definition of L and the matrix multiplications.

LEMMA 11.4.2: $|W| \equiv b^{s(s-1)/2}$, mod $\mathbb{K}^*$.

PROOF: From the orderings of the two bases and the above lemma, we have

$$|W| = \begin{vmatrix} L(\zeta_0) \\ \vdots \\ L(\zeta_{s-1}) \\ L(\zeta_0\eta) \\ \vdots \\ L(\zeta_{s-1}\eta^{s-1}) \end{vmatrix} = \begin{vmatrix} L(\zeta_0) \\ \vdots \\ L(\zeta_{s-1}) \\ L(\zeta_0) \\ \vdots \\ L(\zeta_{s-1}) \end{vmatrix} \begin{vmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \eta & & \\ & & & & \ddots & \\ & & & & & \eta^{s-1} \end{vmatrix}$$

$$= \begin{vmatrix} A_{11} & \cdots\cdots & A_{1s} \\ \vdots & & \vdots \\ (A_{j1}\eta^{j-1}) & \ldots & (A_{js}\eta^{j-1}) \\ \vdots & & \vdots \\ (A_{ss}\eta^{s-1}) & \ldots & (A_{ss}\eta^{s-1}) \end{vmatrix},$$

with $A_{ij} \in M(s, \mathbb{K})$.

Note that for any $X \in M(s, \mathbb{K})$, we have

$$X\eta = \eta\eta^{-1}X\eta = \eta(\beta\tau)^{-1}X(\beta\tau) = \eta\tau^{-1}X\tau = \eta X_1,$$

with $X_1 = \tau^{-1}X\tau \in M(s, \mathbb{K})$.

Thus, by putting $A_{jk}\eta^{j-1} = \eta^{j-1}B_{jk}$, with $B_{jk} \in M(s, \mathbb{K})$, we have

$$|W| = |\eta^{j-1}B_{jk}| = \begin{vmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \eta & & & \\ & & & & \ddots & & \\ & & & & & \eta & \\ & & & & & & \ddots \\ & & & & & & & \eta^{s-1} \end{vmatrix} |B_{jk}| \equiv |\eta|^{s(s-1)/2}$$

$$\equiv |\eta\beta^{-1}|^{s(s-1)/2}|\beta|^{s(s-1)/2} \equiv |\tau|^{s(s-1)/2}b^{s(s-1)/2} \equiv b^{s(s-1)/2}, \bmod \mathbb{K}^*.$$

Lemma 11.4.3: $v_1 \equiv \Delta^{s^2}N_{\mathbb{K}/\mathbb{Q}}(-\theta^2)^{s^2/2}$, mod $\mathbb{Q}^*$.

Proof: This follows directly from (1) and Lemma 11.4.2.

11.5. Now, we turn to the calculation of the volume of the lattice of a basis of the set $\mathscr{H}((R_{\mathbb{K}/\mathbb{Q}}\mathbb{K})_{\mathbb{R}})$ of the symmetric elements of $(R_{\mathbb{K}/\mathbb{Q}}\mathbb{K})_{\mathbb{R}}$.

Let $\{{}^{\sigma}w_i, 1 \leqq i \leqq s^2\}$ be the ${}^{\sigma}\mathbb{K}$-basis of $\mathscr{H}(\mathbb{D}_{\sigma_{\mathbb{K}}} \otimes \mathbb{R})$ and $\{{}^{\sigma}u_i, 1 \leqq i \leqq s^2\}$ be the standard $\mathbb{R}$-basis of $\mathscr{H}(\mathbb{D}_{\sigma_{\mathbb{K}}} \otimes \mathbb{R})$ when it is identified with the hermitian symmetric elements $\mathscr{H}(s, \mathbb{C})$ of $M(s, \mathbb{C})$. As before, let $\{{}^{\sigma}\xi_i, 1 \leqq i \leqq d\}$ be the $\mathbb{Q}$-basis of ${}^{\sigma}\mathbb{K}$ over $\mathbb{Q}$. Then the $\mathbb{Q}$-basis of $\mathscr{H}(R_{\mathbb{K}/\mathbb{Q}}\mathbb{K})$ is $\{\sum_{\sigma}{}^{\sigma}\xi_i{}^{\sigma}\mathrm{w}_j\}$.

Let ${}^{\sigma}\mathrm{w}_j = \sum {}^{\sigma}\mathrm{c}_{kj}{}^{\sigma}\mathrm{u}_k$, ${}^{\sigma}\mathrm{c}_{kj} \in \mathbb{C}$, then $\sum {}^{\sigma}\xi_i{}^{\sigma}w_i = \sum_{\sigma,k}{}^{\sigma}c_{kj}{}^{\sigma}\xi_i{}^{\sigma}u_k$.

Thus the lattice of the $\mathbb{Q}$-basis of $\mathscr{H}(R_{\mathbb{K}/\mathbb{Q}}\mathbb{K})$ has a volume

$$(2)\qquad v_2 = |{}^{\sigma}\xi_i{}^{\sigma}c_{kj}|_{(i,j),(\sigma,k)} = |{}^{\sigma}\xi_i{}^{\sigma}W| \equiv \Delta^{s^2/2}\prod_{\sigma}|{}^{\sigma}W|, \bmod \mathbb{Q}^*,$$

where, again, ${}^{\sigma}W = ({}^{\sigma}c_{kj})_{k,j}$.

11.6. In general, let $S^{(i)} = (s_{jk}^{(i)}) \in \mathscr{H}(s, \mathbb{C})$, $1 \leqq i \leqq s^2$, be s^2 linearly independent matrices, let the $\mathbb{R}$-basis of $\mathscr{H}(s, \mathbb{C})$ be ordered by

$$e_{11}, e_{22}, \ldots, e_{ss}, e_{12}+e_{21}, e_{13}+e_{31}, \ldots, e_{(s-1)s} + e_{s(s-1)}, (e_{12}-e_{21})\sqrt{-1}, \ldots, (e_{(s-1)s}-e_{s(s-1)})\sqrt{-1}.$$

Then the volume of $\{S^{(i)}\}$ with respect to this basis is

$$(3)\qquad \left|\begin{matrix} s_{11}^{(1)}, \ldots, s_{ss}^{(1)}, \ldots, \mathbb{R}(s_{jk}^{(1)}), \ldots, \mathbb{I}(s_{jk}^{(1)}), \ldots \\ * \end{matrix}\right|_{j<k}$$

$$\begin{aligned}
&= |\ldots, s_{jj}^{(i)}, \ldots, s_{jk}^{(i)}, \ldots, \mathbb{I}(s_{jk}^{(i)}), \ldots|_{j<k} \\
&= (-2i)^{-s(s-1)/2}|\ldots, s_{jj}^{(i)}, \ldots, s_{jk}^{(i)}, \ldots, -2i\mathbb{I}(s_{jk}^{(i)}), \ldots| \\
&= (-2i)^{-s(s-1)/2}|\ldots, s_{jj}^{(i)}, \ldots, s_{jk}^{(i)}, \ldots, s_{kj}^{(i)}, \ldots| \\
&= \pm(2i)^{-s(s-1)/2}\left|\begin{matrix} L(S^{(1)}) \\ \vdots \\ L(S^{(s^2)}) \end{matrix}\right|.
\end{aligned}$$

11.7. We are ready to calculate $|W|$ of § 11.5. First, we deal with the case when s is odd. Then the $\mathbb{K}$-basis of $\mathscr{H}(\mathbb{D}_{\mathbb{R}})$ is:

$$\zeta_i, 0 \leqq i \leqq s-1; \quad \zeta_i\eta^j + b^{-1}a^{(j)}\zeta_{s+i-j}\eta^{s-j}, \quad (\zeta_i\eta^j - b^{-1}a^{(j)}\zeta_{s+i-j}\eta^{s-j})\theta,$$
$$1 \leqq j \leqq (s-1)/2, \quad 0 \leqq i \leqq s-1, \quad \text{where} \quad a^{(j)} = {}^{\tau^s}a\,{}^{\tau^{s-1}}a \ldots {}^{\tau^{j+1}}a.$$

From (3), we have

$$\begin{aligned}
|W| &= \pm(2i)^{-s(s-1)/2}\left|\begin{matrix} L(\zeta_i) \\ L(\zeta_i\eta^j + b^{-1}a^{(j)}\zeta_{s+i-j}\eta^{s-j}) \\ L(\zeta_i\eta^j - b^{-1}a^{(j)}\zeta_{s+i-j}\eta^{s-j})\theta \end{matrix}\right| \\
&= \pm(2i)^{-s(s-1)/2}\left|\begin{matrix} L(\zeta_i) \\ L(\zeta_i\eta^j + b^{-1}a^{(j)}\zeta_{s+i-j}\eta^{s-j}) \\ 2L(\zeta_i\eta^j)\theta \end{matrix}\right|
\end{aligned}$$

$$= \pm(\theta i)^{s(s-1)/2} \begin{vmatrix} L(\zeta_i) \\ L(\zeta_i \eta^j) \\ L(b^{-1}a^{(j)}\zeta_{s+i-j}\eta^{s-j}) \end{vmatrix}$$

$$= \pm(\theta i b^{-1})^{s(s-1)/2} \begin{vmatrix} L(\zeta_i) \\ L(\zeta_i \eta^j) \\ L(a^{(j)}\zeta_{s+i-j}\eta^{s-j} \end{vmatrix}$$

$$\equiv (\theta i b^{-1})^{s(s-1)/2} |\eta|^{s(s-1)/2}, \bmod \mathbb{K}^*.$$

The last step follows by the same argument used in the proof of Lemma 11.4.2. We know that

$$|\eta|^{s(s-1)/2} \equiv b^{s(s-1)/2}, \bmod \mathbb{K}^*.$$

Thus

$$|W| \equiv (-\theta^2)^{s(s-1)/4}, \bmod \mathbb{K}^*,$$

from which we have

LEMMA 11.7: *If s is odd,* $v_2 \equiv \Delta^{s^2/2} N_{\mathbb{K}/\mathbb{Q}}(-\theta^2)^{s(s-1)/4}, \bmod \mathbb{Q}^*$.

11.8. For s even, the $\mathbb{K}$-basis of $\mathscr{H}(\mathbb{D}_{\mathbb{R}})$ is:

$$\zeta_i, 0 \leqq i \leqq s-1; \quad \zeta_i\eta^j + b^{-1}a^{(j)}\zeta_{s+i-j}\eta^{s-j}, \quad (\zeta_i\eta^j - b^{-1}a^{(j)}\zeta_{s+i-j}\eta^{s-j})\theta,$$
$$1 \leqq j \leqq (s-2)/2, \quad 0 \leqq i \leqq s-1; \quad \zeta_i\eta^{s/2} + b^{-1}a^{(s/2)}\zeta_{s/2+i}\eta^{s/2},$$
$$(\zeta_i\eta^{s/2} - b^{-1}a^{(s/2)}\zeta_{s/2+i}\eta^{s/2})\theta, \quad 0 \leqq i \leqq (s-2)/2.$$

By a similar calculation we have

LEMMA 11.8: *For s even, we also have*

$$v_2 \equiv \Delta^{s^2/2}(N_{\mathbb{K}/\mathbb{Q}}(-\theta^2))^{s(s-1)/4}, \bmod \mathbb{Q}^*.$$

11.9. We have $m(m-1)/2$ off diagonal elements of $\mathscr{H}(R_{\mathbb{K}/\mathbb{Q}}\mathbb{D})$, each of which contributes to $v(\Lambda)$, by Lemma 11.4.3, a factor $\Delta^{s^2} N_{\mathbb{K}/\mathbb{Q}}(-\theta^2)^{s^2/2}$, mod $\mathbb{Q}^*$, and m diagonal elements of $\mathscr{H}(R_{\mathbb{K}/\mathbb{Q}}\mathbb{D})$, each of which contributes to $v(\Lambda)$, by Lemma 11.7 and Lemma 11.8, a factor

$$\Delta^{s^2/2} N_{\mathbb{K}/\mathbb{Q}}(-\theta^2)^{s(s-1)/4}, \bmod \mathbb{Q}^*.$$

The total contribution is $N(-\theta^2)^{n(n-1)/4}\Delta^{n\mathbb{N}/4}$, mod $\mathbb{Q}^*$, which is the volume of Λ with respect to the $\mathbb{R}$-basis of the product of $\mathscr{H}(\mathbb{C}_n, J_{AB})$ in § 2.9.

We have another factor of $v(\Lambda)$ when we go from $\mathscr{H}(\mathbb{C}_n, J_{AB})$ to $\mathscr{H}(\mathbb{C}_n, J_I)$. This factor is $N(|AB|)^{\mathbb{N}/2} = N(|A|)^{\mathbb{N}/2}$. Thus, for type A, we also have

THEOREM 11.9: *Assumptions as in* § 2.9,

$$v(\Lambda) \equiv (N(|A|))^{\mathbb{N}/2}(N(-\theta^2))^{n(n-1)/4}\Delta^{n\mathbb{N}/4}, \text{ mod } \mathbb{Q}^*.$$

Combining Theorem 11.2 and Theorem 11.9, we have

THEOREM 11.10: *For any type, we have*

$$v(\Lambda) \equiv (N(|A|))^{\mathbb{N}/2}(N(|f|))^{n(n-1)/4}\Delta^{n\mathbb{N}/4}, \text{ mod } \mathbb{Q}^*,$$

where f is the norm form of $\mathscr{A}_{\mathbb{K}}$.

12. Rationality of the Fourier coefficients

Having done the calculations of the exponential sums and the volume of the lattice, we are ready to use the values of L-functions to prove the rationality of the Fourier coefficients.

12.1. Let $\mathbb{K}$ be a totally real algebraic number field of degree d over $\mathbb{Q}$, Δ be its discriminant. Let $\mathfrak{p}$ be a prime ideal of $\mathbb{K}$, then denote the number of elements in the residue field of $\mathbb{K}_{\mathfrak{p}}$ by $N\mathfrak{p}$. If $a \in \mathbb{K}_{\mathfrak{p}}$, then define $(a/\mathfrak{p}) = 1$ or -1 according as a is a square or a non-square element of $\mathbb{K}_{\mathfrak{p}}$. Define

$$L_a(k) = \prod_{\mathfrak{p}}\left(1 - \left(\frac{a}{\mathfrak{p}}\right)(N\mathfrak{p})^{-k}\right), \qquad k > 1, \tag{1}$$

and

$$L_a^+(k) = \prod_{\mathfrak{p}}\left(1 + \left(\frac{a}{\mathfrak{p}}\right)(N\mathfrak{p})^{-k}\right), \quad k > 1. \tag{2}$$

If a is totally positive, then we know [15, § 23] that

$$L_a(k) \equiv \Delta^{\frac{1}{2}}(N(a))^{\frac{1}{2}}\pi^{dk}, \text{ mod } \mathbb{Q}^*, \text{ if } k \text{ is even,} \tag{3}$$

and

$$(4) \qquad L_{-a}(k) \equiv \Delta^{\frac{1}{2}}(N(a))^{\frac{1}{2}}\pi^{dk}, \bmod \mathbb{Q}^*, \text{ if } k \text{ is odd.}$$

Observing that $L_a(k)L_a^+(k) = L_1(2k)$, we also have

$$(5) \qquad L_a^+(k) \equiv N(a)^{\frac{1}{2}}\pi^{dk}, \bmod \mathbb{Q}^*, \text{ if } k \text{ is even,}$$

and

$$(6) \qquad L_{-a}^+(k) \equiv \equiv N(a)^{\frac{1}{2}}\pi^{dk}, \bmod \mathbb{Q}^*, \text{ if } k \text{ is odd.}$$

THEOREM 12.2: *$a_l(\mathbb{T})$ are rational numbers.*

PROOF: By § 8.7, we may write $a_l(\mathbb{T}) = v(\Lambda)\psi S$, where

$$\psi = (2\pi i)^{dn\mathbb{N}l}\pi^{-dn(\mathbb{N}-2)/4}\prod_{j=0}^{n-1}\gamma(\mathbb{N}l - jn_0/2)^{-d}|\mathbb{T}|^{\mathbb{N}l-\mathbb{N}/2},$$

and

$$S = \sum_{X\in\mathscr{J}_{\mathbb{Q}}/\Lambda'} \varepsilon((\mathbb{T}, X))\chi(X)^l.$$

If $Y \in \mathscr{J}'_{\mathbb{K}}$, we also use Y to denote its image in $\mathscr{J}'_{\mathbb{K}_\mathfrak{p}}$. If $Y \in \mathscr{J}'_{\mathbb{Z}^n}$, we use the same symbol Y to denote its image in $\mathscr{J}'_{\mathbb{F}_{N\mathfrak{p}}}$, when its coordinates are projected canonically into the residue field of $\mathbb{K}_\mathfrak{p}$.

We have $\mathbb{T} = ({}^\sigma A {}^\sigma \mathbb{T}') \in \prod_\sigma \mathscr{H}(\mathscr{A}_n, J_{\sigma_A})$. Then $\forall'\mathfrak{p}$, we have [§ 9.7]

$$\begin{aligned} S_\mathfrak{p} &= \sum_{X\in\mathscr{J}'_{\mathbb{F}_{p^\alpha}}} \varepsilon_p(p^{-1}\operatorname{Tr}(A\mathbb{T}', AX))p^{-\alpha R(X)\mathbb{N}l} \\ &= \sum_{X\in\mathscr{J}'_{\mathbb{F}_{p^\alpha}}} \varepsilon_p(p^{-1}\operatorname{Tr}(A\mathbb{T}'A, X))p^{-\alpha R(X)\mathbb{N}l} \\ &= S_{p^\alpha}(A\mathbb{T}'A)(w), \end{aligned}$$

where $N\mathfrak{p} = p^\alpha$, $w = (N\mathfrak{p})^{-\mathbb{N}l} = p^{-\alpha\mathbb{N}l}$.

Thus, by the corollary of § 9.6, we have

$$(7) \qquad S = \prod_\mathfrak{p} S_\mathfrak{p} \equiv \prod_\mathfrak{p} S_{N\mathfrak{p}}(A\mathbb{T}'A)(w), \bmod \mathbb{Q}^*.$$

Using the results of § 10, we know that S is a product of L-functions, mod $\mathbb{Q}^*$; then by the values provided by 12.1, we may write

$$S \equiv \Delta^{a_1}(N(|A|))^{b_1}(N(|\mathbb{T}'|))^{c_1}(N(|f|))^{d_1}\pi^{e_1}, \bmod \mathbb{Q}^*,$$

where $a_1, \ldots, e_1$ are rational numbers.

Likewise $\psi \equiv (N(|\mathbb{T}'|))^{c_2}\pi^{e_2}$, mod $\mathbb{Q}^*$, and by Theorem 11.10,

$$v(A) \equiv \Delta^{a_3}(N(|A|))^{b_3}(N(|f|))^{d_3}, \bmod \mathbb{Q}^*.$$

It is routine to check that $\sum a_i \equiv \sum b_i \equiv \sum c_i \equiv \sum d_i \equiv 0$, mod $\mathbb{Z}$, and $\sum e_i = 0$, which implies the rationality of the Fourier coefficients.

COROLLARY: *If Γ is maximal discrete in G_h, then the Satake compactification of $\mathfrak{T}/\Gamma$ has a biregularly equivalent projective model defined over the rational number field.*

This follows immediately from the theorem and from the main result of [4].

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