ROBERT E. DRESSLER

A property of the $\varphi$ and $\sigma_j$ functions

Compositio Mathematica, tome 31, n° 2 (1975), p. 115-118

<http://www.numdam.org/item?id=CM_1975__31_2_115_0>
A PROPERTY OF THE \( \varphi \) AND \( \sigma_j \) FUNCTIONS

Robert E. Dressler

1. Introduction

As usual, \( \varphi \) stands for Euler's function and \( \sigma_j \) stands for the sum of the \( j^{th} \) powers of the divisors function. The purpose of this note is to answer the following very natural question: If \( t \) is a positive integer and \( f \) is \( \varphi \) or \( \sigma_j \), when does \( t \) divide \( f(n) \) for almost all positive integers \( n \)? We also answer this question for the Jordan totient function, \( \varphi_j \), a generalization of the \( \varphi \) function.

We will use the well known formulas:

\[
\varphi(n) = n \prod_{p|n} \frac{p-1}{p}
\]

\[
\sigma_j(n) = \prod_{p^e||n} (p^{ej} + p^{(e-1)j} + \ldots + p^j + 1).
\]

Here \( p^e||n \) means \( p^e|n \) and \( p^{e+1} \not| n \).

2. Results

Our first theorem concerns the \( \varphi \) function.

**Theorem (1):** For any prime \( p_0 \) and any positive integer \( k \) we have \( p_0^k|\varphi(n) \) for almost all \( n \). That is, the set of integers \( n \) for which \( p_0^k|\varphi(n) \) has natural density zero.

**Proof:** If \( p_0^k|\varphi(n) \) then by (1), no prime divisor \( p \) of \( n \) satisfies \( p \equiv 1 \pmod{p_0^k} \). In the case of \( \varphi_j, j \) arbitrary, and \( \sigma_j, j \) odd, somewhat stronger results than the ones we give may be obtained by much deeper methods, cf. \cite[pg. 167]{2}. In the case of \( \sigma_j, j \) even, our results are new. In all cases, our methods appear to be much simpler than those of \cite{2}.

\[1\]
(mod $p_0^k$). Now, if $N$ and $M$ satisfy $N > p'_1 p'_2 \cdots p'_M$ where the $p'_i$ ($i = 1, \ldots, M$) are the first $M$ primes congruent to 1 (mod $p_0^k$), then the number of positive integers not exceeding $N$, none of whose prime divisors is congruent to 1 (mod $p_0^k$) is

$$\leq 2N \prod_{i=1}^{M} \frac{p'_i - 1}{p'_i}.$$

If we let $N$ and $M$ vary together to infinity, then we have, by a strong form of Dirichlet's theorem, that

$$\left(2N \prod_{i=1}^{M} \frac{p'_i - 1}{p'_i}\right)/N \to 0.$$

This establishes our result.

Since the finite union of sets of natural density zero is a set of natural density zero we may state the following:

**Corollary (1):** Let $t$ be any positive integer. Then $t|\phi(n)$ for almost all $n$.

It is also worth noting that the $\phi_j$ function where

$$\phi_j(n) = n^j \prod_{p|n} (1 - p^{-j})$$

also satisfies the conclusions of Theorem 1 and Corollary 1. To see this, observe that if $p \equiv 1$ (mod $p_0^k$) then also $p^j \equiv 1$ (mod $p_0^k$).

The situation for the $\sigma_j$ functions is more complicated. We first need the following two lemmas:

**Lemma (1):** [3, pg. 58]. Let $(c, q) = 1$ where $q$ is any integer having primitive roots. The congruence $x^j \equiv c$ (mod $q$) is solvable if and only if

$$c^{\phi(q)/(\phi(q), j)} \equiv 1 \pmod{q}.$$

**Lemma (2):** Given any prime $p_0$ and $r$ such that $(r, p_0) = 1$ and any positive integer $k$, then almost all $n$ are such that $n$ is divisible by only the first power of some prime congruent to $r$ (mod $p_0^k$).

**Proof:** Let $p'_1, p'_2, \ldots, p'_M$ be the first $M$ primes congruent to $r$ (mod $p_0^k$).
Let $N$ be greater than $(p'_1p'_2 \cdots p'_M)^2$. Now for any subset

$$\{p'_1, p'_2, \ldots, p'_T\} \text{ of } \{p'_1, p'_2, \ldots, p'_M\}$$

the number of integers $\leq N$ which are not divisible by any $q \in \{p'_1, \ldots, p'_M\} \setminus \{p'_1, \ldots, p'_T\} = \{q_1, \ldots, q_{M-T}\}$ and are divisible by $p'_{i_1}^2 \cdots p'_{i_T}^2$ is less than

$$2N \cdot \frac{1}{p'_{i_1}^2 \cdots p'_{i_T}^2} \left(\frac{q_1-1}{q_1}\right) \cdots \left(\frac{q_{M-T}-1}{q_{M-T}}\right).$$

Thus, the number of integers $\leq N$ which are divisible by some $p'_i$ ($i = 1, \ldots, M$) only to the first power is greater than

$$N - 2N \sum_{\text{all subsets } \{p'_1, \ldots, p'_T\} \text{ of } \{p'_1, \ldots, p'_M\}} \frac{1}{p'_{i_1}^2 \cdots p'_{i_T}^2} \left(\frac{q_1-1}{q_1}\right) \cdots \left(\frac{q_{M-T}-1}{q_{M-T}}\right)$$

$$= N - 2N \prod_{i=1}^{M} \left(1 + \frac{p'_i - 1}{p'_i}\right).$$

If we now let $M, N \to \infty$ then by a strong form of Dirichlet’s theorem we have

$$(N - 2N) \prod_{i=1}^{M} \left(1 + \frac{p'_i - 1}{p'_i}\right) / N \to 1.$$  

This completes the proof.

**Theorem (2):** Let $p_0$ be an odd prime and let $k$ and $j$ be any positive integers. Then $p_0^k | \sigma_j(n)$ for almost all $n$ if and only if $\varphi(p_0^k)/(\varphi(p_0^k), j)$ is even.

**Proof:** Since $p_0$ is odd, $p_0^k$ has primitive roots. If $\varphi(p_0^k)/(\varphi(p_0^k), j)$ is even then, by Lemma 1, $x^j \equiv -1 \pmod{p_0^k}$ is solvable. Thus we can find an $x_0$ such that $x_0^j \equiv -1 \pmod{p_0^k}$. If a prime $p$ satisfies $p \equiv x_0 \pmod{p_0^k}$ and if $p|n$, then by (2) we have $p_0^k | \sigma_j(n)$. To complete this proof we apply Lemma 2 with $r = x_0$.

Now, suppose $\varphi(p_0^k)/(\varphi(p_0^k), j)$ is odd. Since $\varphi(p_0^k) = p_0^{k-1}(p_0-1)$, it follows that $\varphi(p_0^k)/(\varphi(p_0^k), j)$ is odd, for an odd prime $p_0$, if and only if

$$\frac{p_0 - 1}{(p_0-1, j)} = \frac{\varphi(p_0)}{(\varphi(p_0), j)}.$$
is odd. Thus, by Lemma 1, \( x^l \equiv -1 \pmod{p_0} \) is not solvable. Thus for any square-free integer \( n \) (since \( \sigma_j(n) = \prod_{p|n}(p^j + 1) \)) we have \( p_0 \nmid \sigma_j(n) \). Since the square-free integers have natural density \( 6/\pi^2 > 0 \) we are done.

In addition, we have

**Theorem (3):** For any positive integers \( k \) and \( j \), \( 2^k|\sigma_j(n) \) for almost all \( n \).

**Proof:** It is known \([1]\) that for any positive integer \( k \), almost all integers \( n \) have the property that they are divisible only to the first degree by at least \( k \) distinct odd primes. For these integers \( n \) it follows, from (2), that \( 2^k|\sigma_j(n) \) and the proof is complete.

We may now encapsulate Theorems 2 and 3 with

**Theorem (4):** Let \( p_0 \) be any prime and let \( k \) and \( j \) be any positive integers. Then \( p_0^{k}|\sigma_j(n) \) for almost all integers \( n \) if and only if

\[
\frac{p_0(p_0 - 1)}{(p_0 - 1, j)}
\]

is even.

Finally, we state

**Corollary (2):** Let \( t \) and \( j \) be any positive integers. Then \( t|\sigma_j(n) \) for almost all \( n \) if and only if for each prime divisor \( p \) of \( t \) we have \( p(p - 1)/(p - 1, j) \) is even.

REFERENCES