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1. Introduction

As usual, φ stands for Euler's function and σ_j stands for the sum of the j^{th} powers of the divisors function. The purpose of this note is to answer the following very natural question: If t is a positive integer and f is φ or σ_j, when does t divide $f(n)$ for almost all positive integers n? We also answer this question for the Jordan totient function, φ_j, a generalization of the φ function.

We will use the well known formulas:

$$\varphi(n) = n \prod_{p|n} \frac{p-1}{p} \tag{1}$$

$$\sigma_j(n) = \prod_{p^e||n} (p^{ej}+p^{(e-1)j}+ \dots +p^j+1). \tag{2}$$

Here $p^e||n$ means $p^e|n$ and $p^{e+1}\nmid n$.

2. Results

Our first theorem concerns the φ function.

THEOREM (1): *For any prime p_0 and any positive integer k we have $p_0^k|\varphi(n)$ for almost all n. That is, the set of integers n for which $p_0^k\nmid\varphi(n)$ has natural density zero.*

PROOF: If $p_0^k\nmid\varphi(n)$ then by (1), no prime divisor p of n satisfies $p \equiv 1$

[1] In the case of φ_j, j arbitrary, and σ_j, j odd, somewhat stronger results than the ones we give may be obtained by much deeper methods, cf. [2, pg. 167] In the case of σ_j, j even, our results are new. In all cases, our methods appear to be much simpler than those of [2].

(mod p_0^k). Now, if N and M satisfy $N > p'_1 p'_2 \cdot \ldots \cdot p'_M$ where the p'_i $(i = 1, \ldots, M)$ are the first M primes congruent to 1 (mod p_0^k), then the number of positive integers not exceeding N, none of whose prime divisors is congruent to 1 (mod p_0^k) is

$$\leqq 2N \prod_{i=1}^{M} \frac{p'_i - 1}{p'_i}.$$

If we let N and M vary together to infinity, then we have, by a strong form of Dirichlet's theorem, that

$$\left(2N \prod_{i=1}^{M} \frac{p'_i - 1}{p'_i}\right) \bigg/ N \to 0.$$

This establishes our result.

Since the finite union of sets of natural density zero is a set of natural density zero we may state the following:

COROLLARY (1): *Let t be any positive integer. Then $t | \varphi(n)$ for almost all n.*

It is also worth noting that the φ_j function where

$$\varphi_j(n) = n^j \prod_{p|n} (1 - p^{-j})$$

also satisfies the conclusions of Theorem 1 and Corollary 1. To see this, observe that if $p \equiv 1$ (mod p_0^k) then also $p^j \equiv 1$ (mod p_0^k).

The situation for the σ_j functions is more complicated. We first need the following two lemmas:

LEMMA (1): [3, pg. 58]. *Let $(c, q) = 1$ where q is any integer having primitive roots. The congruence $x^j \equiv c$ (mod q) is solvable if and only if*

$$c^{\varphi(q)/(\varphi(q), j)} \equiv 1 \pmod{q}.$$

LEMMA (2): *Given any prime p_0 and r such that $(r, p_0) = 1$ and any positive integer k, then almost all n are such that n is divisible by only the first power of some prime congruent to r (mod p_0^k).*

PROOF: Let $p'_1, p'_2, \ldots, p'_M$ be the first M primes congruent to r (mod p_0^k).

Let N be greater than $(p'_1 p'_2 \cdot \ldots \cdot p'_M)^2$. Now for any subset

$$\{p'_{i_1}, p'_{i_2}, \ldots, p'_{i_T}\} \text{ of } \{p'_1, p'_2, \ldots, p'_M\}$$

the number of integers $\leqq N$ which are not divisible by any

$$q \in \{p'_1, \ldots, p'_M\} \backslash \{p'_{i_1}, \ldots, p'_{i_T}\} = \{q_1, \ldots, q_{M-T}\}$$

and are divisible by $p'^2_{i_1} \cdot \ldots \cdot p'^2_{i_T}$ is less than

$$2N \cdot \frac{1}{p'^2_{i_1} \cdot \ldots \cdot p'^2_{i_T}} \left(\frac{q_1 - 1}{q_1}\right) \cdot \ldots \cdot \left(\frac{q_{M-T} - 1}{q_{M-T}}\right).$$

Thus, the number of integers $\leqq N$ which are divisible by some p'_i $(i = 1, \ldots, M)$ only to the first power is greater than

$$N - 2N \sum_{\substack{\text{all subsets} \\ \{p'_{i_1}, \ldots, p'_{i_T}\} \text{ of } \{p'_1, \ldots, p'_M\}}} \frac{1}{p'^2_{i_1} \cdot \ldots \cdot p'^2_{i_T}} \left(\frac{q_1 - 1}{q_1}\right) \cdot \ldots \cdot \left(\frac{q_{M-T} - 1}{q_{M-T}}\right)$$

$$= N - 2N \prod_{i=1}^{M} \left(\frac{1}{p'^2_i} + \frac{p'_i - 1}{p'_i}\right).$$

If we now let $M, N \to \infty$ then by a strong form of Dirichlet's theorem we have

$$(N - 2N) \prod_{i=1}^{M} \left(\frac{1}{p'^2_i} + \frac{p'_i - 1}{p'_i}\right) \Big/ N \to 1.$$

This completes the proof.

THEOREM (2): *Let p_0 be an odd prime and let k and j be any positive integers. Then $p_0^k | \sigma_j(n)$ for almost all n if and only if $\varphi(p_0^k)/(\varphi(p_0^k), j)$ is even.*

PROOF: Since p_0 is odd, p_0^k has primitive roots. If $\varphi(p_0^k)/(\varphi(p_0^k), j)$ is even then, by Lemma 1, $x^j \equiv -1 \pmod{p_0^k}$ is solvable. Thus we can find an x_0 such that $x_0^j \equiv -1 \pmod{p_0^k}$. If a prime p satisfies $p \equiv x_0 \pmod{p_0^k}$ and if $p \| n$, then by (2) we have $p_0^k | \sigma_j(n)$. To complete this half of the proof we apply Lemma 2 with $r = x_0$.

Now, suppose $\varphi(p_0^k)/(\varphi(p_0^k), j)$ is odd. Since $\varphi(p_0^k) = p_0^{k-1}(p_0 - 1)$, it follows that $\varphi(p_0^k)/(\varphi(p_0^k), j)$ is odd, for an odd prime p_0, if and only if

$$\frac{p_0 - 1}{(p_0 - 1, j)} = \frac{\varphi(p_0)}{(\varphi(p_0), j)}$$

is odd. Thus, by Lemma 1, $x^j \equiv -1 \pmod{p_0}$ is not solvable. Thus for any square-free integer n (since $\sigma_j(n) = \prod_{p|n}(p^j+1)$) we have $p_0 \nmid \sigma_j(n)$. Since the square-free integers have natural density $6/\pi^2 > 0$ we are done.

In addition, we have

THEOREM (3): *For any positive integers k and j, $2^k|\sigma_j(n)$ for almost all n.*

PROOF: It is known [1] that for any positive integer k, almost all integers n have the property that they are divisible only to the first degree by at least k distinct odd primes. For these integers n it follows, from (2), that $2^k|\sigma_j(n)$ and the proof is complete.

We may now capsulize Theorems 2 and 3 with

THEOREM (4): *Let p_0 be any prime and let k and j be any positive integers. Then $p_0^k|\sigma_j(n)$ for almost all integers n if and only if*

$$\frac{p_0(p_0-1)}{(p_0-1, j)}$$

is even.

Finally, we state

COROLLARY (2): *Let t and j be any positive integers. Then $t|\sigma_j(n)$ for almost all n if and only if for each prime divisor p of t we have $p(p-1)/(p-1, j)$ is even.*

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