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A NOTE ON THE ZEROS OF EXPONENTIAL POLYNOMIALS

A. J. van der Poorten

1. Introduction

In a recent paper The Zeros of Exponential Polynomials [1], C. J. Moreno gave precise information as to the location of the strips containing the zeros of exponential sums of the shape

\[ \sum_{j=1}^{m} p_j e^{a_j z} \quad p_j \in \mathbb{C}, \ a_j \in \mathbb{R}. \]

In particular he showed that the real parts of the zeros of exponential sums of the shape (1) are dense in the intervals of the real line which lie entirely inside a strip of zeros. Moreno conjectured that the appropriate generalisation of this density result would hold for exponential sums with complex frequencies \( a_1, \ldots, a_m \). He remarks that it seems difficult to obtain generalisations of his result for exponential polynomials

\[ F(z) = \sum_{j=1}^{m} p_j(z) e^{a_j z} \quad p_j(z) \in \mathbb{C}[z], \ a_j \in \mathbb{C}. \]

It is the purpose of this brief note to describe how the ideas employed by Moreno in [1] can be applied to obtain the generalisation of the results of [1] for exponential polynomials \( F(z) \) of the shape (2). For brevity we avoid a detailed repetition of the results of [1] and also refer the reader to [1] for references to the relevant literature.

2. A simplification of the general case

It will be sufficient to suppose that we may write

\[ p_j(z) = p_j z^{\varepsilon_j}(1 + \varepsilon_j(z)), \quad p_j \neq 0, \quad (j = 1, 2, \ldots, m) \]

where \( \varepsilon_j(z) \to 0 \) as \( |z| \to \infty \).
Our remarks therefore apply to a somewhat wider class of functions (2) than just the class with polynomial coefficients.

We now recall the well-known result 1 that zeros of (2) with large absolute value lie in strips in the complex plane where at least two of the terms $p_k(z)e^{z\alpha_k}$, $p_l(z)e^{z\alpha_l}$, are of similar size, dominating the size of the remaining terms. Indeed such strips lie perpendicular to the convex hull determined in the complex plane by the points $\alpha_1, \alpha_2, \ldots, \alpha_m$, the complex conjugates of the frequencies.

For convenience write for $j = 1, 2, \ldots, m$

$$\alpha_j = a_j + ib_j \quad a_j, b_j \in \mathbb{R}$$

We make our remark precise as follows:

*All but finitely many of the zeros of (2) lie in logarithmic strips of the shape*

$$I_{\theta, c} = \left\{ z = x + t \exp \left( \theta + \frac{c \log t}{t} \right) : x_0 < x < x_1, t_0 < t < \infty \right\},$$

where, $\theta$, $0 \leq \theta < 2\pi$ and $c$ are such that:

(a) there exist $k, l$ ($k \neq l$) in \{1, 2, \ldots, m\} such that

$$a_k \cos \theta - b_k \sin \theta = 0, \quad b_l \cos \theta - a_l \sin \theta = 0,$$

and

$$a_k \cos \theta - b_k \sin \theta \geq \left( a_j \cos \theta - b_j \sin \theta \right) \quad \text{for all } j = 1, 2, \ldots, m.$$ 

Write

$$s_\theta = \{ j \in \{1, 2, \ldots, m\} : (a_k - a_j) \cos \theta - (b_k - b_j) \sin \theta = 0 \}.$$

(b) $c((a_k - a_l) \sin \theta + (b_k - b_l) \cos \theta) - (\mu_k - \mu_l) = 0$, and

$$c(a_k \sin \theta + b_k \cos \theta) - \mu_k \leq c(a_j \sin \theta + b_j \cos \theta) - \mu_l, \quad \text{for all } j \in S_\theta.$$

We write

$$S_{\theta, c} = \{ l \in S_\theta : c((a_k - a_l) \sin \theta + (b_k - b_l) \cos \theta) - (\mu_k - \mu_l) = 0 \}.$$

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With $k$ as above, we now write

$$F_k(z) = z^{-\mu_k}e^{-\alpha_k z}F(z) = \sum_{l \in S_{\theta, c}} p_l z^{\mu_l - \mu_k} e^{(\alpha_l - \alpha_k) z(1 + \varepsilon_l(z)) + \delta_k(z)},$$

and observe that by virtue of the inequalities (4) and (5) if

$$z = x + t \exp \left( \theta + \frac{c \log t}{t} \right),$$

then $|\delta_k(z)| \to 0$ as $t \to \infty$. Similarly, for $l \in S_{\theta, c}$ and $z \in I_{\theta, c}$ as above, we see by virtue of the equations (4) and (5) and some simple manipulation that

$$|\exp ((\alpha_l - \alpha_k) z + (\mu_l - \mu_k) \log z)| = |\exp (\alpha_l - \alpha_k) x(1 + v_l(z))|,$$

where $|v_l(z)| \to 0$ as $t \to \infty$. Recalling that $|\varepsilon_l(z)| \to 0$ as $|z| \to \infty$ and hence as $t \to \infty$, we can summarise the situation as follows:

Write

$$F_k \left( x + t \exp \left( \theta + \frac{c \log t}{t} \right) \right) = f_k(x; t).$$

Then for every $\delta > 0$ there is a $t_0 = t_0(\delta)$ such that for $t > t_0$, $x_0 < x < x_1$,

$$|f_k(x; t) - \sum_{l \in S_{\theta, c}} p_l \exp (x(\alpha_l - \alpha_k) + it\beta_l)| < \delta,$$

where the $p_l$ are given by $p_l \exp i(\mu_l - \mu_k) \theta$ so that $|p_l| = |p_l|$, and a simple calculation shows that the $\beta_l$ are given by

$$\beta_l = (\alpha_l - \alpha_k) \sin \theta + (b_l - b_k) \cos \theta + \gamma_l(t) = \pm |x_l - \alpha_k| + \gamma_l(t),$$

where $t\gamma_l(t) \to 0$ as $t \to \infty$.

3. Main result

It is now convenient to state the main result of this note.
THEOREM: Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be distinct complex numbers and
an exponential polynomial; suppose that the $p_j(z)$ are of exact degree $\mu_j$
respectively with leading coefficient $p_j$. Let $j, c$ and the index $k$ be defined
by the conditions (a) and (b) of section 2. Let the numbers $|\alpha_l - \alpha_k|$, $l \neq k$,
$\alpha_j \in \mathbb{Q}$ be irrationals linearly independent over $\mathbb{Q}$. Then a necessary and
sufficient condition for $F(z)$ to have infinitely many zeros near any curve

$$C_{\theta, c, x} = \left\{ z = x + t \exp i \left( \theta + \frac{c \log t}{t} \right) : t_0 < t < \infty \right\}$$

is that

$$|p_h e^{xh}| \leq \sum_{l \in S_{\theta, c}, l \neq h} |p_l e^{xk}|, \quad \text{all } h \in S_{\theta, c}$$

REMARKS:
(a) The linear independence condition on the $\alpha_j$ is stronger than is
required for the truth of the theorem; but it avoids degenerate cases.
(b) By 'a zero near any curve $C_{\theta, c, x}$' we mean that if $x \in \mathbb{R}$ satisfies (8)
then given any $\epsilon > 0$ there exists a $x'$ satisfying $x - \epsilon < x' < x + \epsilon$ and
a $t' > t_0$ such that

$$F \left( x' + t' \exp i \left( \theta + \frac{c \log t'}{t'} \right) \right) = 0.$$ 

PROOF: We can apply our argument to the function $F_k(z) = z^{-\mu_k} e^{-azF(z)} F(z)$ which has the same zeros as $F(z)$ in the region under
consideration. Our proof consists of indicating that the proof of [1],
pages 73–74, can be applied mutatis mutandis to this case. For the
sufficiency argument we notice that the lemma of [1], pages 73–74,
requires only a reformulation to apply to the function $F_k(z)$ and to the
curves $C_{\theta, c, x}$ within the logarithmic strips $I_{\theta, c}$ mentioned in section 2.
Moreover in section 2 we 'simplified' $F_k(z)$ to show in (6) that $F_k(z)$
behaves on the curves $C_{\theta, c, x}$ like the exponential sum

$$\sum_{l \in S_{\theta, c}} p_l \exp (x(\alpha_l - \alpha_k) + it\beta_l).$$

We may therefore apply the Kronecker-Weyl theorem so as to construct
a sequence \(\{t_n\}\) with \(t_n \to \infty\) as \(n \to \infty\) so that the sum (9) \(\to 0\) as \(n \to \infty\) when we replace \(t\) by \(t_n\) in (9); it follows that

\[
F_k \left( x + t_n \exp i \left( \theta + \frac{c \log t_n}{t_n} \right) \right) \to 0
\]

as \(n \to \infty\) as required. The linear independence condition guarantees the required linear independence of the \(\beta_i\) as given by (7).

The necessity argument of [1], page 73, applies similarly since by (6) \(F_k(z)\) can vanish infinitely many times near the curve \(C_{\theta,c,x}\) only if the sum (9) is arbitrarily small for some \(t > t_0\) for all \(t_0\) sufficiently large. Finally, the sufficiency argument requires that \(F_k(z)\) be bounded below on segments of the curves \(C_{\theta,c,x}\) and again we can apply the argument of [1] page 76 to the sum (9). Alternatively the bound is available directly for \(F(z)\) from the results of Tijdeman [2].

4. Remarks

The theorem of course includes the assertion of section 2 since, in all but degenerate cases, (8) will be satisfied by \(x\) in some intervals \(x_0 < x < x_1\). We have also proved the conjecture of [1], page 71, to the effect that when the coefficients \(p_j(z)\) are constants \(p_j\) then near every line parallel to the sides of a strip of zeros of \(F(z)\) lie infinitely many zeros of \(F(z)\). Indeed the statement of the theorem is the natural generalisation of this conjecture to the case of coefficients \(p_j(z)\) satisfying (3).

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REFERENCES


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