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WEIERSTRASS POINTS AND MODULI OF CURVES

Enrico Arbarello *

Introduction

This paper is devoted to the study of a particular class of subvarieties of the moduli space of stable curves of genus g. These varieties are the loci of moduli of special curves distinguished by the presence of a Weierstrass point of a particular type. These subloci of $\bar{M}_g$ (cf. sec. 3) can be viewed as a natural generalization of the hyperelliptic locus. They are defined in the following way. For each n, $2 \leqq n \leqq g$, we let $\bar{W}_{n,g} =$ (closure in $\bar{M}_g$ of the sublocus of moduli of curves C possessing a Weierstrass point x such that $\dim H^0(\mathcal{O}_C(nx)) \geqq 2$). These subvarieties of $\bar{M}_g$ (more precisely their preimages in the Teichmüller space) were studied by Rauch [9] and Farkas [4]. Their main result is the following. (cf. sec. 3 for the definition of Weierstrass sequence).

THEOREM (1) (Rauch): *The totality of closed Riemann (Torelli, Teichmüller) surfaces of genus g possessing Weierstrass points whose Weierstrass sequences begin with a fixed $n \leqq g$ form a complex analytic (possibly disconnected) submanifold of dimension $2g+n-3$ of the modulus (Torelli, Teichmüller) space in the (general) case when $n+1$ is a gap. When $n+1$ is not a gap, they form a complex analytic subvariety of dimension $2g+n-4$ of a complex analytic submanifold of dimension $2g+n-3$.*

Rauch observes that '*the distinction of cases in Theorem* 1 *represents the discovery of a new sort of "singularity" in the set of Riemann surfaces*' and he adds, '*It must be mentioned that it is not altogether clear that the singularity in the exceptional case of Theorem* 1 *is genuine and not the product of the method of proof*'.

We shall analyze the above subloci from a point of view which differs substantially from the one of Rauch and Farkas. Our approach has been inspired by Fulton's construction of the Hurwitz space $H^{n,w}$, whose points parametrize the set of n-sheeted coverings of $\mathbb{P}^1$ with w simple ramification points. We will in fact introduce a Weierstrass-Hurwitz

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space $WH^{n,w}$ whose points parametrize the set of *simple Weierstrass coverings of type* (n, w), (cf. Definition 2.1). The analytic manifold $WH^{n,w}$ defines then an analytic subvariety of $\bar{M}_g$ whose closure we denote by $\bar{W}_{n,g}$. In Theorem 3.11 we prove that $\bar{W}_{n,g}$ is irreducible and that *dim* $W_{n,g} = 2g+n-3$. In this theorem the irreducibility statement plays a fundamental role in the proof of the dimensionality statement. It is from this point of view that it seems more convenient to work with the moduli space $\bar{M}_g$ rather than the Teichmüller space T^g (for instance it is not at all clear that the preimages of the $\bar{W}_{n,g}$'s in T^g are connected). We also notice that the dimensionality statement on $\bar{W}_{n,g}$ implicitly proves that, indeed, the first case in Rauch's theorem is the "generic" one.

In Theorem 3.18 we then prove that $\bar{W}_{n,g}$ is the n^{th} level of a filtration

$$\bar{W}_{2,g} \subset \bar{W}_{3,g} \subset \cdots \subset \bar{W}_{g-1,g} \subset \bar{W}_{g,g} = \bar{M}_g.$$

The proof of this fact is based on a degeneration argument which shows that any simple Weierstrass covering of type $(n-1, w-1)$ can be thought of as the 'limit' of simple Weierstrass coverings of type (n, w).

A second use of such a degeneration argument comes in the proof of Theorem 3.27. There we prove that given any algebraic curve S in $\bar{W}_{n,g}$ then either some points of S correspond to a curve of genus less than g or S has non-trivial intersection with $\bar{W}_{n-1,g}$. The proof of this fact is based on the following observation. Given any non trivial one-parameter, algebraic, family of n-sheeted Weierstrass coverings, then there are points in the parameter space for which the corresponding n-sheeted covering is degenerate, in the sense that the point of total ramification and another ramification point '*come together*'. One then proves that, when this phenomenon occurs, then either the corresponding curve acquires a node or the Weierstrass point increases its weight. When $n = g-1$, Theorem 3.27 asserts that the divisor $D = \bar{W}_{g-1,g} \cup$ (*boundary of* $\bar{M}_g$) is pseudo-ample in $\bar{M}_g$. The pseudo-ampleness of D acquires a particular significance in light of the fact that the boundary of $\bar{M}_g$ is not pseudo-ample in $\bar{M}_g$, nor is, in general, $\bar{W}_{g-1,g}$ as we shall prove in Theorem 3.28.

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1. The Hurwitz Space

Let X be a compact Riemann surface. Let $f: X \to \mathbb{P}^1$ be an analytic map of X onto the Riemann sphere. Given any point $x \in X$ there exists a neighborhood U of x on which f is conformally equivalent to a map $z \to z^{e(x)}$ of the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ onto itself. The integer $e(x)$ is called the *ramification index of* f *at* x. The set $R = \{x \in X : e(x) \geqq 2\}$ is called the *ramification locus of* f. The set $f(R) \subset \mathbb{P}^1$ is called the *branch locus of* f and it is denoted by $\Lambda(f)$. The branch locus is the support of the divisor

$$(1.1) \qquad \delta(f) = \sum_{y \in \mathbb{P}^1} \Big(\sum_{x \in f^{-1}(y)} (e(x)-1)\Big) y$$

which is called the *discriminant of* f. If $\sum_{x \in f^{-1}(y)} e(x) = n$, $y \in \mathbb{P}^1$, we say that f is an *n-sheeted branched covering* of $\mathbb{P}^1$. A point $x \in X$ is a point of *simple ramification* for f if $e(x) = 2$ and if x is the only point of $f^{-1}f(x)$ at which f ramifies. A point $x \in X$ is a point of *total ramification* for the n-sheeted branched covering f if $e(x) = n$. If $f : X \to \mathbb{P}^1$ is an n-sheeted branched covering and if $g =$ *genus* (X), then the well known *Hurwitz formula* gives

$$(1.2) \qquad 2n+2g-2 = \text{degree } (\delta(f)).$$

Two n-sheeted branched coverings $f : X \to \mathbb{P}^1$ and $g : X' \to \mathbb{P}^1$ are said to be *equivalent* if there exists a biholomorphic map $\varphi : X \to X'$ such that $g \cdot \varphi = f$. The equivalence class containing f will be denoted by (f).

Let $(\mathbb{P}^1)^{(w)}$ denote the w-fold symmetric product of $\mathbb{P}^1$. Let $\Delta \subset (\mathbb{P}^1)^{(w)}$ be the *discriminant locus*, i.e., the subset of $(\mathbb{P}^1)^{(w)}$ formed by those w-tuples which contain fewer than w distinct points. Since $(\mathbb{P}^1)^{(w)}$ may be identified with the complex projective w-space $\mathbb{P}^w$, instead of $(\mathbb{P}^1)^{(w)} - \Delta$ we shall write $\mathbb{P}^w - \Delta$.

For each $A \in \mathbb{P}^w - \Delta$, let $H(n, A)$ denote the set of equivalence classes of n-sheeted branched coverings of $\mathbb{P}^1$ whose branch locus is equal to A. Let $H(n, w)$ denote the set of equivalence classes of n-sheeted branched coverings of $\mathbb{P}^1$ with w branch points, and let

$$(1.3) \qquad \Lambda : H(n, w) \to \mathbb{P}^w - \Delta$$

be the function defined by $\Lambda((f)) = \Lambda(f)$. We then have $\Lambda^{-1}(A) = H(n, A)$. It is an easy matter to introduce a topology on $H(n, w)$ in such a way that Λ becomes a *topological covering*. For a proof of this fact we refer the reader to ([5], page 545). Via the covering Λ the topological space $H(n, w)$ can be equipped with the analytic structure induced from that of $\mathbb{P}^w - \Delta$.

DEFINITION (1.4): *The complex manifold* $H(n, w)$ *is called the Hurwitz space of type* (n, w).

We shall also use the following notation. Let A be a finite subset of $\mathbb{P}^1$, let $y \in \mathbb{P}^1 - A$. Let $\mathfrak{S}_n$ be the symmetric group on n letters acting on the set $\{1, \cdots, n\}$. Let $\overline{\text{Hom}}\,(\pi_1(\mathbb{P}^1 - A, y), \mathfrak{S}_n)$ denote the set of equivalence classes of homomorphisms from $\pi_1(\mathbb{P}^1 - A, y)$ to $\mathfrak{S}_n$, where two homomorphisms are equivalent if they differ by an inner automorphism. For each $f : X \to \mathbb{P}^1$, $(f) \in H(n, A)$ we define

$$\Phi((f)) \in \overline{\text{Hom}}\,(\pi_1(\mathbb{P}^1 - A, y), \mathfrak{S}_n) \tag{1.5}$$

in the following manner. Let $\alpha : \{1, \cdots, n\} \to f^{-1}(y)$ be a numbering of $f^{-1}(y)$. Let $\gamma \subset \mathbb{P}^1 - A$ be a loop with initial point at y. Let $\tilde{\gamma}_k$ be the lifting of γ to $X - f^{-1}(A)$ with initial point at $\alpha(k) \in f^{-1}(y)$. Then one can easily see (cf. [5], (1.2)) that setting $\Phi((f))(\gamma)(k) = \alpha^{-1}$ (*end point of* $\tilde{\gamma}_K$), for each $k \in \{1, \cdots, n\}$, gives a well-defined element of

$$\overline{\text{Hom}}\,(\pi_1(\mathbb{P}^1 - A, y), \mathfrak{S}_n).$$

We finish this section by fixing, once and for all, a standard way to choose a system of generators for $\pi_1(\mathbb{P}^1 - A, y)$, (cf. [5]). Let us fix an orientation on the complex manifold $\mathbb{P}^1$.

DEFINITION (1.6): *Let A be a finite subset of $\mathbb{P}^1$. Let $y \in \mathbb{P}^1 - A$. Let L be a simple, closed, oriented arc containing the points of A. Let G (resp. G') be the region of $\mathbb{P}^1$ on the right (resp. left) side of L. Let L be oriented in such a way that $y \in G$. Let $(a_1, \cdots, a_w) = A$ be an ordering of the points of A such that L passes successively through $a_1, \cdots, a_w$ and returns to a_1. Choose non intersecting simple arcs ℓ_i, in G, from y to a_i. Let G_i be the simply connected region of G enclosed by the lines ℓ_i, ℓ_{i+1} and the arc of L form a_i to a_{i+1}. Let σ_i be a loop which begins at y, travels along ℓ_i to a point near a_i, makes a small clockwise loop around a_i and returns to y along ℓ_i. The loops $\sigma_1, \cdots, \sigma_w$ generate $\pi_1(\mathbb{P}^1 - A, y)$ and satisfy $\Pi_i \sigma_i = id$. Such a system of generators is called a standard system of generators for $\pi_1(\mathbb{P}^1 - A, y)$.*

2. The Weierstrass-Hurwitz space

We shall now restrict our attention to a particular component of $H(n, w)$.

We recall that a point x on a Riemann surface X is called a Weierstrass point if $\dim H^0(\mathcal{O}_X(nx)) \geqq 2$ for some positive integer n such that $n \leqq genus\,(X)$. The following terminology is therefore suggested.

DEFINITION (2.1): *An n-sheeted covering $f: X \to \mathbb{P}^1$ with w branch points is called a simple Weierstrass covering of type (n, w) if there exists a point $x \in X$ such that $e(x) = n$ and such if every ramification point $x' \in X - \{x\}$ is simple.*

We observe that if $f: X \to \mathbb{P}^1$ is a simple Weierstrass covering of type (n, w) then the Hurwitz formula (1.2) determines the genus of X, namely

$$\text{(2.2)} \qquad \text{genus}(X) = (\tfrac{1}{2})(w-n).$$

We denote by $WH^{n,w}$ the set of equivalence classes of simple Weierstrass coverings of type (n, w). It is easy to see that $WH^{n,w}$ is both open and closed in $H(n, w)$ and it is therefore a union of connected components of $H(n, w)$. It follows that the covering (1.3) restricts to a covering

$$\text{(2.3)} \qquad \Lambda : WH^{n,w} \to \mathbb{P}^w - \Delta.$$

DEFINITION (2.4): *The complex manifold $WH^{n,w}$ is called the Weierstrass Hurwitz space of type (n, w).*

By using an argument which is essentially due to Lüroth and Clebsch one can prove

THEOREM (2.5): *$WH^{n,w}$ is connected.*

For a proof of this fact we refer the reader to ([1], (2.7); [5], (1.5)). We just observe that given a point $A \in \mathbb{P}^w - \Delta$ and considering the covering (2.3) one proves that $WH^{n,w}$ is connected by showing that the action of $\pi_1(\mathbb{P}^w - \Delta, A)$ on $\Lambda^{-1}(A)$ is transitive. The proof of this fact turns out to be of a combinatorial nature. It consists, in fact, in showing that, given $(f) \in \Lambda^{-1}(A)$, there is a way of assigning to each ramification point x, of f, a permutation among the sheets 'coming together' at x. The result can be stated in the following manner.

LEMMA (2.6): *Let $f: X \to \mathbb{P}^1$ be a simple Weierstrass covering of type (n, w). Then it is possible to find a standard system of generators $\sigma_1, \cdots, \sigma_w$ for $\pi_1(\mathbb{P}^1 - A(f), y)$, (see Def. (1.6)), such that*

$$\text{(2.7)} \qquad \Phi((f))(\sigma_1, \cdots, \sigma_w) = ((123\cdots n), \underbrace{(12), \cdots, (12)}_{2g+1 \text{ times}}, (23), \cdots, (n-1\ n)),$$

(*where $\Phi((f))$ is defined as in* (1.5)).

REMARK (2.8): With the notation of the preceding lemma, let γ_i be the only connected component of $f^{-1}(\sigma_{i+1}\cdot\sigma_{i+2})$, $1 \leqq i \leqq 2g$, which is a two-sheeted covering of $\sigma_{i+1}\cdot\sigma_{i+2}$. Then the 1-cycles $\gamma_1,\cdots,\gamma_{2g}$ form a basis for $H_1(X)$. Also observe that Lemma 2.6 gives the classical result that any Riemann surface can be obtained from a hyperelliptic surface of the same genus by 'attaching spheres'.

Given the connected manifold $WH^{n,w}$, parametrizing simple Weierstrass coverings of type (n, w), it is easy to construct, analytically, when $n > 2$, a '*universal family*' of Riemann surfaces parametrized by $WH^{n,w}$ (cf. [1], (3.10)). By this we mean that there exists a pair (Y, h) of an analytic manifold Y and an analytic map

$$h : Y \to \mathbb{P}^1 \times WH^{n,w} \tag{2.9}$$

having the following properties

(t) *for each* $s \in WH^{n,w}$ *the fiber* $Y_s = h^{-1}(\mathbb{P}^1 \times \{s\})$ *is a Riemann surface of genus* $g = (\frac{1}{2})(w-n)$.

(ii) *for each* $s \in WH^{n,w}$, $h_s = h|_{Y_s}$ *is an n-sheeted covering such that* $(h_s) = s$.

The universal family (2.9) is obtained by first constructing families $f_\alpha : Y_\alpha \to \mathbb{P}^1 \times \mathrm{V}_\alpha$ for an open cover $\{V_\alpha\}_{\alpha\in I}$ of $WH^{n,w}$ (see [1], (3.1)) and then by patching together these local data.

We finish this section by observing that the Weierstrass-Hurwitz space $WH^{n,w}$ satisfies a 'universal property' in the following sense. If V and Y are connected analytic manifolds and $h : Y \to \mathbb{P}^1 \times V$ an analytic map such that, for each $s \in V$, $h_s : Y_s \to \mathbb{P}^1 \times \{s\}$ is a *simple Weierstrass covering of type* (n, w), then there exists an analytic map $\varphi : V \to WH^{n,w}$ such that $\varphi(s) = (h_s)$, for each $s \in V$. For a proof of this fact we refer the reader to ([5], (1.7)) where the word 'simple covering' should be replaced by 'simple Weierstrass covering'. In exactly the same way one can prove that $WH^{n,w}$ is an algebraic variety ([5], (6.4), (7.3)).

3. A filtration of the moduli space

We denote by M_g the *moduli space* of smooth, irreducible algebraic curves of genus g defined over the complex numbers. If C is a smooth *algebraic curve* of genus g we let $m(C) \in M_g$ denote the point in M_g corresponding to C. Also we let $\bar{M}_g$ denote the Mumford-Deligne [3] compactification of M_g, i.e., the moduli space of *stable curves* of genus g (see Def. 3.13 below).

For an irreducible algebraic curve C and a positive divisor D, of degree n, on C, we shall denote by $|D|$ the *complete linear series* determined by D, and by $\ell(D)$ the dimension of $H^0(\mathcal{O}_C(D))$, so that dim $|D| = \ell(D)-1$. Also we shall occasionally denote by $g_n^r(D)$ (or simply g_n^r) a linear series of dimension r contained in $|D|$.

Consider now, for $n > 2$, the Weierstrass-Hurwitz space $WH^{n,w}$ together with the universal family

$$h : Y \to \mathbb{P}^1 \times WH^{n,w} \tag{3.1}$$

described in section 2. Let $g = (\frac{1}{2})(w-n)$. Then the *universal property* of M_g, (cf. [8] page 99), gives a morphism

$$m : WH^{n,w} \to M_g \tag{3.2}$$

such that, for each $s \in WH^{n,w}$, $m(s) = m(Y_s)$.

For $n > 2$ we set $W_{n,g} = m(WH^{n,w})$ and we let $W_{2,g} =$ (*hyperelliptic locus*) $\subset M_g$. Also we let $\bar{W}_{n,g}$ denote the *closure* of $W_{n,g}$ in $\bar{M}_g$.

Definition (3.3): *The subvariety $\bar{W}_{n,g}$ of M_g is called the Weierstrass space of type (n, g).*

We recall that given a point x on a curve of genus g, a positive integer n, $n \leqq 2g-1$, is said to be a *gap* for x if $\ell(nx) = \ell((n-1)x)$ and that the sequence $n_1, n_2, \cdots$ of *non-gaps* is called *Weierstrass sequence* of x. As we mentioned in the Introduction, the spaces $\bar{W}_{n,g}$ (more precisely their preimages in the Teichmüller space) were analyzed by Rauch and Farkas. Following their language we can say that

(3.4) $\bar{W}_{n,g} \supset \{$*space of moduli of curves of genus g having a W. pt. whose first non-gap is n*$\}$

The relation (3.4) is intuitively clear since, again intuitively, any n-sheeted Weierstrass covering of genus g, $f_0 : Y_0 \to \mathbb{P}^1$ (not necessarily simple), is the '*limit*' of *simple* Weierstrass coverings. To be more precise one should show that there exists a smooth surface Y and an analytic map $h : Y \to \mathbb{P}^1 \times V$, $V = \{t \in \mathbb{C} : |t| < 1\}$, such that for all $t \in V - \{0\}$, $(h_t) \in WH^{n,w}$ and $h_0 = f_0$. In the proof of Theorem 3.18 we will prove the existence of such families in a very similar setting. That proof, with only minor changes, can be used to prove (3.4).

We shall use the following result due to Rauch [9]. Let T^g be the *Teichmüller space* of genus g.

Theorem (3.5) (Rauch): *The sublocus of T^g consisting of Riemann* *surfaces having a Weierstrass point x for which ν and ν+1 are respectively the first and second non-gap, is a subvariety of T^g of dimension $2g+\nu-4$.*

Consider now the universal family (3.1). For each $s \in WH^{n,w}$ we denote by $y(s) \in Y_s$ the point of total ramification of h_s. Also consider the finite topological covering (2.3). Let (X_0, X_1) be projective coordinates in $\mathbb{P}^1$ and denote by $\infty \in \mathbb{P}^1$ the point of coordinates (0, 1). Let

$$\mathbb{P}^w_\infty - \Delta = \{(a_1, \cdots, a_w) \in \mathbb{P}^w - \Delta : a_i = \infty \text{ for some } i\}.$$

We let

$$WH^{n,w}_\infty = \{s \in \Lambda^{-1}(\mathbb{P}^w_\infty - \Delta) : h_s(y(s)) = (\infty, s)\}. \tag{3.6}$$

One can easily see that $WH^{n,w}_\infty$ is both open and closed in $\Lambda^{-1}(\mathbb{P}^w_\infty - \Delta)$ and it is therefore a $(w-1)$-*dimensional* (possibly disconnected) *submanifold* of $WH^{n,w}$. Moreover Λ restricts to a finite topological covering $\Lambda : WH^{n,w}_\infty \to \mathbb{P}^w_\infty - \Delta$. We shall denote by

$$h : Y \to \mathbb{P}^1 \times WH^{n,w}_\infty \tag{3.7}$$

the restriction to $WH^{n,w}_\infty$ of the universal family (3.1), and by

$$m : WH^{n,w}_\infty \to M_g \tag{3.8}$$

the induced map.

LEMMA 3.9: $m(WH^{n,w}_\infty) = m(WH^{n,w}) = W_{n,g}$.

PROOF: Let $p \in m(WH^{n,w})$, so that $p = m(s) = m(Y_s)$ for some $s \in WH^{n,w}$. Let $\chi \in \mathbb{P}GL(1)$ be such that $\chi(h_s(y(s))) = \infty$. Then $s' = (\chi \cdot h_s) \in WH^{n,w}_\infty$ and $m(s) = m(s')$.

LEMMA (3.10): *Let $g \geqq n > 2$. Let $p \in W_{n,g} = m(WH^{n,w}_\infty)$. Let C be a smooth curve of genus g such that $p = m(C)$. Let $x_1, \cdots, x_N$ be the Weierstrass points of C which satisfy the following conditions*
(i) $\ell(nx_i) \geqq 2$, $i = 1, \cdots, N$
(ii) *there exists $f_i \in H^0(\mathcal{O}_C(nx_i))$ such that $(f_i) \in WH^{n,w}_\infty$, $i = 1, \cdots, N$.*
Let $\ell = \max \ell(nx_i)$. Then $\dim m^{-1}(p) = \ell$.

PROOF: Let us fix our attention on one of the vector spaces $H^0(\mathcal{O}_C(nx_i))$, and let, for simplicity, $x_i = x$ and $\ell(nx) = r$. Let $g_1, \cdots, g_r$ be a basis for

$H^0(\mathcal{O}_C(nx))$ and let $V_x = \{(\lambda_1, \cdots, \lambda_r) \in \mathbb{C}^r : \Sigma\lambda_j g_j \in WH^{n,w}_\infty\}$. By assumption $V_x \neq \emptyset$ and it is easy to see that V_x is a *Zariski open* in $\mathbb{C}^r$. Also we define a family of simple Weierstrass coverings on C, $F : C \times V_x \to \mathbb{P}^1 \times V_x$, by letting, for each $\lambda = (\lambda_1, \cdots, \lambda_r) \in V_x$, $F_\lambda = \Sigma\lambda_j g_j : C \times \{\lambda\} \to \mathbb{P}^1 \times \{\lambda\}$. Let $V = m^{-1}(p) = m^{-1}(m(C)) \subset WH^{n,w}_\infty$. Then by the universal property of $WH^{n,w}$ we get an analityc map $\varphi : V_x \to V$ defined by $\varphi(\lambda) = (F_\lambda)$, for each $\lambda \in V_x$. Since the automorphism group of C is finite it follows that the map φ is finite to one, so that dim $\varphi(V_x) = r$. By repeating this argument for all the points x_i, $i = 1, \cdots, N$, we conclude, with an obvious notation, that $V \supset \cup_i \varphi_i(V_{x_i})$ and that dim $\varphi_i(V_{x_i}) = \ell(nx_i)$. It now suffices to prove that $V = \cup_i \varphi_i(V_{x_i})$. Let $s \in V$. Consider $h_s : Y_s \to \mathbb{P}^1 \times (s)$, (3.7). Since $s \in V$ there exists an isomorphism $\psi_s : Y_s \to C$ so that $h_s \cdot \psi_s^{-1} \in H^0(\mathcal{O}_C(nx_i))$ for some i. But $(h_s \cdot \psi_s^{-1}) = (h_s) = s$ proving that $s \in \varphi_i(V_{x_i})$ for some i.

THEOREM (3.11): *$\bar{W}_{n,g}$ is an irreducible subvariety of $\bar{M}_g$ of dimension* $2g+n-3$.

PROOF: Consider the morphism $m : WH^{n,w} \to M_g$. Let $\overline{WH^{n,w}}$ be a non-singular compactification of $WH^{n,w}$. It follows then from a theorem of Borel (cf. [2]: Theorem A and 3.10) that m extends to a morphism $m : \overline{WH^{n,w}} \to \bar{M}_g$, so that $\bar{W}_{n,g} = m(\overline{WH^{n,w}})$. The irreducibility of $\bar{W}_{n,g}$ now follows from Theorem 2.5.

As is well known dim $\bar{W}_{2,g} = 2g-1$. We shall prove the theorem by induction. Suppose the theorem true for $v \leqq n-1$. Consider the map (3.8) and let p be a *generic* point in $\bar{W}_{n,g}$. We have

$$\dim \bar{W}_{n,g} = \dim WH^{n,w}_\infty - \dim m^{-1}(p).$$

Since dim $WH^{n,w}_\infty = w-1$, (2.2) gives dim $\bar{W}_{n,g} = 2g+n-1-\ell$, where $\ell = \dim m^{-1}(p)$. Let C be a smooth curve of genus g such that $m(C) = p$. It follows from Lemma 3.10 that there exists a Weierstrass point $x \in C$ with $\ell(nx) = \ell$. In view of the same lemma it now suffices to prove that $\ell = 2$. Suppose, on the contrary, that $\ell > 2$. Let v be the smallest positive integer (first non gap) such that $\ell(vx) = 2$. We then have $v \leqq n-(\ell-2)$. Since v is the *first non gap* for $x \in C$ and since $m(C)$ is a *generic* point in $\bar{W}_{n,g}$ it follows from (3.4) that $\bar{W}_{n,g} \subset \bar{W}_{v,g}$. The induction hypothesis gives dim $\bar{W}_{v,g} = 2g+v-3$, and therefore

$$2g+n-1-\ell = \dim \bar{W}_{n,g} \leqq 2g+v-3.$$

This inequality together with the preceding one gives

$$(3.12) \qquad 2g+n-1-\ell = 2g+\nu-3.$$

But $\bar{W}_{n,g}$ and $\bar{W}_{\nu,g}$ are both *irreducible* so that $\bar{W}_{n,g} = \bar{W}_{\nu,g}$. Also (3.12) gives $\nu = n-(\ell-2)$ which in turn implies $\ell((\nu+1)x) = 3$. Therefore ν and $\nu+1$ are both non gaps for x. The conclusion is that there exists a $(2g+\nu-3)$*-dimensional* subvariety of M_g consisting of birational classes of curves having a Weierstrass point for which ν *and* $\nu+1$ are, respectively, the first and the second non gap. But this contradicts Theorem 3.5.

Before proving our next result we recall the following, (cf. [3] page 76).

Definition (3.13): *Let V be a variety. Let $g \geqq 2$. A stable curve of genus g over V is a proper flat morphism $\pi : Y \to V$ such that the fibers Y_s of π, $s \in V$, are reduced connected curves such that*
(i) *Y_s has only ordinary double points.*
(ii) *no non-singular rational component of Y_s meets other components at less than 3 points.*
(iii) $\dim H^1(\mathcal{O}_{Y_s}) = g$.

We will use the following result (cf. [7] Proposition 1).

Theorem 3.14: *Let $S_1 \subset \bar{M}_g$ be an irreducible curve. Then there exists a non-singular curve S, a finite morphism $p : S \to S_1$, and a stable curve of genus g over S, $\pi : Y \to S$, such that*

(i) *Y is a non-singular surface.*
(ii) *for all $s \in S$, $p(s) = m(\pi^{-1}(s))$.*

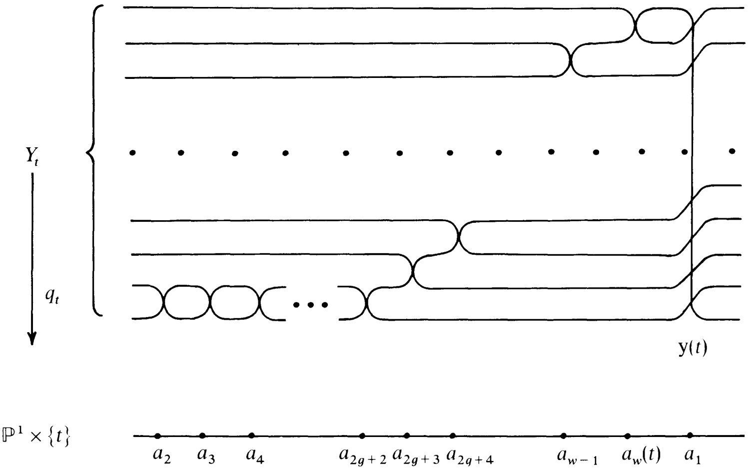
We shall now prove

(3.15)
$$\text{(Hyperelliptic locus)} = \bar{W}_{2,g} \subset \bar{W}_{3,g} \subset \cdots \subset \bar{W}_{g-1,g} \subset \bar{W}_{g,g} = \bar{M}_g.$$

We obtain in this way a *filtration* of $\bar{M}_g$ by means of *irreducible* subvarieties such that $\dim \bar{W}_{n,g} = \dim \bar{W}_{n-1,g}+1$.

In order to prove (3.15) we will show that any simple Weierstrass covering of type $(n-1, w-1)$ can be thought of as a '*limit*' of simple Weierstrass coverings of type (n, w). We will first construct a *stable curve* $\pi : Y \to V$, $V = \{t \in \mathbb{C} : |t| < 1\}$ and then a family of *simple Weierstrass coverings* $q : Y-\pi^{-1}(0) \to \mathbb{P}^1 \times V - \{0\}$ which can be represented, by the following picture (cf. (2.7)).

(3.16)



Here $q_t : Y_t \to \mathbb{P}^1 \times \{t\}$ is a *simple Weierstrass covering of type* (n, w) varying *analytically* with t and such that $\Lambda(q_t) = (a_1, \cdots, a_{w-1}, a_w(t))$. As t tends to 0, $a_w(t)$ tends to a_1, and, as this picture suggests, the curve Y_t tends to a (non stable) curve $\tilde{Y}_0 = \mathbb{P}^1 \cup Y_0'$, while the covering q_t tends to a map $q_0 : \tilde{Y}_0 \to \mathbb{P}^1$ such that $q_0 : Y_0' \to \mathbb{P}^1$ is a *simple Weierstrass covering of type* $(n-1, w-1)$. This phenomenon should be interpreted in the following way. If we denote by $y(t) \in Y_t$, $t \in V$, the varying Weierstrass point, then, when $t = 0$, the linear series $|ny(t)|$ acquires a fixed point at $y(0)$ and the curve $\tilde{Y}_0$ is obtained by *blowing up* Y_0 at $y(0)$.

REMARK (3.17): *Let S be a reduced irreducible curve. Let $\pi : Y \to S$ be a stable curve of genus g over S. Let Γ be a reduced curve contained in Y and such that the restriction of π to Γ is an N-sheeted branched covering. It is then easy to see that there exists a finite branched covering*

$$\varphi : \tilde{S} \to S$$

and a stable curve

$$\tilde{\pi} : \tilde{Y} \to \tilde{S}$$

such that

(i) *the pull back $\tilde{\Gamma}$ of Γ to $\tilde{Y}$ consists of N distinct components $\tilde{\Gamma}_1, \cdots, \tilde{\Gamma}_N$, such that $\tilde{\pi} : \tilde{\Gamma}_i \to \tilde{S}$ is an isomorphism.*

(ii) *for each $\tilde{s} \in \tilde{S}$ the curves $\tilde{\pi}^{-1}(s)$ and $\pi^{-1}(\varphi(s))$ are isomorphic.*

The stable curve $\tilde{\pi} : \tilde{Y} \to \tilde{S}$, is said to be obtained by *unwinding* Y *along* Γ.

THEOREM (3.18): $\bar{W}_{n-1,g} \subset \bar{W}_{n,g}$.

PROOF: Let p be a *generic* point in $\bar{W}_{n-1,g}$. The Theorem will be proved by constructing a *stable curve*

$$\pi : Y \to S \tag{3.19}$$

over a non singular curve S and a neighborhood V of a point $s_0 \in S$ such that, for all $s \in V - \{s_0\}$, $m(\pi^{-1}(s)) \in W_{n,g}$ and $m(\pi^{-1}(s_0)) = p$. Let

$$f : X \to \mathbb{P}^1 \tag{3.20}$$

be such that $(f) \in WH_\infty^{n-1,w-1}$ and $m((f)) = p$ (cf. (3.8)). Let

$$\Lambda(f) = A = \{a_1, \cdots, a_{w-1}\} \in \mathbb{P}_\infty^{w-1} - \Delta,$$

with $a_1 = \infty$. Let $y \in \mathbb{P}^1 - A$ and $\sigma_1, \cdots, \sigma_{w-1}$ a *standard system of generators for* $\pi_1(\mathbb{P}^1 - A, y)$, (cf. (1.6)), such that
$\Phi((f))(\sigma_1, \cdots, \sigma_{w-1}) = ((123 \cdots n-1), (12), \cdots, (12), (23), \cdots, (n-2 \;\; n-1))$, (cf. (2.6), (2.7)). Let $D \subset \mathbb{P}_1 - \{a_2, \cdots, a_{w-1}\}$ be a small disc centered at a_1. Let $z_1 \in D - \{a_1\}$, and $\sigma_w \subset \mathbb{P}^1$ a loop such that $\sigma_1, \cdots, \sigma_{w-1}, \sigma_w$ is a *standard system of generators for* $\pi_1(\mathbb{P}^1 - A \cup \{z_1\}, y)$. Let

$$f_1 : X_1 \to \mathbb{P}^1 \tag{3.21}$$

be such that $(f_1) \in WH_\infty^{n,w}, \Lambda(f_1) = A \cup \{z_1\}$ and such that $\Phi((f_1))(\sigma_1, \cdots, \sigma_w)$ satisfies (2.7). Let $R \subset \mathbb{P}_\infty^w - \Delta$ be the curve

$$R = \{(a_1, \cdots, a_{w-1}, z) \in \mathbb{P}_\infty^w - \Delta : z \in \mathbb{P}^1 - A\},$$

and let $\tilde{R} \subset WH_\infty^{n,w}$ be the connected component of $\Lambda^{-1}(R)$ containing (f_1). Let $S_1 \subset \bar{W}_{n,g}$ be the closure of $m(\tilde{R}) \subset W_{n,g}$. It easily follows from the definition of S_1 and from Lemma 3.10 that $\dim S_1 = 1$ and that $S_1 \not\subset \bar{W}_{n-1,g}$. Let now $p : S \to S_1$ be the *finite* morphism and

$$\pi : Y \to S \tag{3.22}$$

the *stable* curve of genus g given by Theorem 3.14. Consider the Zariski open $S' \subset S$ defined by $S' = p^{-1}(m(\tilde{R}) - m(\tilde{R}) \cap \bar{W}_{n-1,g})$. We claim that *after unwinding* Y, if necessary, one can assume that there exist *distinct* sections

$$(3.23)\qquad y_1 = r_1, r_2, \cdots, r_w : S \to Y$$

such that for each $s \in S'$

(i) $\ell(ny_1(s)) = 2$

(ii) *there exists* $z \in \mathbb{P}^1 - A$ *and* (*a unique*) $f_s \in H^0(\mathcal{O}_{Ys}(ny_1(s)))$ *with* $\Lambda(f_s) = (a_1, \cdots a_w, z)$

(iii) $r_1(s), \cdots, r_w(s)$ *are the ramification points of the* $g_n^1(ny_1(s))$.

It follows in fact from the definition of S' that the set

$$\Gamma = (\text{closure in } Y \text{ of } \{y \in Y : y \text{ satisfies (i) and (ii)}\})$$

defines a curve in Y. By unwinding Y along Γ (cf. Remark 3.17) we may assume that there exists a section $y_1 : S \to Y$ satisfying (i) and (ii). It is then easy to check that, by further unwinding Y, one may assume the existence of sections r_i, $1 \leqq i \leqq w$, satisfying (i), (ii), and (iii).

We then define an analytic map

$$(3.24)\qquad q : \pi^{-1}(S') \to \mathbb{P}^1 \times S'$$

by setting $q_s = f_s$ for each $s \in S'$. From the universal property of $WH^{n,w}$ we get a surjective analytic map $\varphi : S' \to \tilde{R}$ such that, for each $s \in S'$, $\varphi(s) = (q_s)$. It follows from our construction that there is an open set $V \subset S$, homeomorphic to the unit disc, and a point $s_0 \in V$, such that

(a) $V - \{s_0\} \subset S'$, i.e., $m(\pi^{-1}(s)) \in W_{n,q}$ for each $s \in V - \{s_0\}$.

(b) $(f_1) \in \varphi(V - \{s_0\})$, (cf. (3.21)).

(c) $\Lambda \cdot \varphi(V - \{s_0\}) = \{(a_1, \cdots, a_{w-1}, z) : z \in D\}$.

We shall prove that $m(Y_{s_0}) = p$. We first show that Y_{s_0} is *non singular.* It follows in fact from (a), (b), (c) and Lemma 2.6 that, for each $s \in V - \{s_0\}$, there exists a *standard system of generators* $\sigma_1(s), \cdots, \sigma_w(s)$ for $\pi_1(\mathbb{P}^1 - \Lambda(q_s), y)$ such that $\sigma_i(s) = \sigma_i$ for $2 \leqq i \leqq w-1$, and such that

$$\Phi((q_s))(\sigma_1(s), \sigma_2, \cdots, \sigma_{2g+2}, \sigma_{2g+3}, \cdots, \sigma_{w-1}, \sigma_w(s))$$
$$= ((123 \cdots n), (12), \cdots, (12), (23), \cdots, (n-2\ \ n-1), (n-1, n)).$$

Let now $\gamma_i(s)$, $1 \leqq i \leqq 2g$ be the only connected component of $q_s^{-1}(\sigma_{i+1} \cdot \sigma_{i+2})$ which is a two-sheeted covering of $\sigma_{i+1} \cdot \sigma_{i+2}$. The 1-cycles $\gamma_1(s), \cdots, \gamma_{2g}(s)$ form a basis of $H_1(Y_s)$ which varies continuously with s (cf. Remark 2.8). Therefore for $s \in V - \{s_0\}$ the *monodromy map* $M: H_1(Y_s) \to H_1(Y_s)$, (cf. [10], VI.6), is the identity map. This fact together with the stability of Y_{s_0} implies that Y_{s_0} is a non singular curve of genus g.

We must now show that $m(Y_{s_0}) = p$. Consider the map (3.24) and let $pr: \mathbb{P}^1 \times S' \to \mathbb{P}^1$ be the projection. Let $C_t = \{\text{closure in } Y \text{ of } (pr \cdot q)^{-1}(t)\}$, for each $t \in \mathbb{P}^1$. Since for $s \in S'$ the algebraic system $\{C_t\}_{t \in \mathbb{P}^1}$ *cuts out* on Y_s the linear series g_n^1 $(ny_1(s))$, it follows ([10], V.1) that the algebraic system $\{C_t\}_{t \in \mathbb{P}^1}$ is *linear*. Therefore $\{C_t\}_{t \in \mathbb{P}^1}$ cuts out on Y_{s_0} a linear series of *degree n* which we denote by g_n^1. We claim that the g_n^1 has *one fixed point*. Suppose not, then one can extend $q: \pi^{-1}(V - \{s_0\}) \to \mathbb{P}^1 \times (V - \{s_0\})$ to an analytic map $q: \pi^{-1}(V) \to \mathbb{P}^1 \times V$. Since for each $s \in V - \{s_0\}$, $q_s(r_i(s)) = a_i \neq a_j$, for $1 \leqq i < j \leqq w-1$, (cf. (iii)), it follows that $r_i(s_0) \neq r_j(s_0)$, $1 \leqq i < j \leqq w-1$. From this and from the fact that $r_1(s_0) = r_w(s_0)$ it is easy to show that

$$\delta(q_{s_0}) = na_1 + \sum_{i=1}^{w-1} a_i.$$

So that *degree* $(\delta(q_{s_0})) = n + w - 1$ contradicting the Hurwitz formula (1.2).

Notice that since $ny_1(s_0) \in g_n^1$, the point $y_1(s_0)$ is the only fixed point of the g_n^1. Also observe that q can be extended to $\pi^{-1}(V) - \{y_1(s_0)\}$. From these two facts it easily follows that $r_i(s_0) \neq y_1(s_0)$ for $2 \leqq i \leqq w-1$. Therefore the g_n^1 gives rise to a v-sheeted simple Weierstrass covering $q_0: Y_{s_0} \to \mathbb{P}^1$ such that $\delta(q_0) = va_1 + \sum_{i=1}^{w-1} a_i$. Since Y_{s_0} is a curve of *genus g* it follows from (1.2) that $v = n - 1$. It now follows from our construction that $Y_{s_0} \cong X$ (cf. 3.20), and the theorem is proved.

REMARK (3.25): By using an argument which is essentially due to Severi, it can be shown that for $2 \leqq v \leqq 5$ and $g \geqq 2$, $\bar{W}_{v,g}$ is *unirational* ([1], (4.63)).

We now proceed to analyze *degenerating families of Weierstrass coverings*. Before proving our next result we recall the following.

DEFINITION (3.26): *Let V be an algebraic variety. An effective divisor $D \subset V$ is said to be pseudo-ample in V if, for every algebraic curve $S \subset V$, $D \cap S \neq \emptyset$.*

We will denote by $\partial \bar{M}_g$ the divisor $(\bar{M}_g - M_g)$ in $\bar{M}_g$. Points in $\partial \bar{M}_g$ correspond, essentially, to curves of genus less than g with 'marked points'. It is well known ([6]) that $\partial \bar{M}_g$ is not pseudo-ample in $\bar{M}_g$. We shall prove that $\bar{W}_{g-1,g} \cup \partial \bar{M}_g$ is pseudo-ample in $\bar{M}_g$. More generally we shall study the problem of degenerating families of curves in each level of the filtration (3.15).

For $2 \leqq n \leqq g$ we set $\partial \bar{W}_{n,g} = \bar{W}_{n,g} \cap \partial \bar{M}_g$.

THEOREM (3.27): *$\bar{W}_{n-1,g} \cup \partial \bar{W}_{n,g}$ is pseudo-ample in $\bar{W}_{n,g}$.*

PROOF: Suppose that the statement of the Theorem is false. Then there exists an irreducible complete curve $S_1 \subset \bar{W}_{n,g}$ such that $S_1 \cap (\bar{W}_{n-1,g} \cup \partial \bar{W}_{n,g}) = \emptyset$. Let $p: S \to S_1$ be the finite morphism and $\pi: Y \to S$ be the *stable* curve of genus g given by Theorem 3.14. It follows from our assumption on S_1 that, for each $s \in S$, Y_s is a *non singular* curve of genus g possessing a Weierstrass point y_s such that $\ell(ny_s) \geqq 2$. Since $m(Y_s) \not\subset \bar{W}_{n-1,g}$ we actually have $\ell(ny_s) = 2$. By proceding as in the proof of Theorem 3.18, we can now assume, after unwinding Y, if necessary, that there exist w sections $y_1 = r_1, \cdots, r_w : S \to Y$ (not necessarily distinct) such that for each $s \in S$, $y_1(s)$ is a Weierstrass point with $\ell(ny_1(s)) = 2$ and such that $r_1(s), \cdots, r_w(s)$ are the ramification points of the $g_n^1(y_1(s))$. We now define a Zariski open $S' \subset S$ by setting $S' = \{s \in S : r_i(s) \neq r_j(s)$ for some fixed i and j, $i \neq j$, $i \neq 1, j \neq 1\}$. We may as well assume that $i = 2$ and $j = 3$. Let f_s be the *unique* element of $H^0(\mathcal{O}_{Y_s}(ny_1(s)))$ such that $f_s(r_2(s)) = 0$, $f_s(r_3(s)) = 1$, $f_s(r_1(s)) = \infty$, $s \in S'$. We then define an analytic map $q: \pi^{-1}(S') \to \mathbb{P}^1 \times S'$ by setting $q_s = f_s$ for each $s \in S'$.

Let $pr: \mathbb{P}^1 \times S' \to \mathbb{P}^1$ be the projection, and let $C_t =$ (closure in Y of $(pr \cdot q)^{-1}(t)$) for each $t \in \mathbb{P}^1$. As in the proof of Theorem 3.18 one can see that the algebraic system $\{C_t\}_{t \in \mathbb{P}^1}$ is *linear*. We now show that $\{C_t\}_{t \in \mathbb{P}^1}$ has *no base point*. Suppose in fact that $y_0 \in Y$ is a base point for $\{C_t\}_{t \in \mathbb{P}^1}$. Let $s_0 = \pi(y_0)$ and denote by g_n^1 the linear series of degree n cut out on Y_{s_0} by $\{C_t\}_{t \in \mathbb{P}^1}$. Then y_0 is a fixed point for the g_n^1. Since $ny_1(s_0) \in g_n^1$ we have $y_0 = y_1(s_0)$. It follows that $m(Y_{s_0}) \subset \bar{W}_{v,g}$ for some $v < n$. This together with Theorem 3.18 contradicts our assumption on S_1. Therefore $\{C_t\}_{t \in \mathbb{P}^1}$ has no base point. We then get an extension of q to a *regular* map $q: Y \to \mathbb{P}^1 \times S$, by setting $q(y) = (t, \pi(y))$, where C_t is the unique curve of $\{C_t\}_{t \in \mathbb{P}^1}$ passing through y. Consider now the rational functions $g_i : S \to \mathbb{P}^1$ defined by $g_i = pr \cdot q \cdot r_i$, $1 \leqq i \leqq w$. Since $\Lambda(q_s) = (g_1(s), \cdots, g_w(s))$ and since the map $m: S \to M_g$ is non constant, it follows that not all the g_i's are constant. Therefore there exist $s_0 \in S$ such that $g_i(s_0) = g_1(s_0)$ for some $i \neq 1$. This in turn implies that $r_i(s_0) = y_1(s_0)$. It is now an easy matter to check that $y_1(s_0)$ is a base point for $\{C_t\}_{t \in \mathbb{P}^1}$. But we just proved that this cannot be the case. This contradiction shows that our assumption on S_1 is absurd and the Theorem is proved.

It is now natural to ask whether the irreducible divisor $\bar{W}_{n-1,g}$ is *pseudo-ample* in $\bar{W}_{n,g}$. We will see that, in general, this is not the case.

The following notation will be used. Let C (resp. C') be an *irreducible stable curve* of genus g such that the only singularity of C (resp. C') is a *double point* $x \in C$ (resp. $x' \in C'$). We shall denote by $\tilde{C}$ (resp. $\tilde{C}'$) the *normalization* of C (resp. C') and by x_1, x_2 (resp. x'_1, x'_2) the preimages in $\tilde{C}$ (resp. $\tilde{C}'$) of the double point $x \in C$ (resp. $x' \in C'$). We shall denote by $(C)^{(2)}$ (resp. $(C')^{(2)}$) the *second symmetric product* of $\tilde{C}$ (resp. $\tilde{C}'$). If $\varphi : C \to C'$ is an isomorphism we shall denote by $\varphi^{(2)} : (C)^{(2)} \to (C')^{(2)}$ the isomorphism induced by φ. We also recall that the definition of $\bar{M}_g$, [3], implies that $m(C) = m(C') \in \partial \bar{M}_g$ *if and only if* there exists an isomorphism $\varphi : \tilde{C} \to \tilde{C}'$ such that $\tilde{\varphi}^{(2)}((x_1, x_2)) = (x'_1, x'_2)$.

THEOREM (3.28): $\bar{W}_{2,3}$ *is not pseudo-ample in* $\bar{W}_{3,3} = \bar{M}_3$.

PROOF: Observe, first of all, that for $g = 3$ we have $\bar{W}_{2,3} =$ (*hyperelliptic locus*). We shall construct a curve $S \subset \bar{W}_{3,3}$ such that $S \cap \bar{W}_{2,3} = \emptyset$. For this purpose let

$$f : Y \to \mathbb{P}^1 \tag{3.29}$$

be a *pencil* of *stable curves* of genus 3 such that for all $t \in \mathbb{P}^1 - \{t_1, \cdots, t_N\}$, $f^{-1}(t) = Y_t$ is a non singular plane quartic and such that Y_{t_i}, $i = 1, \cdots, N$ is an irreducible plane quartic with one *simple node* and no other singularities. Such a '*Lefschetz pencil*' certainly exists. In fact it is not difficult to check that, in the projective space $\mathbb{P}^{14}$, whose points correspond to plane quartics, the subvariety of points corresponding to plane quartics with a non-ordinary double point is of codimension two. The pencil (3.29) induces a morphism $m : \mathbb{P}^1 \to \bar{M}_g$. Let $S = m(\mathbb{P}_1)$. We shall prove that $S \cap \bar{W}_{2,3} = \emptyset$. Notice that for $t \in \mathbb{P}^1 - \{t_1, \cdots, t_N\}$ the plane quartic Y_t is a non singular *canonical* curve of genus 3. Thus Y_t is *non-hyperelliptic*. Therefore we only have to prove that for $i = 1, \cdots, N$

$$m(Y_{t_i}) \notin \partial \bar{W}_{2,3}. \tag{3.30}$$

Let us fix a $t_i \in \{t_1, \cdots, t_N\}$. Set $C = Y_{t_i}$. Let $\tilde{C}$ be a normalization of C and let x_1, x_2 be the preimages in $\tilde{C}$ of the double point $x_0 \in C$. Let $\iota : C \to C$ be the *hyperelliptic involution* on C. Denote by $\tilde{\iota}$ the lifting of ι to $\tilde{C}$. The involution ι, on C, is given by: $\iota(x) =$ (residual intersection between a line, through x and x_0, and C). It easily follows from this that

$$\tilde{\iota}^{(2)}((x_1, x_2)) \neq (x_1, x_2). \tag{3.31}$$

Suppose now that (3.30) is false, and let us then assume that $m(Y_{t_i}) = m(C) \in \partial \bar{W}_{2,3}$. We now construct a stable curve of genus 3

$$g : Y' \to V = \{t \in \mathbb{C} : |t| < 1\} \tag{3.32}$$

such that for all $t \in V - \{0\}$, $m(Y_t') \in \bar{W}_{2,3}$ and $m(Y_0') = m(C)$. Let $h : C \to \mathbb{P}^1$ be the two-sheeted covering induced by the involution ι. Let e_i, $i = 1, \cdots, 6$ be the affine coordinates of the six branch points of h. Let $g_1 : Z \to V$ be the family of hyperelliptic curves defined by letting, for $t \in V$, $Z_t = g_1^{-1}(t)$ be the *plane projective curve* of equation

$$X_2^2 = X_1(X_1 - X_0 t) \prod_{i=1}^{6} (X_1 - e_i X_0).$$

We then obtain a stable curve (3.32) by blowing up Z_t at its 'bad' singularity (point of coordinates (0, 1, 0)). Let $\kappa_1 : Z_0 \to Z_0$ be the hyperelliptic involution. It is then easy to see that if $x \in Z_0 \subset \mathbb{P}^2$ has coordinates (X_0, X_1, X_2) then

$$\kappa_1(x) \equiv (X_0, X_1, -X_2). \tag{3.33}$$

Let now x_0 be the double point of Y_0' and let x_1', x_2' be the preimages of x_0 in a normalization $\tilde{Y}_0'$ of Y'. Let $\tilde{\kappa}$ be the hyperelliptic involution on $\tilde{Y}_0'$. Now the way in which we constructed Y_0' starting from Z_0, the fact that Z_0 has its double point at the point of coordinates (1, 0, 0) and (3.33) easily imply that

$$\tilde{\kappa}^{(2)}((x_1', x_2')) = (x_1', x_2').$$

But this together with (3.31) and the fact that $m(Y_0') = m(C) \in \partial \bar{M}_3$ gives a contradiction. It follows that $m(C) \notin \partial \bar{W}_{2,3}$ and the Theorem is proved.

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