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## LOCAL CONTRACTIBILITY OF SPACES OF HOMEOMORPHISMS

David B. Gauld

### Abstract

In this paper we give a new and simple proof of local contractibility of the space of homeomorphisms of a finite polyhedron. We find that the local contraction preserves PL homeomorphisms so we obtain the new result that the space of PL homeomorphisms of a finite polyhedron is locally contractible.

### 1. Introduction

If  $X$  is a topological space, let  $\mathcal{H}(X)$  denote the group of homeomorphisms of  $X$  with the compact-open topology. If  $X$  is a polyhedron then  $\text{PL}(X)$  is the subgroup of PL homeomorphisms.

**THEOREM (1):** *Let  $K$  be a finite polyhedron. Then for each neighbourhood  $\mathcal{U}$  of 1 in  $\mathcal{H}(K)$  there is a neighbourhood  $\mathcal{V}$  of 1 and a map*

$$\varphi: \mathcal{V} \times [0, 1] \rightarrow \mathcal{U}$$

*satisfying*

- (i)  $\varphi(h, 0) = h$  for each  $h \in \mathcal{V}$ ;
- (ii)  $\varphi(h, 1) = 1$  for each  $h \in \mathcal{V}$ ;
- (iii)  $\varphi(1, t) = 1$  for each  $t \in [0, 1]$ ;
- (iv)  $\varphi(\{h\} \times [0, t])$  is a PL isotopy for each  $h \in \text{PL}(K) \cap \mathcal{V}$  and each  $t \in [0, 1)$ .

**COROLLARY:** *For a finite polyhedron  $K$ , the spaces  $\mathcal{H}(K)$  and  $\text{PL}(K)$  are locally contractible.*

The corollary, together with theorem 1.9 of Geoghegan [3, p. 466] and the theorem of Haver [4, p. 281] tell us that  $\text{PL}(K)$  is an absolute neighbourhood retract.

An alternative proof that  $\mathcal{H}(K)$  is locally contractible appears in [5] and an alternative proof of local contractibility of  $\text{PL}(M)$  for a compact PL manifold  $M$  appears in [2].

We now give an indication of the proof of theorem (1). Suppose that  $h: K \rightarrow K$  is a homeomorphism which is near the identity 1. Using the idea in Section 8 of [1] we modify  $h$  in a neighbourhood of each vertex of  $K$  to  $h_1$  so that  $h_1$  is very close to 1 in a small neighbourhood of the 0-skeleton. The process is then repeated to modify  $h_1$  in a neighbourhood of the barycentre of each 1-simplex to  $h_2$  so that  $h_2$  is very close to 1 in a small neighbourhood of the 1-skeleton. In this way we obtain an isotopy from  $h$  to a homeomorphism which is very close to 1 on the whole of  $K$ . Repeating the whole process, we obtain a sequence of isotopies in which the initial homeomorphism of an isotopy is the end product of the previous isotopy and the final homeomorphism of an isotopy is much closer to 1 than the initial homeomorphism. These isotopies may then be stacked together to give the map  $\varphi$ .

Extensions and variations of theorem (1) are possible. For example, by choosing the auxiliary functions in Proposition (5) to be smooth, one obtains

**THEOREM (2):** *The group of diffeomorphisms of a compact differentiable manifold with the  $C^r$ -topology (any  $r \geq 0$ ) is locally contractible.*

One can also give an alternative proof of Siebenmann's theorem [5, p. 132], as well as obtain relative and non-compact versions.

## 2. Notation and preliminary results

All spaces of embeddings (and homeomorphisms) are assumed to have the compact-open topology. When we say that an embedding  $h'$  constructed from an embedding  $h$  is *canonical* we mean that the function  $h \mapsto h'$  on the embedding spaces is continuous.

If  $X$  is a topological space, by the (*open*) cone over  $X$ , denoted  $cX$ , is meant the quotient space obtained from the disjoint union of  $X \times [0, \infty)$  and the singleton  $\{0\}$  by identifying each point of  $X \times \{0\}$  with 0. The point 0 is the *vertex* of  $cX$ . If  $A \subset X$  then  $cA \subset cX$  in a natural way. If  $\alpha \in [0, \infty)$ , let  $c_\alpha X$  (resp.  $\bar{c}_\alpha X$ ) be the subspace of  $cX$  obtained from  $X \times [0, \alpha)$  (resp.  $X \times [0, \alpha]$ ). If  $y \in cX$  and  $\alpha \in [0, \infty)$ , define  $\alpha y$  as follows:

$y$  is the image of some  $(x, \beta) \in X \times [0, \infty)$ . Let  $\alpha y$  denote the image in  $cX$  of  $(x, \alpha\beta)$ . For  $B \subset cX$  and  $\alpha \in [0, \infty)$ , let

$$\alpha B = \{\alpha y \in cX \mid y \in B\}.$$

If  $y \in cX$  is the image of  $(x, \beta)$ , write  $|y| = \beta$ . Let

$$s_\alpha B = \{y \in cB \mid |y| = \alpha\} = \bar{c}_\alpha B - c_\alpha B.$$

**PROPOSITION (3):** *Let  $X$  be a metrisable space. Then  $X$  is compact if and only if its cone  $cX$  is metrisable by a metric  $d$  satisfying:*

- (i)  $d(\alpha y, \alpha z) = \alpha d(y, z)$  for any  $y, z \in cX$  and  $\alpha \in [0, \infty)$ ;
- (ii)  $d(\alpha y, \beta y) = |\alpha - \beta| \cdot |y|$  for any  $y \in cX$  and  $\alpha, \beta \in [0, \infty)$ .

**PROOF:** If  $X$  is compact then we may embed  $X$  in the unit sphere of the Hilbert space  $l_2$ ;  $cX$  then embeds in  $l_2$  as all rays from the origin through  $X$ . The metric on  $cX$  induced from the norm on  $l_2$  satisfies (i) and (ii).

Conversely, if  $cX$  is metrisable then the vertex 0 has a countable neighbourhood basis, so  $X \times \{0\}$  has a countable neighbourhood basis in  $X \times [0, 1]$ . Thus  $X$  is sequentially compact and hence compact.

In the sequel, whenever we require a metric on  $cX$  for a compact metrisable space  $X$  we will assume that it satisfies conditions (i) and (ii) of Proposition (3).

If  $K$  is a finite polyhedron and  $\sigma$  a simplex of  $K$ , let  $b^\sigma$  denote the barycentre of  $\sigma$  and  $K'$  the first barycentric subdivision of  $K$ . Let  $lk(b^\sigma, K')$  denote the link of  $b^\sigma$  in  $K'$ . Since  $K$  is finite, we may choose any metric on  $K$ ; so suppose  $K$  is metrised by a metric which is linear on each simplex and which assigns to each edge the length 1. Then the closed star of  $b^\sigma$  in  $K'$  may be naturally identified with  $\bar{c}_{\frac{1}{2}} lk(b^\sigma, K')$ .

We require the following result which is 1.7 of [5].

**PROPOSITION (4):** *Let  $h: F \rightarrow F'$  be an open embedding of locally compact locally connected Hausdorff spaces. Let  $C \subset F$  be compact. If  $g: F \rightarrow F'$  is another open embedding sufficiently near  $h$  in the compact-open topology then  $h(C) \subset g(F)$ . If, further,  $g = h$  outside  $C$ , then  $h(F) = g(F)$ .*

### 3. The basic construction

The following result is essentially Edwards' wrapping process, cf. [5, Proposition 4.9] and [1, Section 8], except that we stop short of the final wrapping step.

PROPOSITION (5): Let  $\alpha, \delta$  and  $r$  be positive real numbers with  $r < 1$  and let  $A$  be a closed subset of the compact, locally connected, metrisable space  $X$ . Let  $U$  be an open neighbourhood of  $\bar{c}_\alpha A - c_{\alpha r^8} A$  in  $cX$ . Then there is a neighbourhood  $W$  of  $A$  in  $X$  such that for all sufficiently small  $\varepsilon > 0$  and all open embeddings  $h: c_\alpha X \rightarrow cX$  within  $\varepsilon$  of the inclusion  $i$  and within  $\delta\varepsilon$  of  $i$  on  $U$ , there is an isotopy  $h_t: c_\alpha X \rightarrow cX$  satisfying:

- (i)  $h_t = i$  if  $h = i$ ;
- (ii)  $h_0 = h$ ;
- (iii)  $h_t$  is canonical;
- (iv)  $h_t|_{c_\alpha X - c_{\alpha r^4} X} = h|_{c_\alpha X - c_{\alpha r^4} X}$ ;
- (v)  $h_1(x) = r^3 h(x/r^3)$  if  $x \in \bar{c}_{\alpha r^7} X$ ;
- (vi)  $h_t$  is within  $6\varepsilon$  of  $i$ ;
- (vii)  $h_1|_{c_\alpha W - c_{\alpha r^7} W}$  is within  $6\delta\varepsilon$  of  $i$ ;
- (viii) if  $X$  is a polyhedron and  $h$  is PL then so is  $h_t$ .

PROOF: Let  $W$  be a neighbourhood of  $A$  in  $X$  small enough so that the closure of  $c_\alpha W - c_{\alpha r^8} W$  in  $cX$  lies in  $U$ . Suppose  $\varepsilon$  is a small positive number and  $h: c_\alpha X \rightarrow cX$  is an open embedding within  $\varepsilon$  of  $i$  and within  $\delta\varepsilon$  of  $i$  on  $U$ .

Define  $\bar{\kappa}, \bar{\lambda}: [0, \infty) \rightarrow [0, \infty)$  as follows:  $\bar{\kappa}(y) = y/r^3$ ;  $\bar{\lambda}$  is multiplication by  $r^3$  on  $[0, \alpha r^3]$ , takes  $[\alpha r^3, \alpha r^2]$  linearly onto  $[\alpha r^6, \alpha r^2]$  and is the identity on  $[\alpha r^2, \infty)$ . Let  $\bar{\lambda}_t, \bar{\mu}_t: [0, \infty) \rightarrow [0, \infty)$  be PL isotopies satisfying:

- (a)  $\bar{\lambda}_0 = 1$  and  $\bar{\lambda}_1 = \bar{\lambda}$ ;
- (b)  $\bar{\lambda}_t = 1$  on  $[\alpha r^2, \infty)$ ;
- (c)  $\bar{\mu}_t = (\bar{\lambda}_t \bar{\kappa})^{-1}$ .

Define  $\lambda_t, \mu_t: cX \rightarrow cX$  by  $\lambda_t(0) = \mu_t(0) = 0$  and if  $x \in cX - \{0\}$ , let

$$\lambda_t(x) = \frac{\bar{\lambda}_t(|x|)x}{|x|}, \quad \mu_t(x) = \frac{\bar{\mu}_t(|x|)x}{|x|}.$$

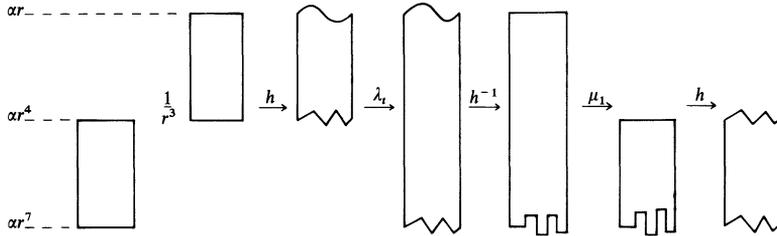
Define the required isotopy  $h_t: c_\alpha X \rightarrow cX$  by

$$h_t(x) = \begin{cases} h\mu_t h^{-1} \lambda_t h(x/r^3) & \text{if } |x| \leq \alpha r^4 \\ h(x) & \text{if } |x| \geq \alpha r^4, \end{cases}$$

see figure 1.

Provided  $\varepsilon$  is small enough, the first line of the definition of  $h_t(x)$  is meaningful by Proposition (4). If  $|x| = \alpha r^4$  then  $x/r^3 \in s_{\alpha r} X$  so if  $\varepsilon$  is small enough,  $h(x/r^3) \in c_\alpha X - c_{\alpha r^2} X$  on which  $\lambda_t = 1$ ; thus

$$h^{-1} \lambda_t h(x/r^3) = x/r^3,$$

Figure 1. Definition of  $h_1$  on  $c_{\alpha r^4}X - c_{\alpha r^7}X$ 

and application of  $\mu_t$  here is multiplication by  $r^3$ , so

$$h\mu_t h^{-1}\lambda_t h(x/r^3) = h(x)$$

when  $|x| = \alpha r^4$ . Thus  $h_t$  is well-defined.

The family  $h_t$  is clearly an isotopy satisfying (i) to (v) and (viii). Proof of satisfaction of (vi) is similar to that of (vii). We will verify the latter.

Suppose  $x \in c_{\alpha}W - c_{\alpha r^7}W$ . If  $|x| \geq \alpha r^4$  then  $h_t(x) = h(x)$  so  $h_t(x)$  is within  $\delta\varepsilon < 6\delta\varepsilon$  of  $x$ . If  $|x| \leq \alpha r^4$  then  $x/r^3 \in c_{\alpha r}W - c_{\alpha r^4}W$ , so

$$d(h(x/r^3), x/r^3) < \delta\varepsilon.$$

Since  $\lambda_1$  expands by at most a factor of 4, this implies

$$d(\lambda_1 h(x/r^3), \lambda_1(x/r^3)) < 4\delta\varepsilon.$$

Thus

$$d(h^{-1}\lambda_1 h(x/r^3), \lambda_1(x/r^3)) < 5\delta\varepsilon,$$

provided  $\varepsilon$  is small enough, so, since  $\mu_1$  does not increase distances, we have

$$d(\mu_1 h^{-1}\lambda_1 h(x/r^3), \mu_1 \lambda_1(x/r^3)) < 5\delta\varepsilon,$$

and hence

$$d(h_1(x), x) < 6\delta\varepsilon,$$

as required by (vii).

REMARKS: There is nothing special about the constant 6 appearing in (vi) and (vii) of Proposition (5) and in Proposition (6) below except that it

is independent of the embedding  $h$ . One could reduce the size of this constant by adjusting the auxiliary functions  $\lambda_t$  and  $\mu_t$ , although it would have to exceed 3, this being the number of applications of  $h$  (or its inverse) in the definition of  $h_t$ . As was pointed out by the referee, the scale  $\alpha r, \alpha r^2, \alpha r^3, \dots$  is not necessary: in fact Edwards and Kirby, and Siebenmann use a linear scale. However the above scale appears most suited to our application as it better respects the cone structure we impose on our polyhedra.

**PROPOSITION (6):** *Let  $\alpha$  and  $\delta$  be positive real numbers and let  $A$  be a closed subset of the compact, locally connected, metrisable space  $X$ . Let  $U$  be an open neighbourhood of  $s_\alpha A$  in  $cX$ . Then there is a neighbourhood  $V$  of  $c_\alpha A$  in  $cX$  such that for each sufficiently small  $\varepsilon > 0$  and each open embedding  $h: c_\alpha X \rightarrow cX$  within  $\varepsilon$  of the inclusion  $i$  and within  $\delta\varepsilon$  of  $i$  on  $U$ , there is an isotopy  $h_t: c_\alpha X \rightarrow cX$  of open embeddings satisfying:*

- (i)  $h_t = i$  if  $h = i$ ;
- (ii)  $h_0 = h$ ;
- (iii)  $h_t$  is canonical;
- (iv)  $h_t$  agrees with  $t$  near  $s_\alpha X$ ;
- (v)  $h_t$  is within  $6\varepsilon$  of  $i$ ;
- (vi)  $h_1|V$  is within  $6\delta\varepsilon$  of  $i$ ;
- (vii) if  $X$  is a polyhedron and  $h$  is PL then so is  $h_t$ .

**PROOF:** Choose  $r < 1$  so that  $\bar{c}_\alpha A - c_{\alpha r^8} A \subset U$ , and let  $n$  be a positive integer so that  $r^{3n} \leq 6\delta$ . Let  $W$  be the neighbourhood of  $A$  given by Proposition (5) and let

$$V = c_\alpha W \cup c_{\alpha r^{3n+4}} X.$$

For  $\varepsilon > 0$  sufficiently small, if  $h: c_\alpha X \rightarrow cX$  is an open embedding within  $\varepsilon$  of  $i$  and within  $\delta\varepsilon$  of  $i$  on  $U$ , we will construct a canonical isotopy  $h_t: c_\alpha X \rightarrow cX$  of open embeddings parametrised by  $[0, n]$  so as to satisfy conditions (i) to (vii) above (but with  $h_n$  in place of  $h_1$  in (vi)). By reparametrising the isotopy we obtain the desired result. Given  $x \in c_\alpha X$ ,  $t \in [0, n]$ , say  $t \in [k, k+1]$ , let

$$h_t(x) = \begin{cases} h_k(x) & \text{if } |x| \geq \alpha r^{3k+4} \\ r^{3k} h_{t-k}(x/r^{3k}) & \text{if } |x| \leq \alpha r^{3k+4}. \end{cases}$$

The isotopy  $h_{t-k}$  in the second line of this definition is that given by Proposition (5). We have used condition (v) of Proposition (5) to enable us

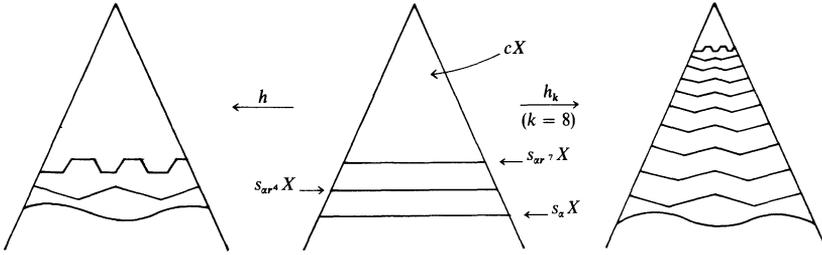


Figure 2. The repetitive nature of  $h_k$

to isotope  $h$  to an embedding which repeats itself as we move toward the vertex of the cone. See Figure 2.

Satisfaction of most of the conditions of Proposition (6) follows from the corresponding condition in Proposition (5). Condition (vi) of Proposition (6) follows from condition (vii) of Proposition (5) for points outside  $c_{ar^{3n+4}}X$  and for  $x \in c_{ar^{3n+4}}X$ , we have

$$h_n(x) = r^{3n}h(x/r^{3n}),$$

so since

$$d(h(x/r^{3n}), x/r^{3n}) < \varepsilon,$$

$$d(h_n(x), x) < r^{3n}\varepsilon \leq 6\delta\varepsilon,$$

by choice of  $n$ .

#### 4. Proof of the main results

PROOF OF THEOREM (1): Suppose  $\dim K = n$ . For a homeomorphism  $h: K \rightarrow K$  sufficiently close to 1, construct a canonical isotopy  $h_t: K \rightarrow K$ ,  $t \in [-1, n]$ , with  $h_{-1} = h$  and  $h_n$  twice as close to 1 as was  $h$ , by induction up the skeleton of  $K$  as follows. For each vertex  $v$  of  $K$ , recall from the end of Section 2 that the closed star of  $v$  in  $K'$  is identified with  $\bar{c}_{\frac{1}{3}}lk(v, K')$ . Applying Proposition (6) to  $h$  on this cone gives us an isotopy  $\mathcal{H}_t$ ,  $t \in [-1, 0]$ , from  $h$  to  $h_0$  which is very close to 1 in a neighbourhood of  $v$ . By constructing this isotopy simultaneously about each vertex of  $K$  we obtain the isotopy  $h_t$ ,  $t \in [-1, 0]$ , from  $h = h_{-1}$  to  $h_0$  which is very close to 1 in a neighbourhood of the 0-skeleton of  $K$ .

Inductively, if  $h_t$  has been constructed for  $t \in [-1, k]$  so that  $h_k$  is very close to 1 in a neighbourhood of the  $k$ -skeleton of  $K$ , and  $\sigma$  is a

$(k+1)$ -simplex of  $K$ , then apply Proposition (6) to  $h_k$  on the cone  $\bar{c}_{\frac{1}{2}}lk(b^\sigma, K')$ , and simultaneously on the corresponding cones for the other  $(k+1)$ -simplices of  $K$ , to obtain an isotopy  $h_t, t \in [k, k+1]$ , so that  $h_{k+1}$  is very close to 1 in a neighbourhood of the  $(k+1)$ -skeleton.

By ‘very close’ above, one might mean the following: if  $h$  is within  $\varepsilon$  of 1, we want  $h_k$  to be within  $\varepsilon/2.6^{n-k}$  of 1 in a neighbourhood of the  $k$ -skeleton. Further details are left to the reader.

If we set  $k = n$  in our definition of ‘very close’, we see that the isotopy  $h_t$  takes  $h_{-1} = h$  to  $h_n$  which is within  $\varepsilon/2$  of 1 on a neighbourhood of the  $n$ -skeleton of  $K$ , i.e. on all of  $K$ . By reparametrising, we obtain an isotopy  $h_t, t \in [0, \frac{1}{2}]$ , from  $h_0 = h$  so that  $h_{\frac{1}{2}}$  is twice as close to 1 as was  $h$ . Repeating the process over and over, we obtain a canonical isotopy  $h_t, t \in [0, 1)$ , from  $h_0 = h$  so that  $h_{1-1/2^k}$  is twice as close to 1 as is  $h_{1-1/2^{k-1}}$ . We can then set  $h_1 = 1$  to obtain the required canonical isotopy  $\varphi(h, t)$ .

**PROOF OF THEOREM (2):** For this we need smooth versions of Propositions (5) and (6). These are easily attained by making  $\lambda_t$  and  $\mu_t$  in the proof of Proposition (5) smooth. Condition (v) of Proposition (5) actually holds in a neighbourhood of  $\bar{c}_{\sigma r} X$  so when we piece together the isotopies in the proof of Proposition (6), this will be done smoothly.

### *Relative versions*

If, for example,  $K$  is a finite polyhedron and  $A, B$  are closed subsets of  $K$  so that  $B$  is a neighbourhood of  $A$  then any homeomorphism of  $K$  sufficiently close to 1 which is already 1 on  $B$  can be deformed to 1 leaving  $A$  fixed. This construction is carried out in the usual way, i.e. choose a subdivision of  $K$  so fine that no simplex meets both  $A$  and  $K - B$ . Proceed as in the proof of theorem (1). Since the homeomorphism is already 1 on the closed star of  $A$ , the deformation will leave this set, and hence also  $A$ , fixed. In the case where  $A$  is a subpolyhedron of  $K$  we can dispense with the set  $B$ . Similarly, in the case where  $A$  is a subpolyhedron of  $K$ , any homeomorphism of the pair  $(K, A)$  sufficiently close to 1 can be deformed through homeomorphisms of  $(K, A)$  to 1. One can formulate analogues in the smooth case.

### *Siebenmann’s Deformation theorem*

(cf. [5], p. 132); We can adapt the above proof to give an alternative proof of Siebenmann’s theorem. For example in the case where (Siebenmann’s notation)  $X$  is a finite polyhedron, subdivide sufficiently so that no simplex meets both  $A$  and  $Cl(X - A')$  and that the closed star of  $C$  lies in  $U$ , where  $C$  consists of the union of all (closed) simplices meeting  $B$ . Barycentrically subdivide  $X$  and apply the ideas of the inductive part of

the proof of theorem (1) to  $st'(C)$  to bring the embedding twice as close to 1 on  $st'(C)$ . Again subdivide  $X$  and repeat the process on  $st''(C)$ . Continuing in this way we obtain an isotopy of embeddings converging to an embedding which is the inclusion on  $C$  and agrees with the old embedding outside  $st(C)$ . Unfortunately this limiting embedding need not be PL even if the original embedding is PL. Thus although we obtain Siebenmann's theorem 2.3, we do not obtain PL or smooth analogues of this result.

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