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A DIVISORIAL CYCLE ACQUIRING AN EMBEDDED COMPONENT UNDER A FLAT SPECIALIZATION

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Introduction

One striking difference between divisors and divisorial cycles appears when they are specialized in a flat family of closed subschemes. On a geometrically normal variety, an effective divisor remains a divisor because of the Ramanujam-Samuel theorem (EGA IV_4, 21.14.1); this remarkable property is equivalent to the completeness theorem for the Picard scheme (cf. [1]) if, in addition, the variety is projective. By contrast, a positive divisorial cycle (not surprisingly) can acquire embedded components and so cease to be a divisorial cycle. Our object is to study an example in detail.

This behavior of divisorial cycles is one reason to feel that divisor-like subschemes should be allowed embedded components located in the singular locus. Here are two other reasons. Let S be a locally noetherian scheme and $f: X \to S$ a flat, projective morphism with geometrically normal fibers. Then, the flat, closed subschemes of X/S that are divisors on the smooth locus are parametrized by an open and closed subscheme of the Hilbert scheme of X/S ([1], (17)). Moreover, those of them that are linearly equivalent in the sense that their ideals are isomorphic locally over S are parametrized by a bundle of projective spaces $\boldsymbol{P}(H)$ with H a coherent $\mathcal{O}_S$-Module ([1], (17)).

The example below involves the theory of cones. EGA II, 8 contains one development of the theory; we found another. Ours seems more concise, less technical and easier to comprehend, and it includes some useful results, which might well have appeared in EGA II. Moreover, our development treats the theory of cones as a special case of the theory of joins. We plan to present it elsewhere. So, the references to the theory of cones are to our development (these references begin with a letter, e.g., (B7)) and, whenever possible, to EGA II, 8.

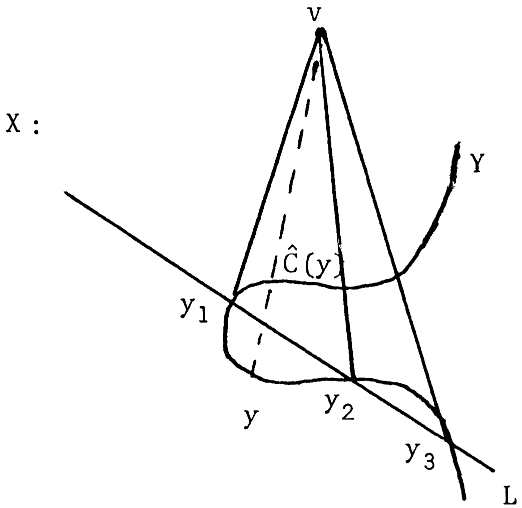
Much of what is done with a smooth plane cubic curve works for any smooth curve; however, there seems to be little additional value and to be some loss in simplicity of notation in doing the general case.

Effective divisors are considered as closed subschemes with invertible ideals (EGA IV_4, 21.2.12), and positive divisorial cycles on normal schemes are considered as closed subschemes with pure codimension one without embedded components (cf. EGA IV_4, 21.7.2).

1. The example

Let k be an algebraically closed field, Y a smooth plane cubic curve, and $F(T_0, T_1, T_2)$ a homogeneous polynomial defining Y in $\boldsymbol{P}_k^2$. Let X denote the cone over Y in $\boldsymbol{P}_k^3$ with vertex $v = (0, 0, 0, 1)$; that is, X is the surface in $\boldsymbol{P}_k^3$ defined by F. Then $(X - v)$ is nonsingular because the partial derivatives, $\partial F/\partial T_i$, for $0 \leqq i \leqq 2$ do not vanish simultaneously on X except at v. Furthermore, X is normal by Serre's criterion (EGA IV_2, 5.8.6) for it is Cohen-Macaulay because it is cut out by one equation, $F = 0$, and it is regular in codimension one because $(X - v)$ is regular.

Fix a line L in $\boldsymbol{P}_k^2$, and let y_1, y_2, and y_3 denote the three (not necessarily distinct) points of intersection of Y and L. For each closed point y of Y, the cone $\hat{C}(y)$ is obviously equal to the (reduced) line through y and v.



For each closed point y of Y, let D_y denote the following positive divisorial cycle on X:

$$D_y = \hat{C}(y_1) + \hat{C}(y_2) + \hat{C}(y).$$

Then, D_{y_3} is a divisor, for it is the intersection with X of the plane spanned by L and v. By contrast, D_y, for $y \neq y_3$, is not a divisor (see Section 2). Consequently, the D_y are not isomorphic to the closed fibers of a flat family of subschemes of X parametrized by Y; for, if they were, then almost all of them would be divisors (cf. [6], Lemma, p. 108).

In Section 3 we construct a positive divisorial cycle Z on $X \times Y$ such that Z is flat and proper over Y and such that, for each closed point y of Y, the fiber $Z(y)$ is, off v, equal to D_y. In Section 4, we show that the Euler characteristic $\chi(I_y)$ of the ideal I_y of D_y is equal to 0 for $y \neq y_3$ and to 1 for $y = y_3$.

Let y be a closed point of Y. Since D_y is defined as the scheme-theoretic closure of its three maximal points (EGA IV_4, 21.7.1), it is clearly equal to the scheme-theoretic closure of $D_y|(X-v)$. Hence, since $Z(y)$ is closed and contains $D_y|(X-v)$, it also contains D_y. So, there is an exact sequence,

$$0 \to I(Z(y)) \to I_y \to N \to 0,$$

where $I(Z(y))$ denotes the ideal of $Z(y)$ and N is a suitable coherent $\mathcal{O}_X$-Module. Since Z has no embedded components, its generic fiber $Z(\eta)$ obviously has no embedded components. Consequently, since Z/Y is flat and proper, $Z(y)$ has no embedded components for almost all points y of Y (EGA IV_3, 12.2.1). Hence, $Z(y)$ is everywhere equal to D_y for almost all y. Therefore, the Euler characteristic $\chi(I(Z(y)))$ of $I(Z(y))$, which is independent of y (EGA III_2, 7.9.4), is equal to 0 for every y. Consequently, the Euler characteristic $\chi(N)$ is equal to 0 for $y \neq y_3$ and to 1 for $y = y_3$. However, N is equal to 0 off v because $Z(y)$ and D_y are equal off v. Therefore, N is equal to 0 everywhere for $y \neq y_3$, but N is nonzero at v for $y = y_3$. Hence, $Z(y)$ is, everywhere, equal to D_y for $y \neq y_3$, but $Z(y_3)$ has an embedded component at v.

In short, $Z(y)$, for $y \neq y_3$ is a divisorial cycle that acquires an embedded component under a flat specialization as y specializes to y_3.

2. D_y, for $y \neq y_3$, is not a divisor

Set

$$P = \mathbf{P}(\mathcal{O}_Y(1) \oplus \mathcal{O}_Y),$$

and form a commutative diagram (B7.1),

$$\begin{tikzcd} P \arrow[r, "i"] \arrow[d, "b"'] \arrow[dr, "p"] & X \times Y \arrow[d, "p_2"] \arrow[dl, "p_1"'] \\ X \arrow[r, dashed, "r"] & Y \end{tikzcd}$$

where p_1 and p_2 denote the projections, where p denotes the structure map, where b denotes the conjunctive transformation, $p_1 \circ i$, and where i denotes (B7) the conjunctive embedding, $i = \text{Proj}\,(\alpha_Y^\sharp[t])$, where $\alpha_Y^\sharp : \mathscr{S}_Y \to \text{Sym}\,(\mathscr{O}_Y(1))$ is the Serre map of the graded coordinate algebra, $\mathscr{S} = k[T]/F$, of Y. Let $e : Y \to P$ denote the fundamental embedding (B7); it is a section of p defined by the projection $(\mathscr{O}_Y(1) \oplus \mathscr{O}_Y) \to \mathscr{O}_Y$. Let E denote the exceptional locus; it is the scheme-theoretic image of e. Let $r : (X - v) \to Y$ denote the fundamental retraction (B4 or EGA II, 8.3.5.1); it is Proj (a) where $a : \mathscr{S} \to \mathscr{S}[t]$ denotes the inclusion.

Let y be a closed point of Y. Because the fundamental retraction is compatible with linear embedding (B9, (iii, b) or EGA II, 8.5.4), there is an equality of divisorial cycles,

$$\hat{C}(y)|(X-v) = r^{-1}(y). \tag{2.1}$$

Consider the following divisor on Y:

$$d_y = y_1 + y_2 + y.$$

A fundamental retraction is smooth (C2 or cf. EGA II, 8.3.5.3), so flat. Hence r^{-1} induces a homomorphism from the group of divisors on Y to the group of divisors on $(X-v)$ by (EGA IV$_4$, 21.4.5, (i) and 21.4.2). Hence, relation (2.1) yields a relation,

$$D_y|(X-v) = r^{-1}(d_y). \tag{2.2}$$

Consider the following subscheme of P:

$$D'_y = p^{-1}(d_y). \tag{2.3}$$

Since p is flat, D'_y is a divisor (EGA IV$_4$, 21.4, 5, (i)). By (C1, (ii) or EGA II, 8.6.2), b carries $(P-E)$ into $(X-v)$ and the composition, $r \circ (b|(P-E))$, is equal to the restriction, $p|(P-E)$; hence, (2.2) and (2.3) yield the relation,

$$b^{-1}(D_y)|(P-E) = D'_y|(P-E). \tag{2.4}$$

Assume D_y is a divisor. Then, $b^{-1}(D_y)$ is also a divisor (C5, (iii) or EGA IV$_4$, 21.4.5, (iii) as P is integral, X is irreducible and b is dominating). Now, D'_y is a divisor (noted after (2.3)), and E is a divisor (C3, (i) or EGA II, 8.7.7 and 8.1.8) because it is the exceptional locus. Hence, since P is normal, relation (2.4) implies there is a relation, (2.5) $$b^{-1}(D_y) = D'_y + mE,$$

for some integer m (EGA IV_4, 21.6.9, (i)).

Since e is a section of p, there is a relation,

(2.6) $$e^{-1}(D'_y) = d_y.$$

By (C3.2 or EGA II, 8.7.7 and 8.1.8)), there is an isomorphism,

(2.7) $$e^*\mathcal{O}_P(mE) \cong \mathcal{O}_Y(-m).$$

Therefore, (2.5) yields an isomorphism,

(2.8) $$\mathcal{O}_Y(d_y) \otimes \mathcal{O}_Y(-m) \cong e^*b^*\mathcal{O}_X(D_y).$$

By (C1, (i) or EGA II, 8.7.1), the following diagram is commutative (in fact cartesian):

$$\begin{tikzcd} Y \arrow[r, "e"] \arrow[d] & P \arrow[d, "b"] \\ \operatorname{Spec}(k) \arrow[r] & X, \end{tikzcd}$$

where the left-hand map is the structure map and the bottom map is the vertex section; so, there is an isomorphism,

(2.9) $$e^*b^*\mathcal{O}_X(D_y) \cong \mathcal{O}_Y.$$

Combining (2.8) with (2.9) yields an isomorphism,

$$\mathcal{O}_Y(d_y) \cong \mathcal{O}_Y(m).$$

Since Y is a cubic curve, deg $(c_1(\mathcal{O}_Y(m))$ is equal to $3m$ by Bézout's theorem; obviously, deg $(c_1(\mathcal{O}_Y(d_y)))$ is equal to 3. Hence, m is equal to 1.

By Serre's explicit computation (EGA III_1, 2.1.12, (i)), the linear system of line sections of Y is complete; that is, the canonical map,

$$\Gamma(\boldsymbol{P}^2_k, \mathcal{O}_{\boldsymbol{P}^2_k}(1)) \to \Gamma(Y, \mathcal{O}_Y(1)),$$

is surjective. Therefore, d_y is obtained as the intersection of a line in $\boldsymbol{P}^2_k$ with Y. This line must be L because, if y_1 and y_2 are distinct points, then L is the only line through them both and, if they are the same point, then L is the only line through this point and tangent to Y there. Consequently, y is equal to y_3. Thus, for $y \neq y_3$, the cycle D_y is not a divisor.

3. Construction of the family Z

Since X and Y are normal, $X \times Y$ is also normal (EGA IV_2, 6.14.3) because k is algebraically closed. For each closed point y of Y, the cone $\hat{C}(y)$ is clearly integral; hence, $\hat{C}(y) \times Y$ is also integral (EGA IV_2, 4.6.5, (ii)) because k is algebraically closed. Moreover, P is integral because it is a projective bundle over Y. Clearly $\hat{C}(y) \times Y$ and $i(P)$ have codimension one in $X \times Y$. So, we may define Z as the following positive divisorial cycle on $X \times Y$:

$$Z = (\hat{C}(y_1) \times Y) + (\hat{C}(y_2) \times Y) + i(P).$$

Obviously, each component of Z intersects the generic fiber $X(\eta)$ of $X \times Y/Y$. Hence, Z is equal to the scheme-theoretic closure of its generic fiber $Z(\eta)$ in $X \times Y$ because Z is a divisorial cycle. So, by (EGA IV_2, 2.8.5), Z is flat over Y. It is obviously proper over Y.

Let y be a closed point of Y. We are going to establish a canonical isomorphism,

$$Z(y)|((X-v)\times y) = D_y|(X-v). \tag{3.1}$$

Obviously, there is a relation,

$$(\hat{C}(y_i) \times Y)(y) = \hat{C}(y_i) \times y, \qquad \text{for} \quad i = 1, 2. \tag{3.2}$$

Now, the conjunctive embedding, $i: P(y) \to \hat{C}(y) \times y$ is an isomorphism (B8, (ii)). So, since i is compatible with linear embedding (B9, (iv)), there is a relation,

$$i(P)(y) = \hat{C}(y) \times y. \tag{3.3}$$

The scheme $(X-v) \times Y$ is nonsingular (EGA IV_2, 6.8.5, (ii)) because both factors are nonsingular and k is algebraically closed; so, $(X-v) \times Y$ is locally factorial (EGA IV_4, 21.11.1). Hence, the divisorial cycles, $\hat{C}(y_i) \times Y$ for $i = 1, 2$ and $i(P)$, are divisors on $(X-v) \times Y$ (EGA IV_4, 21.6.9, (ii)). Since each is obviously flat over Y, they are relative effective divisors (EGA IV_4, 21.15.3.3). Consequently, on $(X-v) \times Y$, formation of their sum commutes with base change (EGA IV_4, 21.15.9). Therefore, (3.2) and (3.3) imply (3.1) by additivity. (The key fact is that the components of Z are divisors on $(X-v)\times Y$; they are not divisors at $v\times y_3$, and, as noted at the end of Section 1, the fiber $Z(y_3)$ is not equal to D_{y_3}.)

4. Computation of the Euler characteristics

Keep the notation of Sections 1 and 2. We first verify that the comorphism of b is an isomorphism,

$$b^c : \mathcal{O}_X \xrightarrow{\sim} b_*\mathcal{O}_P. \tag{4.1}$$

Since X is integral (Section 1) and $b : P \to X$ is birational (C5, (iv) or cf. EGA II, 8.6.2), P and X have the same function field; denote it by K. Let U be an affine open subset of X. Then, there are natural inclusions,

$$\Gamma(U, \mathcal{O}_X) \subset \Gamma(b^{-1}(U), \mathcal{O}_P) \subset K. \tag{4.2}$$

Since P is projective over k and since X is separated over k, the morphism b is projective (EGA II, 5.5.5, (v)). Therefore, $\Gamma(b^{-1}(U), \mathcal{O}_P)$ is a finitely generated $\Gamma(U, \mathcal{O}_X)$-module by Serre's theorem (EGA III_1, 2.2.1, (i)). Hence, since X is normal, the first inclusion in (4.2) is an equality. Thus, the comorphism of b is an isomorphism.

We next establish a canonical isomorphism,

$$I_y \xrightarrow{\sim} b_*(I'_y), \tag{4.3}$$

where I'_y is the ideal of D'_y. By (2.4), the two closed subschemes, $b^{-1}(D_y)$ and D'_y, of P coincide on $(P-E)$. Obviously, no component of the divisorial cycle D'_y lies entirely in E; so, D'_y is equal to the scheme-theoretic closure of its restriction to $(P-E)$. Hence, D'_y is a subscheme of $b^{-1}(D_y)$. Therefore, b induces a morphism b' from D'_y to D_y. Consider the commutative diagram with exact rows,

$$\begin{tikzcd} 0 \arrow[r] & I_y \arrow[r] \arrow[d, dashed] & \mathcal{O}_X \arrow[r] \arrow[d, "\cong"] & \mathcal{O}_{D_y} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & b_*(I'_y) \arrow[r] & b_*\mathcal{O}_P \arrow[r] & b'_*\mathcal{O}_{D'_y}, & \end{tikzcd} \tag{4.4}$$

where the two right-hand vertical maps are the comorphisms. The center vertical map is the isomorphism (4.1). The right-hand vertical map is an isomorphism on $(X-v)$ because b carries $(P-E)$ isomorphically onto

$(X-v)$ by (C1, (ii) or EGA II, 8.6.2) and because $b^{-1}(D_y)$ and D'_y coincide on $(P-E)$ by (2.4); hence it is injective everywhere because its kernel can have no associated point since the associated points of $\mathcal{O}_{D_y}$ are all $(x-v)$. Therefore, by the five lemma, the induced left-hand vertical map in (4.4) is an isomorphism, as desired.

Let m be an integer, and set

$$J_m = \mathcal{O}_P(mE-D'_y).$$

Consider the exact sequence,

$$0 \to J_{m-1} \to J_m \to J_m|E \to 0,$$

obtained by tensoring the canonical exact sequence,

$$0 \to \mathcal{O}_P(-E) \to \mathcal{O}_P \to \mathcal{O}_E \to 0,$$

with the invertible $\mathcal{O}_X$-Module J_m. It yields an exact sequence,

$$\begin{aligned}(4.5)\quad R^jb_*(J_m|E) \to R^{j+1}b_*(J_{m-1}) \to R^{j+1}b_*(J_m)& \\ \to R^{j+1}b_*(J_m|E) \to R^{j+2}b_*(J_{m-1}),&\end{aligned}$$

for each $j \geqq 0$.

Let M be an $\mathcal{O}_P$-Module. Since b carries $(P-E)$ isomorphically onto $(X-v)$ (C1, (ii) or EGA II, 8.6.2), it is clear from the construction of R^jb_*M as a derived functor that it is concentrated at v for $j>0$. Therefore, there is a relation,

$$(4.6)\qquad H^i(X, R^jb_*M) = 0 \quad \text{for} \quad i>0 \quad \text{and} \quad j>0.$$

For every $j \geqq 0$ on the other hand, $H^j(b^{-1}(U), M|E)$ is equal to $H^j(E, M|E)$ for each open set U containing v ([5], II, 4.9.1(a)), and $H^j(b^{-1}(U), M|E)$ is obviously equal to zero for each open set U not containing v. Hence, since $R^jb_*(M|E)$ is equal to the sheaf associated to the presheaf, $U \mapsto H^j(b^{-1}(U), M|E)$, on X, its stalk at v is equal to $H^j(E, M|E)$. Therefore we have the following formula and statement:

$$(4.7)\qquad H^0(X, R^jb_*(M|E)) = H^j(E, M|E) \quad \text{for every} \quad j \geqq 0;$$

$$(4.8)\quad H^j(E, M|E) = 0 \quad \text{implies} \quad R^jb_*(M|E) = 0 \quad \text{for every} \quad j \geqq 0.$$

Since E is a curve, $H^j(E, M|E)$ is equal to zero for $j \geqq 2$ ([5], II, 4.15.2);

so, (4.8) yields the formula,

$$R^j b_*(M|E) = 0 \qquad \text{for} \quad j \geqq 2. \tag{4.9}$$

The invertible sheaf $\mathcal{O}_P(-E)$ is relatively ample for b, (C3, (iii) or EGA 8.7.7 and 8.1.7) because the inverse of an exceptional divisor is. So there is a formula,

$$R^j b_*(J_m) = 0 \qquad \text{for} \quad j \geqq 1 \quad \text{and} \quad m \ll 0, \tag{4.10}$$

by Serre's theorem (EGA III_1, 2.2.1, (ii)). Therefore, by induction on m, starting with $m \ll 0$, using the exactness of (4.5), and applying (4.9) with J_m for M, we obtain the formula,

$$R^j b_*(J_m) = 0 \qquad \text{for} \quad j \geqq 2 \quad \text{and all } m. \tag{4.11}$$

In particular, since J_0 is equal to I'_y, there is a formula,

$$R^j b_*(I'_y) = 0 \qquad \text{for} \quad j \geqq 2. \tag{4.12}$$

Consider the Leray spectral sequence (GD IV, 2.10),

$$H^i(X, R^j b_*(I'_y)) \Rightarrow H^{i+j}(P, I'_y).$$

By (4.12) and by (4.6), the only nonzero terms on the left occur for $j = 0$ or for $j = 1$ and $i = 0$. So, by (EGA 0_{III}, 11.10.3), there is a formula,

$$\chi(b_*(I'_y)) - h^0(X, R^1 b_*(I'_y)) = \chi(I'_y).$$

Thus, since $b_*(I'_y)$ is isomorphic to I_y (4.3), there is a formula,

$$\chi(I_y) = \chi(I'_y) + h^0(X, R^1 b_*(I'_y)). \tag{4.13}$$

In general, let S be a scheme, and let M and N be two locally free $\mathcal{O}_S$-Modules with a finite rank. Then, using Serre's explicit calculation (EGA III_1, 2.1.12) and an appropriate Leray spectral sequence, it is easy to obtain an isomorphism,

$$H^n(S, N) \xrightarrow{\sim} H^n(\boldsymbol{P}(M), q^*N), \tag{4.14}$$

for each integer n, where $q : \boldsymbol{P}(M) \to S$ denotes the structure morphism. Applying (2.3) and (4.14) with Y for S, with $(\mathcal{O}_Y(1) \oplus \mathcal{O}_Y)$ for M, and with

$\mathcal{O}_Y(-d_y)$ for N yields an isomorphism,

$$H^n(Y, \mathcal{O}_Y(-d_y)) \xrightarrow{\sim} H^n(P, I'_y),$$

for each integer n. Therefore, there is a formula,

$$\chi(\mathcal{O}_Y(-d_y)) = \chi(I'_y). \tag{4.15}$$

Let D be any divisor on Y. Then Riemann's theorem (GD VIII, 1.4) asserts the formula,

$$\chi(\mathcal{O}_Y(D)) = \deg(D) + \chi(\mathcal{O}_Y).$$

For a plane curve C with degree n, there are formulas,

$$\chi(\mathcal{O}_C) = \chi(\mathcal{O}_{\boldsymbol{P}^2}) - \chi(\mathcal{O}_{\boldsymbol{P}^2}(-n)) = 1 - \frac{(n-1)(n-2)}{2}.$$

Since Y has degree 3, we therefore have the formula,

$$\chi(\mathcal{O}_Y(D)) = \deg(D). \tag{4.16}$$

Combining the formulas (4.15) and (4.16) with $-d_y$ for D yields the formula,

$$\chi(I'_y) = -3. \tag{4.17}$$

Thus, to compute $\chi(I_y)$, it remains in view of (4.13) to compute $h^0(X, R^1b_*(I'_y))$.

By (2.6) and (2.7), there is an isomorphism,

$$e^*(J_m) \cong \mathcal{O}_Y(-m) \otimes \mathcal{O}_Y(-d_y).$$

So, the section e induces an isomorphism,

$$H^j(E, J_m|E) \cong H^j(Y, \mathcal{O}_Y(-d_y)(-m)), \qquad \text{for each } j \text{ and } m. \tag{4.18}$$

For any divisor D on Y, there is a formula,

$$h^1(Y, \mathcal{O}_Y(D)) = h^0(Y, \mathcal{O}_Y(-D)), \tag{4.19}$$

by duality (GD I, 1.3) because the dualizing sheaf is trivial by (GD I, 2.4

with Y for D and I, 3.1 with 2 for n). Moreover, $h^0(Y, \mathcal{O}_Y(D))$ is obviously equal to zero if the inequality $\deg(D) < 0$ holds. Since, for each m, the degree of $c_1(\mathcal{O}_Y(-d_y)(-m))$ is equal to $-3m-3$, we therefore have the formulas,

$$H^0(Y, \mathcal{O}_Y(-d_y)(-m)) = 0, \qquad \text{for} \quad m \geqq 0, \tag{4.20}$$

$$H^1(Y, \mathcal{O}_Y(-d_y)(-m)) = 0, \qquad \text{for} \quad m \leqq -2. \tag{4.21}$$

In view of (4.21), (4.18), and (4.8), there is a formula,

$$R^1 b_*(J_m|E) = 0, \qquad \text{for} \quad m \leqq -2.$$

Therefore, since $R^1 p_*(J_m)$ is equal to zero for $m \ll 0$ by (4.10), the exactness of (4.5) with $j = 0$ yields the formula,

$$R^1 b_*(J_m) = 0 \qquad \text{for all} \quad m \leqq -2, \tag{4.22}$$

by induction on m.

Since $R^2 b_*(J_{-2})$ is equal to zero (4.11), formula (4.22) and the exactness of (4.5) imply that $R^1 b_*(J_{-1})$ is isomorphic to $R^1 b_*(J_{-1}|E)$. Therefore, by (4.18) and (4.7), there is an isomorphism,

$$H^0(X, R^1 b_*(J_{-1})) \cong H^1(Y, \mathcal{O}_Y(-d_y)(1)). \tag{4.23}$$

In general, let M be an invertible sheaf on a complete, integral curve C over k, and s, a nonzero global section of M. It is evident that, if s has the value zero at some point, then the degree of $c_1(M)$ is strictly positive, and that, if s has no zeroes, then it defines an isomorphism from $\mathcal{O}_C$ onto M. Hence, if $c_1(M)$ has degree zero, then either M is nontrivial and $H^0(C, M)$ is equal to 0, or M is isomorphic to $\mathcal{O}_C$ and $H^0(C, M)$ is equal to k (the k-vector space $H^0(C, \mathcal{O}_C)$ is an integral domain with a finite k-dimension, so equal to k).

Obviously, $c_1(\mathcal{O}_Y(-d_y)(1))$ has degree zero and is isomorphic to $\mathcal{O}_Y$ for $y = y_3$. Recall from the end of Section 2 that, if $\mathcal{O}_Y(-d_y)(1)$ is isomorphic to $\mathcal{O}_Y$ (equivalently, if $\mathcal{O}_Y(d_y)$ is isomorphic to $\mathcal{O}_Y(1)$), then y is equal to y_3. Hence, by the general observations above, $H^0(Y, \mathcal{O}_Y(-d_y)(1))$ is equal to 0 for $y \neq y_3$ and to k for $y = y_3$. Therefore, by Riemann's theorem (4.16), clearly $h^1(Y, \mathcal{O}_Y(-d_y)(1))$ is equal to 0 for $y \neq y_3$ and to 1 for $y = y_3$. Consequently, by (4.23), there are formulas,

$$h^0(X, R^1b_*(J_{-1})) = 0, \qquad \text{for} \quad y \neq y_3 \tag{4.24}$$

$$h^0(X, R^1b_*(J_{-1})) = 1, \qquad \text{for} \quad y = y_3.$$

By (4.20), there is a formula,

$$H^0(Y, \mathcal{O}_Y(-d_y)) = 0. \tag{4.25}$$

So, by Riemann's Theorem (4.16), there is a formula,

$$h^1(Y, \mathcal{O}_Y(-d_y)) = 3.$$

Hence, (4.18) and (4.7) yield the formula,

$$h^0(X, R^1b_*(J_0|E)) = 3. \tag{4.26}$$

Moreover, (4.25), (4.18), and (4.8) yield the formula,

$$b_*(J_0|E) = 0. \tag{4.27}$$

So, since $R^2b_*(J_{-1})$ is equal to zero by (4.11), the exact sequence (4.5) with $j = 0$ and $m = 0$ becomes the exact sequence,

$$0 \to R^1b_*(J_{-1}) \to R^1b_*(J_0) \to R^1b_*(J_0|E) \to 0.$$

Since $H^1(X, R^1b_*(J_{-1}))$ is equal to 0 by (4.6), there is an exact sequence,

$$0 \to H^0(X, R^1b_*(J_{-1})) \to H^0(X, R^1b_*(J_0)) \to H^0(X, R^1b_*(J_0|E)) \to 0.$$

Hence, since J_0 is equal to I'_y, formulas (4.24) and (4.26) yield the formulas,

$$h^0(X, R^1b_*(I'_y)) = 3, \qquad \text{for} \quad y \neq y_3, \tag{4.28}$$

$$h^0(X, R^1b_*(I'_{y_3})) = 4. \tag{4.29}$$

Finally, combining (4.28) and (4.29) with (4.17) and (4.13) yields the formulas,

$$\chi(I_y) = 0, \qquad \text{for} \quad y \neq y_3,$$

$$\chi(I_{y_3}) = 1.$$

REFERENCES

[1] A. ALTMAN and S. KLEIMAN: Algebraic systems of linearly equivalent divisor-like subschemes. *Compositio Math.*, vol. 29 (1974) 113–139.

[2] A. ALTMAN and S. KLEIMAN: *Introduction to Grothendieck Duality Theory*. Lecture Notes in Math., Vol. 146, Springer-Verlag (1970) (cited GD).

[3] A. ALTMAN and S. KLEIMAN: Joins of schemes, linear projections (to appear).

[4] A. GROTHENDIECK and J. DIEUDONNÉ: *Eléments de Géométrie Algébrique*. Publ. Math. No. 8, 11, 17, 24, 28, 32. IHES, (1961, 1961, 1963, 1965, 1966, 1967) (cited EGA II, III_1 or 0_{III}, III_2, IV_2, IV_3, IV_4).

[5] R. GODEMENT: *Topologie Algébrique et Théorie des Faisceaux*. Hermann, Paris (1958).

[6] D. MUMFORD: *Lectures on Curves on an Algebraic Surface*. Annals of Math. Studies, No. 59, Princeton University Press (1966).

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