COMPOSITIO MATHEMATICA

EDUARD LOOIJENGA

A period mapping for certain semi-universal deformations

Compositio Mathematica, tome 30, nº 3 (1975), p. 299-316

<http://www.numdam.org/item?id=CM_1975__30_3_299_0>

COMPOSITIO MATHEMATICA, Vol. 30, Fasc. 3, 1975, pag. 299–316
Noordhoff International Publishing
Printed in the Netherlands

A PERIOD MAPPING FOR CERTAIN SEMI-UNIVERSAL DEFORMATIONS

Eduard Looijenga

Let (X_0, x_0) be germ of a n-dimensional complex isolated hypersurface singularity with Milnor number μ and let $p : X \to S$ be a suitable representative of a semi-universal deformation of X_0. Denote by $\Delta \subset S$ the discriminant variety of p. In this paper we construct a kind of period mapping from the universal cover of $S\backslash\Delta$ to $\mathbb{C}^\mu$. We prove that this mapping is nonsingular if X_0 is quasi-homogeneous and 1 is not eigenvalue of its monodromy automorphism (actually, this last hypothesis can be weakened). The proof hinges on an explicit description of the Gauss-Manin connection for such deformations, which is due to Brieskorn and Greuel.

Using this result we recover in the last section Brieskorns description of the discriminant variety of a rational singularity.

I wish to mention Brieskorns name once more to thank him for his comments on an earlier draft of this paper, leading to corrections of several mistakes.

1. Formulation of the main result

(1.1) Let (X_0, x_0) be a germ of a n-dimensional complex hypersurface which has an isolated singularity at x_0. Such a singularity admits a semi-universal deformation. This is a cartesian diagram of holomorphic mapgerms

$$\begin{array}{ccc} (X_0, x_0) & \subset & (X, x_0) \\ \downarrow & & \downarrow p \\ \{s_0\} & \subset & (S, s_0) \end{array}$$

and characterized by certain properties (see for example [10]).

A representative of p can be obtained as follows. Suppose that x_0 is the origin of $\mathbb{C}^{n+1}$ and that X_0 is defined by a holomorphic function $f: V \to \mathbb{C}$, where V is a neighborhood of $0 \in \mathbb{C}^{n+1}$. Let $\phi_1, \cdots, \phi_l$ be monomials which project onto a $\mathbb{C}$-basis of the artinian ring

$$\mathscr{m}_{V,0}/\left(f, \frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n}\right)\mathcal{O}_{V,0}$$

and define $g: V \times \mathbb{C}^l \to \mathbb{C}$ by $g(z, u) = f(z) + u_1\phi_1(z) + \cdots + u_l\phi_l(z)$ and $F: V \times \mathbb{C}^l \to \mathbb{C} \times \mathbb{C}^l$ by $F(z, u) = (g(z, u), u)$. Let Σ denote the critical set of F. Choose a polycylindrical neighborhood $\Delta_{n+1} \times \Delta_l$ of $(0, 0)\, \hat{o}\ V \times \mathbb{C}^l$ such that

(i) $(\partial\Delta_{n+1} \times \Delta_l) \cap \Sigma = \emptyset$
(ii) $(\Delta_{n+1} \times \{0\}) \cap \Sigma = \{(0, 0)\}$
(iii′) $F^{-1}(0, u)$ intersects $\partial\Delta_{n+1} \times \Delta_l$ transversally for all $u \in \Delta_l$. Then there is a disc Δ_1 in $\mathbb{C}$ centered at 0 such that (iii′) can be strengthened to
(iii) For all $(t, u) \in \Delta_1 \times \Delta_l$, $F^{-1}(t, u)$ intersects $\partial\Delta_{n+1} \times \Delta_l$ transversally.

Put $X = (\mathring{\Delta}_{n+1} \times \mathring{\Delta}_l) \cap F^{-1}(\mathring{\Delta}_1 \times \mathring{\Delta}_l)$, $S = \mathring{\Delta}_1 \times \mathring{\Delta}_l$ and let $p: X \to S$ be the restriction of F. Note that the sets X and S obtained in this way form neighborhood basis of $(0, 0) \in \mathbb{C}^{n+1} \times \mathbb{C}^l$ and $(0, 0) \in \mathbb{C} \times \mathbb{C}^l$ respectively. Hence both X and S may be taken smaller if future operations make this necessary. In such a case we will do this without further comment.

Since $p|\Sigma$ is a proper map, it follows from Grauert's theorem that $\Delta := p(\Sigma)$ is a subvariety of S. For a generic choice of $u \in \Delta_l$, $\Delta_1 \times \{u\}$ intersects Δ in

$$\mu = \dim_{\mathbb{C}} \mathcal{O}_{V,0}/\left(\frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n}\right)\mathcal{O}_{V,0} \tag{1.1.1}$$

distinct simple points [3]. Hence Δ may be defined by an element $h \in \Gamma(\mathcal{O}_S)$ of the form $h(t, u) = t^\mu + a_1(u)t^{\mu-1} + \cdots + a_\mu(u)$.

The ideal $(\partial g/\partial z_0, \cdots, \partial g/\partial z_n)$ defines Σ as a smooth subvariety of X, and since $p^*(h)$ vanishes on Σ there exist $\xi_0, \cdots, \xi_n \in \Gamma(\mathcal{O}_X)$ such that

$$p^*(h) = \sum_{i=0}^{n} \xi_i \frac{\partial g}{\partial z_i}. \tag{1.1.2}$$

We also introduce $S' := S\backslash\Delta$, $X' := p^{-1}(S')$ and let $p': X' \to S'$ denote the restriction of p. p' defines a C^∞-fibrebundle, and it is known that a typical fibre $X_s := p^{-1}(s)$ $(s \in S')$ has the homotopy type of a wedge of

μ n-spheres; in particular $H_n(X_s, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank μ[7]. $\pi_1(S', s)$ acts on $H_n(X_s, \mathbb{Z})$. Let $\pi : \tilde{S}' \to S'$ denote the regular covering of S' which is associated to the kernel of this representation of $\pi_1(S', s)$. We so obtain a Cartesian diagram

$$\begin{array}{ccccc} \tilde{X}' := X' \times_{S'} \tilde{S}' & \longrightarrow & X' & \subset & X \\ \downarrow & & \downarrow p' & & \downarrow p \\ \tilde{S}' & \xrightarrow{\pi} & S' & \subset & S \end{array}$$

Let $\omega \in \Gamma(\Omega_X^{n+l+1})$ and $\alpha \in \Gamma(\Omega_S^{1+l})$ and assume that α vanishes nowhere on S. The 'quotient' of ω and α defines a family of n-forms on $X_s (s \in S')$ as follows. Let U be a neighborhood of s in S' such that $p^{-1}U$ admits a retraction $r : p^{-1}U \to X_s$ coming from a trivialisation. Then there is a unique n-form $\omega(s)$ on X_s such that $\omega = p^*(\alpha) \wedge r^*(\omega(s))$ on $p^{-1}U$.

It is easily verified that $\omega(s)$ doesn't depend on a specific choice of r and U, and that $\omega(s)$ is holomorphic (in particular closed) on X_s. This family of n-forms pulls back to a family $\omega(\tilde{s}) \in \Gamma(\Omega^n_{X_{\tilde{s}}})$, $\tilde{s} \in \tilde{S}'$.

Now fix a $\tilde{s}_0 \in \tilde{S}'$ and choose a basis $(\gamma_1(\tilde{s}_0), \cdots, \gamma_\mu(\tilde{s}_0))$ of $H_n(X_{\tilde{s}_0}, \mathbb{Z})$. By the absence of monodromy over $\tilde{S}'$, $(\gamma_1(\tilde{s}_0), \cdots, \gamma_\mu(\tilde{s}_0))$ displaces canonically to basis $(\gamma_1(\tilde{s}), \cdots, \gamma_\mu(\tilde{s}))$ of $H_n(X_{\tilde{s}}, \mathbb{Z})$ $(\tilde{s} \in \tilde{S}')$. Then the map $P_k : \tilde{S}' \to \mathbb{C}^\mu$ which calculates the periods of $h^k(\pi(\tilde{s})) \cdot \omega_{(\tilde{s})}$:

$$P_k(\tilde{s}) := \left(\int_{\gamma_1(\tilde{s})} h^k(\pi(\tilde{s})) \cdot \omega(\tilde{s}), \cdots, \int_{\gamma_\mu(\tilde{s})} h^k(\pi(\tilde{s})) \cdot \omega(\tilde{s}) \right) \tag{1.1.3}$$

is holomorphic.

If we suppose that X_0 is quasi homogeneous, that is

$$f = \sum_{i=0}^{n} c_i z_i \frac{\partial f}{\partial z_i}$$

for certain positive rational numbers $c_0, \cdots, c_n$, then

$$l = \dim_{\mathbb{C}} \mathfrak{m}_{V,0} / \left(f, \frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n} \right) = \dim_{\mathbb{C}} \mathfrak{m}_{V,0} / \left(\frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n} \right) = \mu - 1.$$

So $\dim_{\mathbb{C}} S' = \mu$ in that case. For convenience we abbreviate $r = c_0 + \cdots + c_n$ and we define $d_\lambda \in \mathbb{Q}$ by

$$d_\lambda \phi_\lambda = \sum_i c_i z_i \frac{\partial \phi_\lambda}{\partial z_i} (\lambda = 1, \cdots, l).$$

Our main result is

(1.2) THEOREM: *Suppose* (X_0, x_0) *quasi homogeneous,* $\omega(x_0) \neq 0$ *and* $\mu k + r, d_1 + r, \cdots, d_l + r$ *all different from* 1. *Then there exists a neighborhood* W *of* s_0 *in* S *such that* $P_k|\pi^{-1}(W)$ *is locally biholomorphic.*

(1.3) REMARK: It is known that $\exp(2\pi i r)$, $\exp 2\pi i(d_\lambda + r)$ $(\lambda = 1, \cdots, l)$ are the eigen values of the monodromy transformation of f [3].

By iterated suspension of f (i.e. replacing $f(z)$ by $f(z) + z_{n+1}^2 + \cdots + z_{n+m}^2$ for some $m \in \mathbb{N}$) we can always attain that $r, d_1 + r, \cdots, d_l + r$ all differ from 1. Suspension doesn't change S and Δ.

(1.4) The condition $\alpha = dt \wedge du_1 \wedge \cdots \wedge du_l$ doesn't restrict the generality of (1.2). We shall therefore assume this in the sequel.

2. The Gauss-Manin connection and some preliminaries

(2.1) For the moment we drop the assumption that (X_0, x_0) is weighted homogeneous and start with recalling the definition of the Gauss-Manin connection in several sheaves, as it has been given by Greuel in his thesis under more general conditions. Our main reference for this will be [6].

Over S' we have a presheaf defined by $U \mapsto H^n(p^{-1}U, \mathbb{C})$. The sheaf $\mathscr{H}_0$ of its local sections admits the canonical algebraic description as the cohomology of a relative de Rham complex:

$$\mathscr{H}_0 \cong \mathscr{H}^n(p'_* \Omega_{X'/S'}).$$

In this context, $\mathscr{H}_0$ admits the canonical extension

$$\mathscr{H} := \mathscr{H}^n(p_* \Omega_{X/S})$$

over S. For making explicit calculations two auxiliary sheaves of $\mathcal{O}_S$-modules are very useful:

$$\mathscr{H}' := (p_* \Omega^n_{X/S})/d(p_* \Omega^{n-1}_{X/S})$$

and

$$\mathscr{H}'' := p_* \Omega_X^{n+1+l}/p^* \Omega_S^{l+1} \wedge d(p_* \Omega_X^{n-1}).$$

Clearly $\mathscr{H}$ embeds in $\mathscr{H}'$. Following Greuel, we define a $\mathcal{O}_S$-homomorphism $\mathscr{H}' \to \mathscr{H}''$ by $\xi \mapsto p^*(\alpha) \wedge \xi$. It follows from a generalized de Rham lemma that this map is injective. We shall consider this injection as an inclusion. $\mathscr{H}$, $\mathscr{H}'$ and $\mathscr{H}''$ turn out to be coherent sheaves of rank μ and the last two are free as well. Using (1.1.2) one easily verifies that $h\mathscr{H}' \subset \mathscr{H}$ and $h\mathscr{H}'' \subset \mathscr{H}'$. The presheaf $U \mapsto H^n(p^{-1}U, \mathbb{Z})$ determines an integral lattice in $\mathscr{H}_0$. It is clear that there is then a unique integrable connection $\nabla_0 : \mathscr{H}_0 \to \mathscr{H}_0 \otimes \Omega^1_{S'}$ whose horizontal sections are generated by the integral lattice. ∇_0 may be characterized more intrinsically by the following property. If $\xi \in \mathscr{H}_0$ and γ is a local section of the presheaf $U \mapsto H_n(p^{-1}U, \mathbb{Z})$, then $d(\int_\gamma \xi) = \int_\gamma \nabla_0 \xi$.

An extension of ∇_0 to a connection ∇ in $\mathscr{H}$ can be obtained at the cost of having the coefficients of ∇ acquire simple poles along Δ. Then ∇ obeys the following algebraic description. Let $\zeta \in p_*\Omega^n_X$ represent $[\zeta] \in \mathscr{H}$. $d[\zeta] = 0$ implies $d\zeta = dg \wedge \alpha_0 + \sum_{\lambda=1}^{l} du_\lambda \wedge \alpha_\lambda$ for certain $\alpha_0, \cdots, \alpha_l \in p_*\Omega^n_X$. Let $\bar{\alpha}_\lambda$ denote the projection of α_λ in $\mathscr{H}'$. Then Greuel defines

$$\text{(2.1.1)} \qquad \nabla([\zeta]) = h \ \bar{\alpha}_0 \otimes \frac{dt}{h} + \sum_{\lambda=1}^{l} h\bar{\alpha}_\lambda \otimes \frac{du_\lambda}{h} \in \mathscr{H} \otimes \frac{1}{h}\Omega^1_S,$$

and he verifies that with this definition of ∇, ∇ is an extension of ∇_0 indeed. The inclusions $h\mathscr{H}' \subset \mathscr{H}$, $h\mathscr{H}'' \subset \mathscr{H}$ and the Leibniz rule allow us to extend ∇ in a canonical way over $\mathscr{H}'$ and $\mathscr{H}''$.

Now we take up the situation of (1.2) again, and we define a $\mathcal{O}_S$-homomorphism $c : p_*\mathcal{O}_X \to \mathscr{H}''$ by sending $\phi \in p_*\mathcal{O}_X$ to the projection of $\phi dz \wedge du$ in $\mathscr{H}''$. We then have

(2.2) LEMMA: *$c(1), c(\phi_1), \cdots, c(\phi_l)$ form a $\mathcal{O}_{S,s_0}$-basis of $\mathscr{H}''_{s_0}$* [1].

PROOF: Since $\mathscr{H}''_{s_0}$ is free, it suffices to show that $c(1), \cdots, c(\phi_l)$ map onto a $\mathbb{C}$-basis of $\mathscr{H}''(s_0)$.

Let $\phi \in \ker(c)$. Then $\phi dz \wedge du \in p^*\Omega^{l+1}_S \wedge d(p_*\Omega^{n-1}_X)$, i.e.

$$\phi dz \wedge du = (\psi dg \wedge du_1 \wedge \cdots \wedge du_l) \wedge \beta$$

with $\psi \in \mathcal{O}_S$ and $\beta \in d(p_*\Omega^{n-1}_X)$. It follows that $\phi \in (\partial g/\partial z_0, \cdots, \partial g/\partial z_n)p_*\mathcal{O}_X$. So the natural projection

$$p_*\mathcal{O}_X \to p_*\mathcal{O}_X / \left(\left(\frac{\partial g}{\partial z_0}, \cdots, \frac{\partial g}{\partial z_n} \right) + p^*(m_{S,s_0}) \right) p_*\mathcal{O}_X$$

[1] For any sheaf $\mathscr{F}$ of $\mathcal{O}_S$-modules, $\mathscr{F}_s$ denotes the stalk of $\mathscr{F}$ at a point $s \in S$ and $\mathscr{F}(s) := \mathscr{F}_s \otimes_{\mathcal{O}_{S,s}} \mathbb{C}$ the fibre of $\mathscr{F}$ over s.

factorizes via c over $\mathscr{H}''(s_0)$. Then the induced map

$$\mathscr{H}''(s_0) \to p_*\mathcal{O}_X / \left(\left(\frac{\partial g}{\partial z_0}, \cdots, \frac{\partial g}{\partial z_n}\right) + p^*(m_{S,s_0})\right) p_*\mathcal{O}_X$$

must be an isomorphism, since it is a surjective homomorphism of $\mathbb{C}$-vectorspaces, whose source and target have the same dimension μ. Since $1, \phi_1, \cdots, \phi_l$ project onto a basis of this target, the lemma follows.

We let K denote the field of fractions of $\mathcal{O}_{S,s_0}$, and we define for any integer k an element $q_k \in K$ as follows. Write

$$h^{-k}\nabla c(h^k) = \omega_0 \otimes dt + \sum_{\lambda=1}^{l} \omega_\lambda \otimes du_\lambda$$

(with $\omega_\lambda \in K \cdot \mathscr{H}''_{s_0}$) and $\omega_\lambda = \sum_{\kappa=0}^{l} f_{\lambda\kappa} c(\phi_\kappa)$ (with $f_{\lambda\kappa} \in K$, and where we have put $\phi_0 := 1$ for notational convenience). We then pose $q_k := \det(f_{\lambda\kappa})$.

(2.3) PROPOSITION: *If $\mu k + r \neq 1$, and $d_\lambda + r \neq 1$ for all λ, then $q_k h$ is a unit of $\mathcal{O}_{S,s_0}$. We postpone the proof of* (2.3) *to Section* 3 *but we show how* (2.3) *implies* (1.2).

Since $h \cdot \mathscr{H}'' \subset \mathscr{H}'$, there exists a $\zeta \in p_* \Omega^n_X$ such that $h\omega = p^*(\alpha) \wedge \zeta$. The restriction of $h^{-1}\zeta$ to a non singular fibre X_s is just $\omega(s)$ as defined in (1.1). Since we have $d(\int_\gamma \zeta) = \int_\gamma \nabla\zeta$, P_k is of maximal rank as a multivalued mapping on S' if and only if

$$\nabla(h^{k-1}\zeta) \in \mathscr{H}' \otimes h^{k-2}\Omega^1_S \cong \operatorname{Hom}(\Omega^{1*}_S, h^{k-2}\mathscr{H}')$$

defines a vector bundle isomorphism outside Δ. And this is the case if and only if $\nabla c(h^k) \in \operatorname{Hom}(\Omega^{1*}_S, h^{k-1}\mathscr{H}'')$ defines a vectorbundle isomorphism outside Δ. Because $q_k \in K$ describes the determinant of $h^{-k}\nabla c(h^k)$ with respect to the basis $(\partial/\partial t, \partial/\partial u_1, \cdots, \partial/\partial u_l)$ and $(c(\phi_0), \cdots, c(\phi_l))$, proposition (2.3) implies (1.2) as stated.

3. Proof of proposition (2.3)

We first derive an explicit description of ∇. We will use abbreviations like du for $du_1 \wedge \cdots \wedge du_l$ and $\widehat{dz}_i$ for $dz_0 \wedge \cdots \wedge dz_{i-1} \wedge dz_{i+1} \wedge \cdots \wedge dz_n$.

(3.1) LEMMA: *Let $\phi \in \Gamma(\mathcal{O}_X)$. Then*

$$\nabla c(\phi) = c\left(-\phi\frac{\partial h}{\partial t} + \sum_i\left(\frac{\partial \xi_i}{\partial z_i}\phi + \xi_i\frac{\partial \phi}{\partial z_i}\right)\right)\otimes\frac{dt}{h}$$
$$+\sum_{\lambda=1}^{l} c\left(h\frac{\partial\phi}{\partial u_\lambda} + \frac{\partial h}{\partial t}\frac{\partial g}{\partial u_\lambda}\phi\right.$$
$$\left.-\sum_i\left(\xi_i\frac{\partial^2 g}{\partial z_i\partial u_\lambda}\phi + \frac{\partial\xi_i}{\partial z_i}\frac{\partial g}{\partial u_\lambda}\phi + \xi_i\frac{\partial g}{\partial u_\lambda}\frac{\partial\phi}{\partial z_i}\right)\right)\otimes\frac{du_\lambda}{h},$$

where we have abusively written h, $(\partial h/\partial t)$ *etc., instead of* $p^*(h)$, $p^*(\partial h/\partial t)$.

PROOF: It follows from (1.1.2) that we can write

$$h^2\phi\, dz\wedge du = dg\wedge du\wedge\sum_i(-1)^{i+nl}\xi_i h\phi\hat{d}z_i.$$

We put $\zeta := \sum_i(-1)^{nl+i}\xi_i h\phi\hat{d}z_i$. By (1.1.2) ζ projects into $\mathscr{H}_{s_0}$ and its image is just $h^2c(\phi)$ under the 'inclusion' $\mathscr{H}\subset\mathscr{H}''$. In order to determine $\nabla h^2c(\phi)$ we compute

$$d\zeta = \sum_i(-1)^{nl}\left(\frac{\partial\xi_i}{\partial z_i}h\phi + \xi_i\frac{\partial h}{\partial t}\frac{\partial g}{\partial z_i}\phi + \xi_i h\frac{\partial\phi}{\partial z_i}\right)dz$$
$$+\sum_{\lambda,i}(-1)^{nl+i}\left(\frac{\partial\xi_i}{\partial u_\lambda}h\phi + \xi_i\left(\frac{\partial h}{\partial u_\lambda} + \frac{\partial h}{\partial t}\frac{\partial g}{\partial u_\lambda}\right)\phi + \xi_i h\frac{\partial\phi}{\partial u_\lambda}\right)du_\lambda\wedge\hat{d}z_i.$$

(Here, as well as below, we use the chain rule:

$$\frac{\partial}{\partial u_\lambda}(p^*h) = \frac{\partial h}{\partial u_\lambda} + \frac{\partial h}{\partial t}\cdot\frac{\partial g}{\partial u_\lambda}.)$$

The term

$$\sum_i\left(\frac{\partial\xi_i}{\partial z_i}h\phi + \xi_i\frac{\partial h}{\partial t}\frac{\partial g}{\partial z_i}\phi + \xi_i h\frac{\partial\phi}{\partial z_i}\right)dz$$

equals

$$\left(\phi\frac{\partial h}{\partial t} + \sum_i\left(\frac{\partial\xi_i}{\partial z_i}\phi + \xi_i\frac{\partial\phi}{\partial z_i}\right)\right)hdz.$$

Since

$$hdz = \sum_j \xi_j \frac{\partial g}{\partial z_j} dz$$
$$= dg \wedge \sum_j (-1)^j \xi_j \hat{d}z_j - \sum_{j,\lambda} du_\lambda \wedge (-1)^j \xi_j \frac{\partial g}{\partial u_\lambda} \hat{d}z_j,$$

we can write $d\zeta = dg \wedge \zeta_0 + \sum_\lambda du_\lambda \wedge \zeta_\lambda$ with

$$\zeta_0 = \sum_j (-1)^{nl+j} \left(\phi \frac{\partial h}{\partial t} + \sum_i \left(\frac{\partial \xi_i}{\partial z_i} \phi + \xi_i \frac{\partial \phi}{\partial z_i} \right) \right) \xi_j \hat{d}z_j$$

and

$$\zeta_\lambda = \sum_i (-1)^{nl+i} \left(\frac{\partial \xi_i}{\partial u_\lambda} h\phi + \xi_i \left(\frac{\partial h}{\partial u_\lambda} + \frac{\partial h}{\partial t} \frac{\partial g}{\partial u_\lambda} \right) \phi + \xi_i h \frac{\partial \phi}{\partial u_\lambda} \right) \hat{d}z_i$$
$$- \sum_j (-1)^{nl+j} \left(\phi \frac{\partial h}{\partial t} + \sum_i \left(\frac{\partial \xi i}{\partial z_i} \phi + \xi_i \frac{\partial \phi}{\partial z_i} \right) \right) \xi_j \frac{\partial g}{\partial u_\lambda} \hat{d}z_j.$$

Then

$$(3.1.1) \qquad dg \wedge du \wedge \zeta_0 = \left(\phi \frac{\partial h}{\partial t} + \sum_i \left(\frac{\partial \xi_i}{\partial z_i} \phi + \xi_i \frac{\partial \phi}{\partial z_i} \right) \right) hdz \wedge du$$

and

$$(3.1.2) \quad dg \wedge du \wedge \zeta_\lambda = \left(\frac{\partial h}{\partial u_\lambda} \phi + h \frac{\partial \phi}{\partial u_\lambda} \right.$$
$$\left. + \sum_i \left(\frac{\partial \xi_i}{\partial u_\lambda} \frac{\partial g}{\partial z_i} \phi - \frac{\partial \xi_i}{\partial z_i} \phi \frac{\partial g}{\partial u_\lambda} - \xi_i \frac{\partial \phi}{\partial z_i} \frac{\partial g}{\partial u_\lambda} \right) \right) hdz \wedge du.$$

Substitution of

$$\sum_i \frac{\partial \xi_i}{\partial u_\lambda} \frac{\partial g}{\partial z_i} = \frac{\partial h}{\partial u_\lambda} + \frac{\partial h}{\partial t} \frac{\partial g}{\partial u_\lambda} - \sum_i \xi_i \frac{\partial^2 g}{\partial z_i \partial u_\lambda}$$

in the righthand side of (3.1.2) yields

$$(3.1.3) \quad \left(2 \frac{\partial h}{\partial u_\lambda} \phi + h \frac{\partial \phi}{\partial u_\lambda} + \frac{\partial h}{\partial t} \frac{\partial g}{\partial u_\lambda} \phi \right.$$
$$\left. - \sum_i \left(\xi_i \frac{\partial^2 g}{\partial z_i \partial u_\lambda} \phi + \frac{\partial \xi_i}{\partial z_i} \phi \frac{\partial g}{\partial u_\lambda} + \xi_i \frac{\partial \phi}{\partial z_i} \frac{\partial g}{\partial u_\lambda} \right) \right) hdz \wedge du$$

Following the discussion in Section 2, $dg \wedge du \wedge \zeta_\lambda$ projects onto the coefficient of du_λ (dt if $\lambda = 0$) in $\nabla h^2 c(\phi)$. Then the lemma follows from (3.1.1), (3.1.3) and the Leibniz formula.

(3.2) LEMMA: *The ξ_i's in* (1.1.2) *can be chosen such that*

$$\xi_i - c_i z_i g^{\mu-1} \in (u_1, \cdots, u_l)\mathcal{O}_X.$$

PROOF: Let $(\xi'_0, \cdots, \xi'_n) \in \Gamma(\mathcal{O}_X)^{n+1}$ satisfy (1.1.2). Then

$$\sum_i (\xi'_i - c_i z_i g^{\mu-1}) \frac{\partial g}{\partial z_i} \in (u_1, \cdots, u_l)\mathcal{O}_X.$$

Since $\{\partial g/\partial z_0, \cdots, \partial g/\partial z_n, u_1, \cdots, u_l\}$ forms a set of parameters of $\mathcal{O}_{X,x_0}$, there exists a skew symmetric matrix (A_{ij}) with coefficients in $\mathcal{O}_{X,x_0}$ such that for all i

$$\xi'_i - c_i z_i g^{\mu-1} + \sum_j A_{ij} \frac{\partial g}{\partial z_j} \in (u_1, \cdots, u_l)\mathcal{O}_{X,x_0}.$$

Then

$$\left\{\xi_i := \xi'_i + \sum_{j=0}^{n} A_{ij} \frac{\partial g}{\partial z_j}\right\}_{i=1}^{n}$$

are as required.

From now on, we suppose that the ξ_i's are as in (3.2).

Let L denote the line in S defined by $(u_1, \cdots, u_l)$. We can use t as a coordinate for L.

(3.3) LEMMA: *Suppose that $\mu k + r \neq 1$ and $d_\lambda + r \neq 1$ for all λ. Then the restriction of $h \cdot q_k$ to L is holomorphic and $hq_k(s_0) \neq 0$.*

PROOF: We first list some congruences

$$p^* \left(\frac{\partial h}{\partial t}\right) \equiv \mu g^{\mu-1} \bmod (u_1, \cdots, u_l)O_{X,x}$$

$$\sum_i \frac{\partial \xi_i}{\partial z_i} \equiv (r + \mu - 1) g^{\mu-1} \bmod (u_1, \cdots, u_l)\mathcal{O}_{X,x} \qquad \text{(from (3.2))}$$

$$\sum_i \xi_i \frac{\partial^2 g}{\partial z_i \partial u_\lambda} \equiv d_\lambda \phi_\lambda g^{\mu-1} \bmod (u_1, \cdots, u_l)\mathcal{O}_{X,x} \qquad \text{(from (3.2))}$$

Since the tangent cone of Δ at s_0 is defined by dt [11], we also have

$$p^*\left(\frac{\partial h}{\partial u_\lambda}\right) \in (u_1, \cdots, u_l)\mathcal{O}_{X,x}.$$

Define $\phi \in \Gamma(\mathcal{O}_X)$ by $\omega = \phi\, dz \wedge du$. By the Leibniz rule we have

$$h^{-k}\nabla c(\phi h^k) = \nabla c(\phi) - kc(\phi) \otimes \frac{dh}{h}.$$

Then a simple computation using (3.1) and the congruences above shows that the coefficient of dt/h in $h^{-k}\nabla c(\phi h^k)$ equals

$$c((r-1-\mu k)\phi g^{\mu-1}) \bmod (u_1, \cdots, u_l)\mathcal{H}''_{s_0} + p_*(\mathfrak{m}_{X,x_0})\phi g^{\mu-1}\mathcal{H}''_{s_0},$$

and the coefficient of du_λ/h in $h^{-k}\nabla c(\phi h^k)$ equals

$$c((-d_\lambda - r + 1)\phi g^{\mu-1}\phi_\lambda) \bmod (u_1, \cdots, u_l)\mathcal{H}''_{s_0} + p_*(\mathfrak{m}_{X,x_0})\phi g^{\mu-1}\phi_\lambda \mathcal{H}''_{s_0}.$$

$\mathcal{H}''_{s_0}$ is in a natural way a $p_*\mathcal{O}_X$-module, and as such z_i $(i = 0, \cdots, n)$ acts in a nilpotent manner on the fibre

$$\mathcal{H}''(s_0) \approx \mathbb{C}\{z_0, \cdots, z_n\}/\left(\frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n}\right).$$

Hence, if we write $h^{-k}\nabla c(\phi h^k) = \sum_\lambda f_{\lambda 0}\, c(\phi_\lambda) \otimes dt + \sum_{\lambda,\kappa} f_{\lambda\kappa}\, c(\phi_\lambda) \otimes du_\kappa$, with $f_{\lambda\kappa}$ in the quotient field of $\mathcal{O}_{L,0}$, then

$$(f_{\lambda\kappa}) = \phi(0)t^{-1}\begin{bmatrix} \mu k - 1 + r & & & 0 \\ & -d_1 - r + 1 & & \\ & & \ddots & \\ 0 & & & -d_1 - r + 1 \end{bmatrix} + t^{-1}N$$

where N is a matrix with coefficients in $\mathcal{O}_{L,0}$ and nilpotent for $t = 0$. It follows that $q_k h|_{\mathrm{L}} = t^\mu \det(f_{\lambda_\kappa})$ is holomorphic in s_0, where it takes the nonzero value $\phi(0)(\mu k - 1 + r)\prod_{\lambda=1}^{l}(-d_\lambda - r + 1)$.

Since $\pi_1(S', s)$ acts via $\pm\mathrm{id}$ on $\wedge^\mu H_n(X_s, \mathbb{Z})$, $\delta := \det^2\left(\int_{\gamma_i(s)} c(\phi_j)\right)$ will be a holomorphic and univalued function on S'. It follows from the regularity theorem [5] that δ extends meromorphically over X. In fact we have

(3.4) PROPOSITION: $\operatorname{div}(\delta) = (n-1)\Delta$.

PROOF: We restrict δ to the line L, and prove that $\delta|L$ vanishes of order $\mu(n-1)$ at s_0. This suffices, since L intersects Δ with multiplicity μ at s_0. It follows from (3.1) that

$$\left(\nabla \frac{\partial}{\partial t}\right) c(\phi_0) = (r-1)c(\phi_0)t^{-1},$$

and

$$\left(\nabla \frac{\partial}{\partial t}\right) c(\phi_\lambda) = (r-1+d_\lambda)c(\phi_\lambda)t^{-1}.$$

Since $\int_\gamma \nabla\omega = d\int_\gamma \omega$, it follows that

$$\int_{\gamma_\kappa} c(\phi_0)|L = \alpha_{\kappa 0} t^{r-1},$$

and

$$\int_{\gamma_\kappa} c(\phi_\lambda)|L = \alpha_{\kappa\lambda} t^{r-1+d_\lambda} \qquad (\lambda = 1, \cdots, l),$$

where the $\alpha_{\kappa\lambda}$'s are constants.

So up to a constant factor $\delta|L(t)$ equals $t^{2(\mu(r-1)+d_1+\cdots+d_l)}$. Since δ does not vanish outside Δ, this constant must be nonzero. In the appendix it is proved that $2(\mu(r-1)+d_1+\cdots+d_l) = \mu(n-1)$, and this will complete the proof.

(3.5) Let s_1 be a simple point of Δ. Then X_{s_1} has only one singular point, x_1 say, and x_1 is an ordinary double point. Let us choose neighborhoods X_1 of $x_1 \in X$ and S_1 of $s_1 \in S_1$ and $z'_0, \cdots, z'_n \in \Gamma(\mathcal{O}_{X_1})$, $v_0, \cdots, v_l \in \Gamma(\mathcal{O}_{S_1})$ such that the following conditions are satisfied.

(i) $pX_1 \subset S_1$ and (X_1, S_1) satisfies the conditions (1.1)–i, ii, iii.

(ii) $\{v_0, \cdots, v_l\}$ is a set of coordinates for S_1

(iii) $\{z'_0, \cdots, z'_n, p^*v_1, \cdots, p^*v_l\}$ is a set of coordinates for X_1, and it maps X_1 onto a subset of $\mathbb{C}^{n+1} \times \mathbb{C}^l$ which contains the polycylinder $\{|z'|^2 \leqq 1, |p^*(v_1)|^2 + \cdots + |p^*v_l|^2 \leqq 1\}$

(iv) $\alpha|S_1 = dv_0 \wedge \cdots \wedge dv_l$.

(v) $p^*v_0 = \sum_{i=0}^n z_i'^2$.

Let Δ_l denote the open unit ball of $\mathbb{C}^l$ and define a mapping $\sigma: [0, 1) \times \Delta_l \to S_1$ by $\sigma(\tau, w) = (\tau, w_1, \cdots, w_l)$.

(3.6) LEMMA: *Let* $\{\Gamma(\tau, w) \in H_n(X_{\sigma(\tau, w)}, \mathbb{Z}) : (\tau, w) \in (0, 1) \times \Delta_l\}$ *be a continuous family of cycles, and put*

$$I(\tau, w) := \int_{\Gamma(\tau, w)} \omega(\sigma(\tau, w)).$$

Then

(a) *I is bounded if* $n > 1$ *and* $I = 0(-\log \tau)$, $\tau \to 0$ *if* $n = 1$
(b) $\partial I / \partial w_\lambda = 0(-\log \tau)$, $\tau \to 0$
(c) $\partial I / \partial \tau = 0(\tau^{-1})$, $\tau \to 0$
(d) *If* $\Gamma(\tau, w)$ *'vanishes' as* $\tau \to 0$, *then I is of the form* $J(\tau, w)\tau^{\frac{1}{2}(n-1)}$, *where J extends to a real analytic function on* $[0, 1) \times \Delta_l$.

PROOF: Suppose first that $\Gamma(\tau, w)$ doesn't intersect the vanishing cycle in $H_n(X_{\sigma(\tau, w)}, \mathbb{Z})$ which corresponds to $x_1 \in X_{s_1}$. Then it is easily seen that $\{\Gamma(\tau, w)\}$ is homologous to a family of cycles which extends over $[0, 1) \times \Delta_l$ and avoids Σ. We continue to denote this family by $\{\Gamma(\tau, w)\}$. Since $\omega(s)$ is a well-defined n-form on $X_s \backslash \Sigma$ for all s, the first three clauses of (3.6) follow immediately in this case.

Now let Y denote the subset of X_1 defined by $|z'|^2 \leqq 1$ and $|p^*v_1|^2 + \cdots + |p^*v_l|^2 < 1$. We put $z'_j = x_j + iy_j$. Then the vanishing cycle in $H_n(X_{\sigma(\tau, w)}, \mathbb{Z})$ can be represented by the oriented n-sphere $\delta(\tau, w)$ defined by $|x|^2 = \tau$, $y = 0$ and $w_\lambda = p^*v_\lambda$ $(\lambda = 1, \cdots, l)$. Its dual, generating $H_n(Y_{\sigma(\tau, w)}, \partial Y_{\sigma(\tau, w)}; \mathbb{Z})$ can be represented by the (suitably oriented) n-disc $\varepsilon(\tau, w)$ which is defined by $x_0^2 = |y|^2 + \tau$, $x_j = 0$ for $j > 0$, $y_0 = 0$, $|y|^2 \leqq (1-\tau)/2$ and $w_\lambda = p^*v_\lambda$ $(\lambda = 1, \cdots, l)$.

Let m be the intersection number of $\Gamma(\tau, w)$ and $\delta(\tau, w)$. Then we may assume that the restriction of $\Gamma(\tau, w)$ to $Y\sigma(\tau, w)$ equals $m\varepsilon(\tau, w)$. The same argument used for the case $m = 0$ proves that the function $\int_{\Gamma(\tau, w)\backslash Y} \omega(\sigma(\tau, w))$ as well as its derivatives is bounded. So we only need to consider the integral $I'(\tau, w) := \int_{\varepsilon(\tau, w)} \omega(\sigma(\tau, w))$. To this end we observe that

$$p^*(\alpha) \wedge v_0^{-1} \sum_j (-1)^j z'_j \hat{d}z'_j = \pm 2dv_1 \wedge \cdots \wedge dv_l dz'.$$

Hence $\omega(v)|Y \cap X'$ is of the form $\psi(v_1, \cdots, v_l, z')v_0^{-1}\sum_j(-1)^j z'_j \hat{d}z'_j$, where ψ is some holomorphic function on Y. If we use $\{y_1, \cdots, y_n\}$ as coordinates for $\varepsilon(\tau, w)$, then the restriction of $\omega(\sigma(\tau, w))$ to $\varepsilon(\tau, w)$ becomes

$$\tau^{-1}\psi_1(\tau, w, y_1, \cdots, y_n)(y_1^2 + \cdots + y_n^2 + \tau)^{-\frac{1}{2}} dy_1 \wedge \cdots \wedge dy_n.$$

We have to integrate this form over the ball $y_1^2 + \cdots + y_n^2 \leqq (1-\tau)/2$.

Hence we have

$$(3.6.1)\qquad I'(\tau, w) = \int_0^{\frac{1}{2}(1-\tau)} dr \int_{S_r} \psi_1(\tau, w, y_1, \cdots, y_n)(r^2+\tau)^{-\frac{1}{2}} d\sigma,$$

where S_r denotes the sphere of radius r in $(y_1, \cdots, y_n)$-space and $d\sigma$ its volume-form.

It follows from (3.6.1) that $I'(\tau, w)$ has the form

$$(3.6.2)\qquad I'(\tau, w) = \int_0^{\sqrt{\frac{1}{2}(1-\tau)}} (r^2+\tau)^{-\frac{1}{2}} r^{n-1} \psi_2(r, w) dr,$$

where ψ_2 is some real-analytic function on $[0, 1) \times \Delta_l$. For $n \geqq 2$, the righthand side of (3.6.2) is clearly bounded. This proves the first part of (a).

Clause (b) follows from

$$\frac{\partial I'}{\partial w_\lambda} = 0\left(\int_0^1 (r^2+\tau)^{-\frac{1}{2}} dr \right) = 0(-\log \tau),\ \tau \to 0,$$

and the last part of (a) is proved in the same way. (c) follows from

$$\begin{aligned} \frac{\partial I'}{\partial \tau} &= 0\left(\int_0^1 (r^2+\tau)^{-\frac{1}{2}} dr \right) + 0\left(\frac{d}{d\tau} \int_0^1 (r^2+\tau)^{-\frac{1}{2}} dr \right) \\ &= 0(-\log \tau) + 0(\tau^{-1}) = 0(\tau^{-1}),\ \tau \to 0. \end{aligned}$$

To prove (d), we note that the restriction of $\omega(\sigma(\tau, w))$ to $\delta(\tau, w)$ equals $\psi(w, x)\tau^{-1}\sum_{j=0}^n (-1)^j x_j \widehat{dx}_j$. So

$$\begin{aligned} I(\tau, w) &= \int_{|x|^2=\tau} \psi(w, x)\tau^{-1} \sum_{j=0}^n (-1)^j x_j \widehat{dx}_j \\ &= \tau^{-1} 0(\tau^{(n+1)\frac{1}{2}}) = 0(\tau^{(n-1)\frac{1}{2}}),\ \tau \to 0. \end{aligned}$$

For notational convenience we shall now write u_0 instead of t, and we put

$$\varepsilon_k(s) := \det{}^2 \left(h^{-k}(s) \frac{\partial}{\partial u_\kappa} \int_{\gamma_\lambda(s)} h^k(s)\omega(s) \right).$$

Since $\pi_1(S', s)$ acts via $\pm\mathrm{id}$ on $\Lambda^\mu H_n(X_s, \mathbb{Z})$, ε_k is a holomorphic univalued function on S'.

(3.7) PROPOSITION: ε_k *extends meromorphically over S and*

$$\operatorname{div}(\varepsilon_k) \geqq (n-3)\Delta.$$

PROOF: The meromorphy property follows from the regularity theorem [5]. Now let s_1 be any simpie point of Δ and choose $\sigma : [0, 1) \times \Delta_l \to S$ as in (3.6) above. Let $\{(\gamma_1(\tau, w), \cdots, \gamma_\mu(\tau, w) : (\tau, w) \in (0, 1) \times \Delta_l\}$ denote a continuous family of basis of $H_n(X_{\sigma(\tau, w)}, \mathbb{Z})$ induced by the multivalued basis $(\gamma_1(s), \cdots, \gamma_l(s))$. For calculating ε_k, we may assume that $\gamma_1(\tau, w) = \delta(\tau, w)$. Up to an invertible real analytic function $h(\sigma(\tau, w))$ equals τ. It then follows from (3.6) that

$$\varepsilon_k(\tau, w) = \det{}^2\, 0 \begin{bmatrix} -\tau^{(n-3)\frac{1}{2}} \log \tau & \tau^{-1} & \cdots & \tau^{-1} \\ -\tau^{(n-1)\frac{1}{2}} \log \tau & -\log \tau & & -\log \tau \\ \vdots & \vdots & & \vdots \\ -\tau^{(n-1)\frac{1}{2}} \log \tau & -\log \tau & & -\log \tau \end{bmatrix}$$
$$= 0(\tau^{(n-3)}(\log \tau)^{2\mu}),\ \tau \to 0.$$

This implies that $\lim_{\tau \to 0} \tau^{-(n-2)} \varepsilon_k(\tau, w) = 0$. Since ε_k is meromorphic along Δ, it follows that $\operatorname{div}(\varepsilon_k) \geqq (n-3)\Delta$.

PROOF OF (2.3). Consider the equality $\varepsilon_k = q_k^2 \delta$. It follows from (3.4) and (3.7) that $\operatorname{div}(q_k) \geqq -\Delta$. Hence $q_k h$ is holomorphic at s_0 and following (3.3), $q_k h(s_0) \neq 0$.

4. Rational singularities

In this section we assume that (X_{s_0}, x_0) is a rational singularity [1] and we take $n = 2$ and $k = 0$. Then g becomes quasi-homogeneous and so we may take $X = \mathbb{C}^3 \times \mathbb{C}^l$ and $S = \mathbb{C} \times \mathbb{C}^l$. Brieskorn has shown how (S, Δ) can be described in terms of the action of $\pi_1(S', s)$ on $H_n(X_s, \mathbb{C})$ [4]. We will show that this can also be derived from the preceding.

(4.1) First note that we actually constructed multivalued mappings P_k from S' to $H^n(X_s, \mathbb{C})$, rather than to $\mathbb{C}^\mu$ (for in the last case we needed a specific basis of $H_n(X_s, \mathbb{Z})$). An alternative way of making P_k univalued is passing to a quotient of its target: Put $V := H^n(X_s, \mathbb{C})$ and $G := \operatorname{Im}(\pi_1(S', s) \to \operatorname{Aut} V)$. We so obtain a univalued mapping

$$\bar{P}_k : S' \to V/G.$$

Since (X_{s_0}, x_0) is rational, (V, G) is a finite Weyl group. Before proceeding, we first recall a few properties of such groups.

(i) There exist algebraically independent G-invariant polynomials $\alpha_1, \cdots, \alpha_\mu$ on V such that $\alpha_1, \cdots, \alpha_\mu$ generate all G-invariant polynomials on V. In particular $(\alpha_1, \cdots, \alpha_\mu): V \to \mathbb{C}^\mu$ realizes $V \to V/G$.

(ii) The number of reflections among the elements of G equals $C\mu/2$, where C denotes the Coxeter number of G.

(iii) Let Δ' denote the discriminant of the natural projection $V \to V/G$ and let D be a polynomial on V that defines the union of hyperplanes fixed by any reflection in G. Then D^2 is G-invariant and if $J \in \mathbb{C}[\alpha_1, \cdots, \alpha_\mu]$ is such that

$$D^2(v) = J(\alpha_1(v), \cdots, \alpha_\mu(v)),$$

then J defines Δ' as a reduced hypersurface in $\mathbb{C}^\mu$.

These properties can be found in [2, Ch. V]. Property (iv) below was already noted in [9] and can be proved by checking cases.

(iv) $C = (r-1)^{-1}$.

It follows from (i) that V/G admits a complex structure making $V \to V/G$ holomorphic and $V/G \approx \mathbb{C}^\mu$. Hence $\bar{P}_0 : S' \to V/G$ is holomorphic as well.

(4.2) Lemma: *$\bar{P}_0$ extends holomorphically over S.*

Proof: Let s_1 be a simple point of Δ and let S_1 be a neighborhood of s_1 in S as in (3.5). Let $\{\gamma(s) \in H_n(X_s, \mathbb{Z}) : s \in S_1 \backslash \Delta\}$ be a multivalued section. Since the automorphism of $H_n(X_s, \mathbb{Z})$ induced by a generator of $\pi_1(S_1 \backslash \Delta, s)$ is a reflection (given by the Picard-Lefschetz formula), $(\int_{\gamma(s)} \omega(s))^2$ is a univalued function on $S_1 \backslash \Delta$. The regularity theorem asserts that it extends meromorphically over S_1. It follows from (3.6–(a)) that this extension is in fact holomorphic. Hence $\bar{P}_0$ admits a holomorphic extension along the simple point-set of Δ. Since the singular part of Δ is of codimension $\geqq 2$ in S, $\bar{P}_0$ extends holomorphically over all of S.

(4.3) Theorem: *$\bar{P}_0$ is an isomorphism which maps Δ' onto Δ.*

We prove this via the following lemma.

(4.4) Lemma: *Let L denote the line in S defined by $(u_1, \cdots, u_l)$. Then $P_0(t, 0, \cdots, 0) = (c_1 t^{r-1}, \cdots, c_\mu t^{r-1})$ on L with $(c_1, \cdots, c_\mu) \in \mathbb{C}^\mu \backslash \{0\}$.*

Proof: Let $(p_1(t), \cdots, p_\mu(t))$ denote the restriction of P_0 to L. It then follows from (3.5) that p_α satisfies a differential equation

$$\frac{dp_\alpha}{dt} = (r-1)t^{-1}p_\alpha \qquad (\alpha = 1, \cdots, \mu)$$

This obviously proves that $p_\alpha = c_\alpha t^{r-1}$ for some $c_\alpha \in \mathbb{C}$. Since P_0 is local isomorphism on $L\backslash\{0\}$, the c_α's cannot all vanish.

Proof of (4.3). It follows from (4.1)-iii and (4.4) above that the multiplicity of $\bar{P}_0^{-1}(\Delta')$ at s_0 will be at most $(r-1)\cdot 2$ {reflections in G}. By (4.1)–ii, iv this equals $(r-1)2.\ C_\mu/2 = \mu$. The multiplicity of Δ at s_0 equals μ. It follows from (3.6)–(d)) and (4.1–iii) that $\bar{P}_0$ maps Δ into Δ'. Since both Δ and Δ' are weighted homogeneous, we then must have that $P_0^{-1}(\Delta')$ defines Δ as a *reduced* variety.

Since $r > 1$, theorem (1.2) applies, and it follows that the branch locus B of P_0 is a union of components of Δ'. If $B \neq \emptyset$, $P_0^{-1}(\Delta')$ does not define Δ as a reduced variety. Hence $B = \emptyset$ and the theorem follows.

Appendix

In this appendix we shall prove that

(A.1) $$2(\mu(r-1)+d_1+\cdots+d_l) = \mu(n-1).$$

We first make some observations.

Since $c_0, \cdots, c_n$ are positive rational numbers, we can write $c_i = w_i/N$ with $w_i, N \in \mathbb{N}$. Since $f = \sum_i c_i z_i(df/\partial z_i)$, f must be a $\mathbb{C}$-linear combination of monomials $z_0^{j_0} \cdots z_n^{j_n}$ with $\sum_i w_i j_i = N$.

By putting deg $z_i = w_i$ we make $\mathbb{C}[z_0, \cdots, z_n]$ into a graded $\mathbb{C}$-algebra. Note that with this grading $d_\lambda = N^{-1}$ deg (ϕ_λ).

For any graded $\mathbb{C}$-module $\mathscr{A} = \sum_{n=0}^{\infty} \mathscr{A}_n$, one has the Poincaré series of A [2, Ch. V, § 5],

$$P_{\mathscr{A}}(T) = \sum_{n=0}^{\infty} \dim(\mathscr{A}_n)T^n \in \mathbb{Z}[[T]]$$

These series have the obvious property that they multiply with respect to tensor products.

Let $\mathscr{C}$ denote the $\mathbb{C}$-module $\mathbb{C}\cdot 1+\mathbb{C}\phi_1+\cdots+\mathbb{C}\phi_l$, equipped with its natural grading. Then it is clear that

$$N(d_1+\cdots+d_l) = \frac{dP_{\mathscr{C}}}{dT}(1) \quad \text{and} \quad \mu = P_{\mathscr{C}}(1).$$

Hence (A.1) may be written as

$$\text{(A.2)} \qquad \frac{\dfrac{dP_{\mathscr{C}}}{dT}(1)}{P_{\mathscr{C}}(1)} = (n+1)N/2 - \sum_{i=0}^{n} w_i$$

Proof of (A.2). Since $\{1, \phi_0, \cdots, \phi_l\}$ projects onto a basis of

$$\mathbb{C}[z_0, \cdots, z_n]/\left(\frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n}\right),$$

it follows from the Weierstrass preparation theorem that $\{1, \phi_0, \cdots, \phi_l\}$ generates $\mathbb{C}[z_0, \cdots, z_n]$ freely as a $\mathbb{C}[\partial f/\partial z_0, \cdots, \partial f/\partial z_n]$-module. So we have

$$\mathbb{C}[z_0, \cdots, z_n] \cong \mathscr{C} \otimes \mathbb{C}\left[\frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n}\right]$$

as graded $\mathbb{C}$-modules, which can also be written as

$$\mathbb{C}[z_0] \otimes \cdots \otimes \mathbb{C}[z_n] \cong \mathscr{C} \otimes \mathbb{C}\left[\frac{\partial f}{\partial z_0}\right] \otimes \cdots \otimes \mathbb{C}\left[\frac{\partial f}{\partial z_n}\right]$$

It follows that

$$P_{\mathscr{C}}(T) = \prod_{i=0}^{n} \left(\frac{1-T^{N-w_i}}{1-T^{w_i}}\right).$$

The right hand side of this formula equals

$$\prod_{i=0}^{n} \left(\frac{1+T+\cdots+T^{N-w_i-1}}{1+T+\cdots+T^{w_i-1}}\right),$$

and if we take the logarithmic derivative of this and evaluate at $T = 1$, we obtain the desired result.

REFERENCES

[1] M. Artin: On isolated rational singularities for surfaces. *Amer. J. of Math. 88* (1966) 129–136.
[2] N. Bourbaki: *Groupes et algèbres de Lie*. Ch. 4, 5 et 6, Hermann, Paris (1968).
[3] E. Brieskorn: Die Monodromie der isolierten Singularitäten von Hyperflächen. *Manuscripta Math. 2* (1970) 103–161.