COMPOSITIO MATHEMATICA

R. TIJDEMAN
H. G. MEIJER

On integers generated by a finite number of fixed primes

Compositio Mathematica, tome 29, nº 3 (1974), p. 273-286
<http://www.numdam.org/item?id=CM_1974__29_3_273_0>

COMPOSITIO MATHEMATICA, Vol. 29, Fasc. 3, 1974, pag. 273–286.
Noordhoff International Publishing
Printed in the Netherlands

ON INTEGERS GENERATED BY A FINITE NUMBER OF FIXED PRIMES

R. Tijdeman and H. G. Meijer

1

Let $p_1, \cdots, p_r$ be different primes, $r \geqq 2$. Denote the multiplicative semigroup generated by them by N. We arrange the elements of N in increasing order, $1 = n_1 < n_2 < n_3 < \cdots$. It was noted by Pólya [3] that $\lim_{i\to\infty} n_{i+1}/n_i = 1$. Later better estimates were obtained for the quotient n_{i+1}/n_i. See [1], [5], [6]. In this paper we investigate the set of quotients n_{i+1}/n_i $(i = 1, 2, 3, \cdots)$. Theorem 1 contains a complete characterization of this set in case $r = 2$. The situation for $r > 2$ is much more complicated. As a first step we made the following conjecture.

Let t be fixed, $1 \leqq t \leqq r-1$. Then there exist infinitely many pairs n_i, n_{i+1} such that one of the numbers n_i, n_{i+1} is composed of $p_1, \cdots, p_t$ and the other is composed of $p_{t+1}, \cdots, p_r$.

We prove this conjecture for $t = 1$ in Theorem 2 and for $t = 2$ in Theorem 3. The case $t > 2$ is still open. Since $t = 1$ and $t = 2$ are equivalent to $t = r-1$ and $t = r-2$ respectively, the conjecture is true for $r \leqq 5$.

2

Let $p_1, \cdots, p_r$ be different primes, $r \geqq 2$. By the *sequence composed of* $p_1, \cdots, p_r$ we mean the monotonically increasing sequence $N = \{n_i\}_{i=1}^{\infty}$ of all numbers which are of the form $p_1^{k_1} \cdots p_r^{k_r}$, where $k_1, \cdots, k_r$ are non-negative integers. We observe that

$$(1) \quad d|n_i \quad \text{and} \quad d|n_{i+1} \Rightarrow \frac{n_i}{d} \quad \text{and} \quad \frac{n_{i+1}}{d} \text{ are consecutive elements of } N.$$

Indeed, $n_i/d < n_j < n_{i+1}/d$ would imply $n_i < dn_j < n_{i+1}$, which is impossible. We denote the G.C.D. of two integers a and b by (a, b).

We shall use the following lemmas

LEMMA 1: *Let $p_1, \cdots, p_r$ be fixed primes, $r \geqq 2$. Let $n_1, n_2, \cdots$ be the sequence composed of these primes. Then there exist positive constants C_1, C_2 and N such that*

$$\frac{n_i}{(\log n_i)^{C_1}} < n_{i+1} - n_i < \frac{n_i}{(\log n_i)^{C_2}} \qquad \text{for } n_i \geqq N. \tag{2}$$

PROOF. The first inequality is a corollary of [5, Theorem 1]. The second can be found in [6].

LEMMA 2: *Let $n_1, n_2, \cdots$ be the sequence composed of the primes $p_1, \cdots, p_r$ with $r \geqq 2$. Then*

$$\lim_{i \to \infty} \frac{(n_i, n_{i+1})}{n_i} = 0.$$

PROOF: Let $d_i = (n_i, n_{i+1})$. If $n_j = n_i/d_i$, then, by (1), $n_{j+1} = n_{i+1}/d_i$. Hence, by (2),

$$\frac{1}{(\log n_j)^{C_1}} < \frac{n_{j+1}}{n_j} - 1 = \frac{n_{i+1}}{n_i} - 1 < \frac{1}{(\log n_i)^{C_2}}.$$

It follows that

$$(\log n_i)^{C_2} < \left(\log \frac{n_i}{d_i}\right)^{C_1}.$$

Since the left hand term tends to ∞ if $i \to \infty$, we also have $n_i/d_i \to \infty$ if $i \to \infty$.

3

We need several elementary results from the theory of continued fractions. Let $\xi > 0$ be an irrational number with simple continued fraction $[a_0, a_1, a_2, \cdots]$. The n-th convergent $[a_0, \cdots, a_n]$ to ξ is denoted by A_n/B_n. It is well known that the denominators B_n form a monotonically increasing sequence of integers for $n \geqq 1$, that the sequence A_0/B_0, A_2/B_2, $A_4/B_4, \cdots$ is monotonically increasing to ξ and A_1/B_1, A_3/B_3, $A_5/B_5, \cdots$ is monotonically decreasing to ξ. The convergents A_n/B_n are the best approximations to ξ in the sense of Lemma 3(a). For our convenience we give a slightly different form of this assertion in Lemma 3(b).

LEMMA 3: (a) *The convergents to ξ are just the fractions A/B having the property that every fraction r/s with $0 < |r-s\xi| < |A-B\xi|$ satisfies $s > B$.*

(b) *If A_n/B_n is a convergent to ξ, then every fraction r/s with $0 < |r-s\xi| < |A_n-B_n\xi|$ satisfies $s \geqq B_{n+1}$.*

PROOF: See [2, Satz 2.18, 2.17].

Apart from the convergents to ξ we shall consider a larger set of fractions. We recall

$$(3) \qquad \begin{aligned} A_{n+1} &= a_{n+1}A_n + A_{n-1}, \\ B_{n+1} &= a_{n+1}B_n + B_{n-1}, \end{aligned} \qquad \text{for } n \geqq 0.$$

We call a fraction

$$\frac{A}{B} = \frac{jA_n + A_{n-1}}{jB_n + B_{n-1}} \qquad \text{with } j \in \{1, 2, \cdots, a_{n+1}\}$$

a *one-sided convergent* to ξ (Näherung). We call it a *left convergent* if $A/B < \xi$ and a *right convergent* if $A/B > \xi$. We can arrange the one-sided convergents to ξ with increasing denominators. Part of this sequence reads as follows

$$\frac{A_n}{B_n}, \frac{A_n + A_{n-1}}{B_n + B_{n-1}}, \cdots, \frac{a_{n+1}A_n + A_{n-1}}{a_{n+1}B_n + B_{n-1}} = \frac{A_{n+1}}{B_{n+1}}, \frac{A_{n+1} + A_n}{B_{n+1} + B_n}.$$

It follows immediately from the construction that

$$(jA_n + A_{n-1})/(jB_n + B_{n-1}) \qquad (j = 1, \cdots, a_{n+1})$$

are on the same side of ξ, but A_n/B_n and $(A_{n+1} + A_n)/(B_{n+1} + B_n)$ are on the opposite side of ξ.

In [2, Satz 2.21, 2.22] a complete characterization of the one-sided convergents is given. The second theorem states the following.

LEMMA 4: *If a fraction A/B with positive denominator has the property that every fraction between ξ and A/B has a denominator greater than B, then A/B is a one-sided convergent to ξ.*

We shall use Lemma 4 to derive a slightly different characterization which is more analogous to Lemma 3(a) and more appropriate for our purposes.

LEMMA 5:

(a) *The left convergents to ξ are just the fractions A/B having the property that every fraction r/s with $A-B\xi < r-s\xi < 0$ satisfies $s > B$.*

(b) *The right convergents to ξ are just the fractions A/B having the property that every fraction r/s with $0 < r-s\xi < A-B\xi$ satisfies $s > B$.*

PROOF: Since the proofs of both parts are almost identical we only prove the second assertion.

Let A/B have the property that every fraction r/s with

$$0 < r-s\xi < A-B\xi$$

satisfies $s > B$. Then every fraction r/s with $\xi < r/s < A/B$ satisfies $s > B$. Indeed, if r/s were a fraction with $s \leqq B$ and $\xi < r/s < A/B$ then it would follow that

$$0 < r-s\xi = s\left(\frac{r}{s}-\xi\right) \leqq B\left(\frac{A}{B}-\xi\right) = A-B\xi,$$

which is a contradiction. It follows from Lemma 4 that A/B is a right convergent.

Let A/B be any right convergent. By definition A/B can be written in the form

$$\frac{A}{B} = \frac{jA_n+A_{n-1}}{jB_n+B_{n-1}}, \qquad j \in \{1, 2, \cdots, a_{n+1}\}, \tag{4}$$

where A_{n-1}/B_{n-1} and A_n/B_n are convergents to ξ with

$$\frac{A_n}{B_n} < \xi < \frac{A_{n-1}}{B_{n-1}}. \tag{5}$$

Define A^*/B^* by

$$A^*-B^*\xi = \min_{\substack{r-s\xi>0\\ s\leqq B}} (r-s\xi) \tag{6}$$

Since ξ is irrational, A^* and B^* are uniquely determined. It is obvious that there does not exist a fraction r/s with $s \leqq B^*$ and $0 < r-s\xi < A^*-B^*\xi$. Hence, by the first part of the proof, A^*/B^* is a right convergent. It follows from (6) and (5) that $0 < A^*-B^*\xi \leqq A_{n-1}-B_{n-1}\xi$. On applying Lemma

3(b) we obtain $B^* \geqq B_{n-1}$. Since A^*/B^* is a right convergent to ξ and $B^* \leqq B$, we obtain

$$\frac{A^*}{B^*} = \frac{iA_n + A_{n-1}}{iB_n + B_{n-1}}, \qquad \text{where } i \in \{0, 1, \cdots, j\}. \tag{7}$$

We have, by (7), (5) and (4),

$$\begin{aligned} A^* - B^*\xi &= i(A_n - B_n\xi) + (A_{n-1} - B_{n-1}\xi) \\ &\geqq j(A_n - B_n\xi) + (A_{n-1} - B_{n-1}\xi) = A - B\xi, \end{aligned}$$

while equality holds if and only if $i = j$. By (6), $A^* - B^*\xi \leqq A - B\xi$. Hence, $i = j$ and $A^*/B^* = A/B$. In view of (6) this completes the proof of Lemma 5(b).

4

Let α and β be real numbers with $\alpha > \beta > 1$. By the *sequence composed of* α and β we mean the monotonically increasing sequence $N = \{n_i\}_{i=1}^{\infty}$ of all numbers of the form $\alpha^k\beta^l$, where k and l are non-negative integers. The following theorem gives a complete characterization of the set of quotients $\{n_{i+1}/n_i\}_{i=1}^{\infty}$.

THEOREM 1: *Let α and β be real numbers with $\alpha > \beta > 1$, and such that $\xi = \log \beta / \log \alpha$ is irrational. Let $n_1, n_2, \cdots$ be the sequence composed of α and β. If $S = \{n_{i+1}/n_i | i = 1, 2, \cdots\}$, then S is the set of all products $\alpha^{-k}\beta^l$ and $\alpha^k\beta^{-l}$ which are greater than* 1 *and such that k/l is a one-sided convergent to ξ.*

REMARK: In view of Theorem 1 one can define a natural generalization of the continued fractions as follows. Let $\alpha_1, \cdots, \alpha_m$ be real numbers all greater than 1. Let $n_1, n_2, \cdots$ be the sequence composed of $\alpha_1, \cdots, \alpha_m$. Put $S = \{n_{i+1}/n_i | i = 1, 2, \cdots\}$. We would be very interested in a characterization of S like Theorem 1 does in case $m = 2$.

PROOF: Let k/l be a one-sided convergent to ξ. We shall prove that α^k and β^l are consecutive elements of N. This implies that k/l belongs to S.

Assume k/l is a left convergent to ξ. Then $\alpha^k < \beta^l$. Suppose there exists an element $\alpha^r\beta^s$ such that $\alpha^k < \alpha^r\beta^s < \beta^l$. Hence, $l > s \geqq 0$. We have

$$k < r + s\xi < l\xi,$$

or, equivalently,

$$k-l\xi < r-(l-s)\xi < 0.$$

This is a contradiction with Lemma 5(a).

If k/l is a right convergent to ξ, then $\beta^l < \alpha^k$ and a similar argument gives that β^l and α^k are consecutive elements of N.

In order to prove that every element of S is of the required form, put $n_i = \alpha^{r_i}\beta^{s_i}$, $n_{i+1} = \alpha^{r_{i+1}}\beta^{s_{i+1}}$. Since $\alpha > \beta$, we have

$$\alpha^{r_i+1}\beta^{s_i} > \alpha^{r_i}\beta^{s_i+1} \geqq n_{i+1},$$

and, hence, either $r_{i+1} \leqq r_i$ or $s_{i+1} < s_i$. Since both cases are treated in similar ways, we only deal with the first. Assume $r_{i+1} \leqq r_i$. Then $s_{i+1} > s_i$. Put $k = r_i - r_{i+1}$, $l = s_{i+1} - s_i$. We have $\alpha^{-k}\beta^l = n_{i+1}/n_i > 1$. We shall prove that k/l is a left convergent to ξ. We have $k/l < \log\beta/\log\alpha = \xi$. Suppose there exists a fraction r/s with $s \leqq l$ and

$$k-l\xi < r-s\xi < 0.$$

Then

$$\alpha^{r-k+r_i}\beta^{l-s+s_i} = n_i e^{(r-k)\log\alpha+(l-s)\log\beta} > n_i. \tag{8}$$

Since $r-k+r_i = r+r_{i+1} > 0$ and $l-s+s_i \geqq s_i > 0$, we obtain

$$\alpha^{r-k+r_i}\beta^{l-s+s_i} \in N. \tag{9}$$

On the other hand,

$$\alpha^{r-k+r_i}\beta^{l-s+s_i} = n_{i+1} e^{r\log\alpha - s\log\beta} < n_{i+1}. \tag{10}$$

The contradiction (8), (9), (10), proves by Lemma 5(a) that k/l is a left convergent to ξ. (In case $s_{i+1} < s_i$ the fraction k/l turns out to be a right convergent to ξ).

5

It would be very valuable to have a characterization like Theorem 1 for sequences composed of r multiplicatively independent positive numbers, $r > 2$. This would solve the conjecture in the introduction immediately. We now prove case $t = 1$ of this conjecture.

THEOREM 2: *Let $n_1, n_2, \cdots$ be the sequence composed of the primes $p_1, \cdots, p_r$ ($r \geqq 2$). Let p be one of these primes. Then there exists an infinite number of pairs n_i, n_{i+1} such that n_i is a pure power of p and n_{i+1} is not divisible by p.*

PROOF: Without loss of generality we may assume $p = p_1$. Let k be a positive integer and $n_{j_k} = p^k$. Let $n_{j_k+1} = p_1^{l_1} \cdots p_r^{l_r}$ and $n_{i_k} = p_1^{k-l_1}$. It follows from (1) that $n_{i_k+1} = p_2^{l_2} \cdots p_r^{l_r}$. Since, by Lemma 2,

$$n_{i_k} = \frac{n_{j_k}}{(n_{j_k}, n_{j_k+1})} \to \infty \qquad \text{for } k \to \infty,$$

we obtain infinitely many different pairs n_{i_k}, n_{i_k+1} with the required property.

REMARK: In the same way one can prove the existence of infinitely many pairs n_i, n_{i+1} such that n_{i+1} is a pure power of p and n_i is not divisible by p.

6

Finally we prove case $t = 2$ of our conjecture.

THEOREM 3: *Let $p_1, \cdots, p_r$ be $r > 2$ different primes. Let $M = \{m_1, m_2, \cdots\}$ be the sequence composed of these primes. Let p and q be two primes from $p_1, \cdots, p_r$. Then there exist infinitely many pairs m_i, m_{i+1} such that one of the numbers m_i, m_{i+1} is composed of p and q and the other is neither divisible by p nor by q.*

The proof is based on two lemmas.

LEMMA 6: *Let $r > 2$. Let $M = \{m_1, m_2, \cdots\}$ be the sequence composed of the different primes $p_1, \cdots, p_r$ and $N = \{n_1, n_2, \cdots\}$ the sequence composed of p_1 and p_2. Suppose there exists an i_0 such that for every $i \geqq i_0$*

$$m_i \in N \Rightarrow (m_{i-1}, p_1 p_2) > 1 \quad \textit{and} \quad (m_{i+1}, p_1 p_2) > 1.$$

Then there exists an i_1 such that for every $i \geqq i_1$

(a) *if* $m_i \in N$ *and* $m_i^2 \leqq m_{i-1}m_{i+1}$, *then* $m_{i-1} \in N$,
(b) *if* $m_i \in N$ *and* $m_i^2 \geqq m_{i-1}m_{i+1}$, *then* $m_{i+1} \in N$.

PROOF: We know from Lemma 2 that

$$\frac{m_{i-1}}{(m_{i-1}, m_i)} \to \infty \qquad \text{as } i \to \infty.$$

We choose i_1 such that

$$\frac{m_{i-1}}{(m_{i-1}, m_i)} > m_{i_0} \quad \text{for } i \geqq i_1.$$

In the sequel we only consider i with $i \geqq \mathrm{i}_1$.

Assume $m_i \in N$. Let $m_i = p_1^a p_2^b$. Put $m_{i-1} = p_1^{k_1} \cdots p_r^{k_r}$ and $m_{i+1} = p_1^{l_1} \cdots p_r^{l_r}$. Then

$$m_{i-1} < \frac{m_{i-1}m_{i+1}}{m_i} < m_{i+1}.$$

Hence, we have either

$$m_{i-1}m_{i+1}/m_i = m_i \tag{11}$$

or

$$m_{i-1}m_{i+1}/m_i \notin M. \tag{12}$$

We note $m_{i-1}m_{i+1}/m_i = p_1^{k_1+l_1-a}p_2^{k_2+l_2-b}p_3^{k_3+l_3} \cdots p_r^{k_r+l_r}$. If (11) holds, then $k_3+l_3 = \cdots = k_r+l_r = 0$, and, hence, $k_3 = \cdots = k_r = 0$ and $l_3 = \cdots = l_r = 0$. In this case both $m_{i-1} \in N$ and $m_{i+1} \in N$. If (12) holds, then

$$k_1+l_1-a < 0 \quad \text{or} \quad k_2+l_2-b < 0. \tag{13}$$

Suppose $k_1 \leqq a$ and $k_2 \leqq b$. By (1), $p_1^{a-k_1}p_2^{b-k_2}$ is preceeded in M by $p_3^{k_3} \cdots p_r^{k_r}$. Since

$$p_1^{a-k_1}p_2^{b-k_2} = \frac{m_i}{(m_{i-1}, m_i)} > m_{i_0},$$

this is a contradiction with the condition of the lemma. Hence, $k_1 > a$ or

$k_2 > b$. Similarly, $l_1 > a$ or $l_2 > b$. Without loss of generality we may assume $k_2 > b$. Then, by (13), $k_1 < a$ and $l_1 < a$. Thus $l_2 > b$. So we obtain

$$(14)\qquad k_1 < a, \quad l_1 < a, \quad k_2 > b, \quad l_2 > b.$$

We define a sequence of positive integers $\{a_j\}_{j=0}^{\infty}$ by

$$m_{a_j} = p_1^a p_2^j \qquad \text{for } j = 0, 1, 2, \cdots.$$

We have, by (1) and (14),

$$m_{a_j - 1} = p_1^{k_1} p_2^{k_2 - b + j} p_3^{k_3} \cdots p_r^{k_r} \quad \text{and} \quad m_{a_j + 1} = p_1^{l_1} p_2^{l_2 - b + j} p_3^{l_3} \cdots p_r^{l_r},$$

for $j = 0, 1, \cdots, b$. Consider the pairs of quotients

$$(15)\qquad \left(\frac{m_{a_j - 1}}{m_{a_j}}, \frac{m_{a_j + 1}}{m_{a_j}}\right) \qquad \text{for } j = 0, 1, 2, \cdots.$$

We know

$$\frac{m_{a_j - 1}}{m_{a_j}} = p_1^{k_1 - a} p_2^{k_2 - b} p_3^{k_3} \cdots p_r^{k_r} \quad \text{and} \quad \frac{m_{a_j + 1}}{m_{a_j}} = p_1^{l_1 - a} p_2^{l_2 - b} p_3^{l_3} \cdots p_r^{l_r}$$

for $j = 0, \cdots, b$. Let J_0 be the smallest value of j for which one of the quotients in (15) assumes another value. This J_0 exists, since, by Lemma 1, m_{i+1}/m_i tends to 1 as $i \to \infty$. We assume that the first quotient changes firstly. Thus

$$(16)\qquad 1 > \frac{m_{a_J - 1}}{m_{a_J}} > \frac{m_{a_{J-1} - 1}}{m_{a_{J-1}}} = \cdots = \frac{m_{a_0 - 1}}{m_{a_0}} = \frac{m_{i-1}}{m_i}$$

and

$$(17)\qquad \frac{m_{a_{J-1} + 1}}{m_{a_{J-1}}} = \cdots = \frac{m_{a_0 + 1}}{m_{a_0}} = \frac{m_{i+1}}{m_i}.$$

Put $m_{a_J - 1} = p_1^{\kappa_1} \cdots p_r^{\kappa_r}$ and $m_{a_J + 1} = p_1^{\lambda_1} \cdots p_r^{\lambda_r}$. The following argument shows $\kappa_2 = 0$. If $\kappa_2 > 0$, then, by (1), $m_{a_J - 1}/p_2$ is the precessor of $m_{a_J}/p_2 = m_{a_{J-1}}$, and, hence, $m_{a_J - 1}/m_{a_J} = m_{a_{J-1} - 1}/m_{a_{J-1}}$ in contradiction with (16). Since we know from the argument preceding formula (14) that both $\kappa_1 \leqq a$ and $\kappa_2 \leqq J$ is impossible, we have

(18) $$\kappa_1 > a.$$

Consider

$$m = \frac{m_{a_{J-1}+1}\, m_{a_J - 1}}{m_{a_J}}.$$

We have, by (17)

$$m = \frac{m_{a_{J-1}+1}}{m_{a_{J-1}}} \cdot m_{a_{J-1}} \cdot \frac{m_{a_J - 1}}{m_{a_J}} = \frac{m_{i+1}}{m_i} \cdot \frac{m_{a_{J-1}}}{m_{a_J}} \cdot m_{a_J - 1}$$
$$= p_1^{l_1+\kappa_1-a} p_2^{l_2+\kappa_2-b-1} p_3^{l_3+\kappa_3} \cdots p_r^{l_r+\kappa_r}.$$

From (18) and (14) we see that $m \in M$. Moreover,

$$m_{a_{J-1}+1} > m > \frac{m_{a_{J-1}+1}\, m_{a_{J-1}-1}}{m_{a_{J-1}}} > m_{a_{J-1}-1}.$$

Hence,

(19) $$m = m_{a_{J-1}}.$$

This implies $l_3+\kappa_3 = \cdots = l_r+\kappa_r = 0$. Thus $l_3 = \cdots = l_r = 0$ and $m_{i+1} \in N$. Furthermore, in view of (17), (19), the definition of m and (16),

$$\frac{m_{i+1}}{m_i} = \frac{m_{a_{J-1}+1}}{m_{a_{J-1}}} = \frac{m_{a_{J-1}+1}}{m}$$
$$= \frac{m_{a_J}}{m_{a_J - 1}} < \frac{m_{a_{J-1}}}{m_{a_{J-1}-1}} = \frac{m_i}{m_{i-1}}.$$

Similarly, the assumption that the second quotient in (15) changes firstly leads to $m_{i-1} \in N$ and $m_{i-1}/m_i > m_i/m_{i+1}$. This completes the proof of the lemma.

LEMMA 7: *Let $r > 2$. Let $M = \{m_1, m_2, \cdots\}$ be the sequence composed of the primes $p_1, \cdots, p_r$. Let p and q be two arbitrary primes from $p_1, \cdots, p_r$ with $p > q$ and let $N = \{n_1, n_2, \cdots\}$ be the sequence composed of p and q. Suppose there exists an i_0 such that for every $i \geqq i_0$*

$$m_i \in N \Rightarrow (m_{i-1}, pq) > 1 \quad \textit{and} \quad (m_{i+1}, pq) > 1.$$

Then there exists a monotonically increasing, unbounded sequence $T_1, T_2, T_3, \cdots$ *such that no interval* $[T_H, qT_H]$ *contains an element of* $M \backslash N$.

PROOF: Let $[a_0, a_1, a_2, \cdots]$ be the continued fraction of $\xi := \log q/\log p$. Put $A_h/B_h = [a_0, \cdots, a_h]$ for $h = 0, 1, 2, \cdots$. It follows from the Gel'fond-Schneider theorem [4, Satz 14], that ξ is transcendental. Hence, the sequence $a_0, a_1, a_2, \cdots$ is not periodical [2, Satz 3.1]. There therefore exist infinitely many values h with $a_h > 1$.

Let H be such that $a_H > 1$. It is no loss of generality to assume $A_H/B_H < \xi$. We consider the subsequence N_1 of N beginning with

$$T_H = p^{A_H} q^{B_{H-1}-1} \quad \text{and ending with} \quad qT_H = p^{A_H} q^{B_{H-1}}.$$

(If $A_H/B_H > \xi$, we may choose $T_H = \mathrm{p}^{A_{H-1}-1} q^{B_H}$ and consider the interval $[T_H, pT_H]$.)

Let $n_i = p^c q^d$ be in N_1, $n_i \neq qT_H$. Since $qT_H < q^{B_H + B_{H-1}} < p^{A_H + A_{H-1}}$, we have

$$c < A_H + A_{H-1} \quad \text{and} \quad d < B_H + B_{H-1}. \tag{20}$$

We distinguish two cases.

(i) $c \geqq A_H$. We assert that $n_{i+1} = p^{c-A_H} q^{d+B_H}$. Since $A_H/B_H < \xi$, we have

$$n_i < p^{c-A_H} q^{d+B_H} \in N_1.$$

Suppose

$$n_{i+1} = p^s q^t < p^{c-A_H} q^{d+B_H}.$$

This implies

$$A_H - B_H \xi < (c-s) - (t-d)\xi < 0.$$

By Lemma 3(b), $|t-d| \geqq B_{H+1}$. Hence, $d \geqq B_{H+1}$ or $t \geqq B_{H+1}$ in contradiction with (20).

(ii) $c < A_H$. Since $d \leqq B_{H-1} - 1$ implies $p^c q^d < p^{A_H} q^{B_{H-1}-1} = T_H$, we have $d \geqq B_{H-1}$. We assert that $n_{i+1} = p^{c+A_{H-1}} q^{d-B_{H-1}}$. Since $A_{H-1}/B_{H-1} > \xi$, we have

$$n_i < p^{c+A_{H-1}} q^{d-B_{H-1}} \in N_1.$$

Suppose

$$n_{i+1} = p^s q^t < p^{c+A_{H-1}} q^{d-B_{H-1}}.$$

Then

$$0 < (s-c)-(d-t)\xi < A_{H-1} - B_{H-1}\xi.$$

By Lemma 3(b), $|d-t| \geqq B_H$. If $d-t<0$, then $t > d+B_H \geqq B_H + B_{H-1}$ and $p^s q^t > q^{B_H+B_{H-1}} > qT_H$, which is false. Hence, $d-t \geqq 0$. This implies $s-c>0$ and $d \geqq B_H + t$. By Lemma 5(b) $(s-c)/(d-t)$ is a right convergent to ξ. Since A_H/B_H is a left convergent to ξ, we obtain $d \geqq d-t \geqq B_H + B_{H-1}$, which is impossible in view of (20). Summarizing we see that among the quotients n_{i+1}/n_i for $n_i \in N_1$ only $p^{-A_H} q^{B_H}$ and $p^{A_{H-1}} q^{-B_{H-1}}$ occur. Note

$$p^{A_{H-1}} q^{-B_{H-1}} > p^{-A_H} q^{B_H} > 1. \tag{21}$$

We now assert that

$$n_{i+1}/n_i = p^{A_{H-1}} q^{-B_{H-1}} \Rightarrow \frac{n_i}{n_{i-1}} = \frac{n_{i+1}}{n_i} \quad \text{or} \quad \frac{n_{i+1}}{n_i} = \frac{n_{i+2}}{n_{i+1}}. \tag{22}$$

Since $n_i = T_H$ implies $n_{i+1} = p^{-A_H} q^{B_H} n_i$, we have $n_i > T_H$. Hence, $n_{i-1} \in N_1$. Suppose

$$\frac{n_i}{n_{i-1}} = \frac{n_{i+2}}{n_{i+1}} = p^{-A_H} q^{B_H}.$$

Then

$$n_{i+2} = p^{-2A_H + A_{H-1}} q^{2B_H - B_{H-1}} n_{i-1}.$$

By $a_H \geqq 2$, it follows that $n_{i+2} \geqq q^{B_H + B_{H-1} + B_{H-2}}$. This is a contradiction.

We now turn our attention to the subsequence M_1 of M starting with T_H and ending with qT_H. Let $m_i \in N_1$, $m_i \neq qT_H$. Put $m_i = n_j$. Hence, $n_{j+1} \in N_1$. Note that $n_{j-1} \leqq m_{i-1} < m_i < m_{i+1} \leqq n_{j+1}$. The condition of Lemma 7 enables us to apply Lemma 6. Hence, $m_{i-1} = n_{j-1}$ if $m_i^2 \leqq m_{i-1} m_{i+1}$ and $m_{i+1} = n_{j+1}$ if $m_i^2 \geqq m_{i-1} m_{i+1}$. It follows that

$$m_{i-1} = n_{j-1} \quad \text{if} \quad n_j^2 \leqq n_{j-1} m_{i+1} \leqq n_{j-1} n_{j+1} \tag{23}$$

and

$$(24) \qquad m_{i+1} = n_{j+1} \quad \text{if} \quad n_j^2 \geqq m_{i-1}n_{j+1} \geqq n_{j-1}n_{j+1}.$$

We can now prove that all elements m_i with $T_H \leqq m_i \leqq qT_H$ belong to N_1. Suppose $T_H = m_I$ and all integers $m_I, m_{I+1}, \cdots, m_i$ belong to N_1, while $m_i < qT_H$. We shall prove that $m_{i+1} \in N_1$. Put $m_i = n_j$. We distinguish two cases.

(i) $n_{j-1}n_{j+1} \leqq n_j^2$. It follows from (24) that $m_{i+1} = n_{j+1} \in N_1$.

(ii) $n_{j-1}n_{j+1} \geqq n_j^2$. It follows from formula (21) and the lines before that

$$\frac{n_j}{n_{j-1}} = p^{-A_H}q^{B_H}, \qquad \frac{n_{j+1}}{n_j} = p^{A_{H-1}}q^{-B_{H-1}}.$$

Since $n_j = T_H$ implies $n_{j+1} = p^{-A_H}q^{B_H}n_j$, we have $n_j > T_H$ and, hence, $n_{j-1} \in N_1$. By (22) we have $n_{j+2}/n_{j+1} = p^{A_{H-1}}q^{-B_{H-1}}$. Let $n_{j+1} = m_{i^*}$. Since $n_{j+2}/n_{j+1} = n_{j+1}/n_j$, we obtain from (23) that $n_j = m_{i^*-1}$. Hence, $m_{i^*-1} = m_i$ and $i^* - 1 = i$. It follows that $m_{i+1} = m_{i^*} = n_{j+1} \in N$.

Since we have constructed an infinite number of T_H's such that all integers $m_i \in [T_H, qT_H]$ belong to N, the lemma has been proved.

We are now going to prove the main result.

Proof of Theorem 3: It is no restriction to assume $p = p_1$, $q = p_2$, $p > q$. Suppose that there are only a finite number of values i for which the statement of the theorem holds. Then the condition of Lemma 7 is fulfilled for some i_0. It follows that there exists an unbounded sequence $T_1, T_2, T_3, \cdots$ such that each element $m_i \in [T_H, qT_H]$ belongs to the sequence N composed of p and q. Let $N = \{n_1, n_2, n_3, \cdots\}$. We know from Lemma 1 that $n_{i+1}/n_i \to 1$ as $i \to \infty$. Consider the sequence $p_3n_1, p_3n_2, p_3n_3, \cdots$. These elements belong to $M \setminus N$. However, $p_3n_{i+1}/p_3n_i \to 1$ as $i \to \infty$. This is a contradiction.

REFERENCES

[1] P. ERDŐS: Some recent advances and current problems in number theory. *Lectures on Modern Mathematics, Vol. III*, 196–244, Wiley, New York, 1965.
[2] O. PERRON: *Die Lehre von den Kettenbrüchen.* Band I, 3rd ed. Teubner, Stuttgart, 1954.
[3] G. PÓLYA: Zur arithmetischen Untersuchung der Polynome. *Math. Z.* 1 (1918) 143–148.
[4] TH. SCHNEIDER: *Einführung in die transzendenten Zahlen.* Springer, Berlin, 1957.
[5] R. TIJDEMAN: On integers with many small prime factors. *Compositio Math.*, *26* (1973) 319–330.
[6] R. TIJDEMAN: On the maximal distance between integers composed of small primes. *Compositio Math.*, *28* (1974) 159–162.

(Oblatum 19-III-1974)

University of Leiden
Mathematical Institute
Leiden, Netherlands

University of Technology
Department of Mathematics
Delft, Netherlands