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On integers generated by a finite number of fixed primes


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Let \( p_1, \cdots, p_r \) be different primes, \( r \geq 2 \). Denote the multiplicative semigroup generated by them by \( N \). We arrange the elements of \( N \) in increasing order, \( 1 = n_1 < n_2 < n_3 < \cdots \). It was noted by Pólya [3] that 
\[
\lim_{i \to \infty} \frac{n_{i+1}}{n_i} = 1.
\]
Later better estimates were obtained for the quotient \( n_{i+1}/n_i \). See [1], [5], [6]. In this paper we investigate the set of quotients \( n_{i+1}/n_i \) (\( i = 1, 2, 3, \cdots \)). Theorem 1 contains a complete characterization of this set in case \( r = 2 \). The situation for \( r > 2 \) is much more complicated. As a first step we made the following conjecture.

Let \( t \) be fixed, \( 1 \leq t \leq r-1 \). Then there exist infinitely many pairs \( n_i, n_{i+1} \) such that one of the numbers \( n_i, n_{i+1} \) is composed of \( p_1, \cdots, p_t \) and the other is composed of \( p_{t+1}, \cdots, p_r \).

We prove this conjecture for \( t = 1 \) in Theorem 2 and for \( t = 2 \) in Theorem 3. The case \( t > 2 \) is still open. Since \( t = 1 \) and \( t = 2 \) are equivalent to \( t = r-1 \) and \( t = r-2 \) respectively, the conjecture is true for \( r \leq 5 \).
LEMMA 1: Let $p_1, \cdots, p_r$ be fixed primes, $r \geq 2$. Let $n_1, n_2, \cdots$ be the sequence composed of these primes. Then there exist positive constants $C_1, C_2$ and $N$ such that

\[
\frac{n_i}{(\log n_i)^{C_1}} < \frac{n_{i+1} - n_i}{\log n_i} < \frac{n_i}{(\log n_i)^{C_2}} \quad \text{for } n_i \geq N.
\]

PROOF. The first inequality is a corollary of [5, Theorem 1]. The second can be found in [6].

LEMMA 2: Let $n_1, n_2, \cdots$ be the sequence composed of the primes $p_1, \cdots, p_r$ with $r \geq 2$. Then

\[
\lim_{i \to \infty} \frac{(n_i, n_{i+1})}{n_i} = 0.
\]

PROOF: Let $d_i = (n_i, n_{i+1})$. If $n_j = n_i/d_i$, then, by (1), $n_{j+1} = n_{i+1}/d_i$. Hence, by (2),

\[
\frac{1}{(\log n_j)^{C_1}} < \frac{n_{j+1} - n_j}{n_j} = \frac{n_{i+1}}{n_i} - 1 < \frac{1}{(\log n_i)^{C_2}}.
\]

It follows that

\[
(\log n_i)^{C_2} < \left(\log \frac{n_i}{d_i}\right)^{C_1}.
\]

Since the left hand term tends to $\infty$ if $i \to \infty$, we also have $n_i/d_i \to \infty$ if $i \to \infty$.

We need several elementary results from the theory of continued fractions. Let $\xi > 0$ be an irrational number with simple continued fraction $[a_0, a_1, a_2, \cdots]$. The $n$-th convergent $[a_0, \cdots, a_n]$ to $\xi$ is denoted by $A_n/B_n$. It is well known that the denominators $B_n$ form a monotonically increasing sequence of integers for $n \geq 1$, that the sequence $A_0/B_0$, $A_2/B_2$, $A_4/B_4$, $\cdots$ is monotonically increasing to $\xi$ and $A_1/B_1$, $A_3/B_3$, $A_5/B_5$, $\cdots$ is monotonically decreasing to $\xi$. The convergents $A_n/B_n$ are the best approximations to $\xi$ in the sense of Lemma 3(a). For our convenience we give a slightly different form of this assertion in Lemma 3(b).
LEMMA 3: (a) The convergents to $\xi$ are just the fractions $A/B$ having the property that every fraction $r/s$ with $0 < |r - s\xi| < |A - B\xi|$ satisfies $s > B$.

(b) If $A_n/B_n$ is a convergent to $\xi$, then every fraction $r/s$ with $0 < |r - s\xi| < |A_n - B_n\xi|$ satisfies $s \geq B_{n+1}$.

PROOF: See [2, Satz 2.18, 2.17].

Apart from the convergents to $\xi$ we shall consider a larger set of fractions. We recall

$$A_{n+1} = a_{n+1} A_n + A_{n-1}, \quad B_{n+1} = a_{n+1} B_n + B_{n-1},$$

for $n \geq 0$.

We call a fraction

$$\frac{A}{B} = \frac{jA_n + A_{n-1}}{jB_n + B_{n-1}}$$

with $j \in \{1, 2, \ldots, a_{n+1}\}$

a one-sided convergent to $\xi$ (Näherung). We call it a left convergent if $A/B < \xi$ and a right convergent if $A/B > \xi$. We can arrange the one-sided convergents to $\xi$ with increasing denominators. Part of this sequence reads as follows

$$\frac{A_n}{B_n}, \frac{A_n + A_{n-1}}{B_n + B_{n-1}}, \ldots, \frac{a_{n+1} A_n + A_{n-1}}{a_{n+1} B_n + B_{n-1}} = \frac{A_{n+1}}{B_{n+1}}, \frac{A_{n+1} + A_n}{B_{n+1} + B_n}.$$ 

It follows immediately from the construction that

$$(jA_n + A_{n-1})/(jB_n + B_{n-1})$$

for $j = 1, \ldots, a_{n+1}$

are on the same side of $\xi$, but $A_n/B_n$ and $(A_{n+1} + A_n)/(B_{n+1} + B_n)$ are on the opposite side of $\xi$.

In [2, Satz 2.21, 2.22] a complete characterization of the one-sided convergents is given. The second theorem states the following.

LEMMA 4: If a fraction $A/B$ with positive denominator has the property that every fraction between $\xi$ and $A/B$ has a denominator greater than $B$, then $A/B$ is a one-sided convergent to $\xi$.

We shall use Lemma 4 to derive a slightly different characterization which is more analogous to Lemma 3(a) and more appropriate for our purposes.
LEMMA 5:
(a) The left convergents to $\xi$ are just the fractions $A/B$ having the property that every fraction $r/s$ with $A - B\xi < r - s\xi < 0$ satisfies $s > B$.
(b) The right convergents to $\xi$ are just the fractions $A/B$ having the property that every fraction $r/s$ with $0 < r - s\xi < A - B\xi$ satisfies $s > B$.

PROOF: Since the proofs of both parts are almost identical we only prove the second assertion.

Let $A/B$ have the property that every fraction $r/s$ with

$$0 < r - s\xi < A - B\xi$$

satisfies $s > B$. Then every fraction $r/s$ with $\xi < r/s < A/B$ satisfies $s > B$. Indeed, if $r/s$ were a fraction with $s \leq B$ and $\xi < r/s < A/B$ then it would follow that

$$0 < r - s\xi = s \left( \frac{r}{s} - \xi \right) \leq B \left( \frac{A}{B} - \xi \right) = A - B\xi,$$

which is a contradiction. It follows from Lemma 4 that $A/B$ is a right convergent.

Let $A/B$ be any right convergent. By definition $A/B$ can be written in the form

$$A = \frac{ja_n + A_{n-1}}{jB_n + B_{n-1}}, \quad j \in \{1, 2, \cdots, a_{n+1}\},$$

where $A_{n-1}/B_{n-1}$ and $A_n/B_n$ are convergents to $\xi$ with

$$\frac{A_n}{B_n} < \xi < \frac{A_{n-1}}{B_{n-1}}.$$  

Define $A^*/B^*$ by

$$A^* - B^*\xi = \min_{r - s\xi > 0, s \leq B} \{r - s\xi\}$$

Since $\xi$ is irrational, $A^*$ and $B^*$ are uniquely determined. It is obvious that there does not exist a fraction $r/s$ with $s \leq B^*$ and $0 < r - s\xi < A^* - B^*\xi$. Hence, by the first part of the proof, $A^*/B^*$ is a right convergent. It follows from (6) and (5) that $0 < A^* - B^*\xi \leq A_{n-1} - B_{n-1}\xi$. On applying Lemma
3(b) we obtain $B^* \geq B_{n-1}$. Since $A^*/B^*$ is a right convergent to $\zeta$ and $B^* \leq B$, we obtain

$$\frac{A^*}{B^*} = \frac{iA_n + A_{n-1}}{iB_n + B_{n-1}}, \quad \text{where } i \in \{0, 1, \ldots, j\}.$$  

We have, by (7), (5) and (4),

$$A^* - B^*\zeta = i(A_n - B_n \zeta) + (A_{n-1} - B_{n-1} \zeta) \geq j(A_n - B_n \zeta) + (A_{n-1} - B_{n-1} \zeta) = A - B \zeta,$$

while equality holds if and only if $i = j$. By (6), $A^* - B^*\zeta \leq A - B \zeta$.

Hence, $i = j$ and $A^*/B^* = A/B$. In view of (6) this completes the proof of Lemma 5(b).

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Let $\alpha$ and $\beta$ be real numbers with $\alpha > \beta > 1$. By the sequence composed of $\alpha$ and $\beta$ we mean the monotonically increasing sequence $N = \{n_i\}_{i=1}^\infty$ of all numbers of the form $\alpha^k \beta^l$, where $k$ and $l$ are non-negative integers. The following theorem gives a complete characterization of the set of quotients $\{n_{i+1}/n_i\}_{i=1}^\infty$.

**THEOREM 1:** Let $\alpha$ and $\beta$ be real numbers with $\alpha > \beta > 1$, and such that $\zeta = \log \beta / \log \alpha$ is irrational. Let $n_1, n_2, \cdots$ be the sequence composed of $\alpha$ and $\beta$. If $S = \{n_{i+1}/n_i|i = 1, 2, \cdots\}$, then $S$ is the set of all products $\alpha^k \beta^l$ and $\alpha^{-k} \beta^{-l}$ which are greater than 1 and such that $k/l$ is a one-sided convergent to $\zeta$.

**REMARK:** In view of Theorem 1 one can define a natural generalization of the continued fractions as follows. Let $\alpha_1, \cdots, \alpha_m$ be real numbers all greater than 1. Let $n_1, n_2, \cdots$ be the sequence composed of $\alpha_1, \cdots, \alpha_m$. Put $S = \{n_{i+1}/n_i|i = 1, 2, \cdots\}$. We would be very interested in a characterization of $S$ like Theorem 1 does in case $m = 2$.

**PROOF:** Let $k/l$ be a one-sided convergent to $\zeta$. We shall prove that $\alpha^k$ and $\beta^l$ are consecutive elements of $N$. This implies that $k/l$ belongs to $S$.

Assume $k/l$ is a left convergent to $\zeta$. Then $\alpha^k < \beta^l$. Suppose there exists an element $\alpha' \beta^s$ such that $\alpha^k < \alpha' \beta^s < \beta^l$. Hence, $l > s \geq 0$. We have

$$k < r + s \zeta < l \zeta,$$
or, equivalently,
\[ k - l \xi < r - (l - s) \xi < 0. \]

This is a contradiction with Lemma 5(a).

If \( k/l \) is a right convergent to \( \xi \), then \( \beta^l < \alpha^k \) and a similar argument gives that \( \beta^l \) and \( \alpha^k \) are consecutive elements of \( N \).

In order to prove that every element of \( S \) is of the required form, put \( n_i = \alpha^r \beta^{s_i}, \ n_{i+1} = \alpha^{r_{i+1}} \beta^{s_{i+1}} \). Since \( \alpha > \beta \), we have
\[ \alpha^{r_{i+1}} \beta^{s_i} > \alpha^{r_i} \beta^{s_{i+1}} \geq n_{i+1}, \]
and, hence, either \( r_{i+1} \leq r_i \) or \( s_{i+1} < s_i \). Since both cases are treated in similar ways, we only deal with the first. Assume \( r_{i+1} \leq r_i \). Then \( s_{i+1} > s_i \).

Put \( k = r_i - r_{i+1}, \ l = s_{i+1} - s_i \). We have \( \alpha^{-k} \beta^l = n_{i+1}/n_i > 1 \). We shall prove that \( k/l \) is a left convergent to \( \xi \). We have \( k/l < \log \beta/\log \alpha = \xi \).

Suppose there exists a fraction \( r/s \) with \( s \leq l \) and
\[ k - l \xi < r - s \xi < 0. \]

Then
\[ \alpha^{r-k+r_i} \beta^{l-s+s_i} = n_i e^{(r-k) \log \alpha + (l-s) \log \beta} > n_i. \]

Since \( r - k + r_i = r + r_{i+1} > 0 \) and \( l - s + s_i \geq s_i > 0 \), we obtain
\[ \alpha^{r-k+r_i} \beta^{l-s+s_i} \in N. \]

On the other hand,
\[ \alpha^{r-k+r_i} \beta^{l-s+s_i} = n_{i+1} e^{r \log \alpha - s \log \beta} < n_{i+1}. \]

The contradiction (8), (9), (10), proves by Lemma 5(a) that \( k/l \) is a left convergent to \( \xi \). (In case \( s_{i+1} < s_i \) the fraction \( k/l \) turns out to be a right convergent to \( \xi \)).

It would be very valuable to have a characterization like Theorem 1 for sequences composed of \( r \) multiplicatively independent positive
numbers, \( r > 2 \). This would solve the conjecture in the introduction immediately. We now prove case \( t = 1 \) of this conjecture.

**Theorem 2**: Let \( n_1, n_2, \cdots \) be the sequence composed of the primes \( p_1, \cdots, p_r \) (\( r \geq 2 \)). Let \( p \) be one of these primes. Then there exists an infinite number of pairs \( n_i, n_{i+1} \) such that \( n_i \) is a pure power of \( p \) and \( n_{i+1} \) is not divisible by \( p \).

**Proof**: Without loss of generality we may assume \( p = p_1 \). Let \( k \) be a positive integer and \( n_{ik} = p^k \). Let \( n_{ik+1} = p_1^{l_1} \cdots p_r^{l_r} \) and \( n_{ik} = p_1^{l_1-1} \). It follows from (1) that \( n_{ik+1} = p_2^{l_2} \cdots p_r^{l_r} \). Since, by Lemma 2,

\[
\lim_{k \to \infty} \frac{n_{ik}}{(n_{ik}, n_{ik+1})} = \infty
\]

we obtain infinitely many different pairs \( n_{ik}, n_{ik+1} \) with the required property.

**Remark**: In the same way one can prove the existence of infinitely many pairs \( n_i, n_{i+1} \) such that \( n_{i+1} \) is a pure power of \( p \) and \( n_i \) is not divisible by \( p \).

Finally we prove case \( t = 2 \) of our conjecture.

**Theorem 3**: Let \( p_1, \cdots, p_r \) be \( r > 2 \) different primes. Let \( M = \{m_1, m_2, \cdots\} \) be the sequence composed of these primes. Let \( p \) and \( q \) be two primes from \( p_1, \cdots, p_r \). Then there exist infinitely many pairs \( m_i, m_{i+1} \) such that one of the numbers \( m_i, m_{i+1} \) is composed of \( p \) and \( q \) and the other is neither divisible by \( p \) nor by \( q \).

The proof is based on two lemmas.

**Lemma 6**: Let \( r > 2 \). Let \( M = \{m_1, m_2, \cdots\} \) be the sequence composed of the different primes \( p_1, \cdots, p_r \), and \( N = \{n_1, n_2, \cdots\} \) the sequence composed of \( p_1 \) and \( p_2 \). Suppose there exists an \( i_0 \) such that for every \( i \geq i_0 \)

\[
m_i \in N \Rightarrow (m_{i-1}, p_2) > 1 \quad \text{and} \quad (m_{i+1}, p_1 p_2) > 1.
\]

Then there exists an \( i_1 \) such that for every \( i \geq i_1 \).
(a) if \( m_i \in N \) and \( m_i^2 \leq m_{i-1} m_{i+1} \), then \( m_{i-1} \in N \),
(b) if \( m_i \in N \) and \( m_i^2 \geq m_{i-1} m_{i+1} \), then \( m_{i+1} \in N \).

**Proof:** We know from Lemma 2 that
\[
\frac{m_{i-1}}{(m_{i-1}, m_i)} \to \infty \quad \text{as } i \to \infty.
\]

We choose \( i_1 \) such that
\[
\frac{m_{i-1}}{(m_{i-1}, m_i)} > m_{i_0} \quad \text{for } i \geq i_1.
\]

In the sequel we only consider \( i \) with \( i \geq i_1 \).
Assume \( m_i \in N \). Let \( m_i = p_1^{k_1} p_2^{k_2} \). Put \( m_{i-1} = p_1^{k_1} \cdots p_r^{k_r} \) and \( m_{i+1} = p_1^{l_1} \cdots p_r^{l_r} \). Then
\[
m_{i-1} < \frac{m_{i-1} m_{i+1}}{m_i} < m_{i+1}.
\]

Hence, we have either
\[(11) \quad m_{i-1} m_{i+1}/m_i = m_i \]
or
\[(12) \quad m_{i-1} m_{i+1}/m_i \notin M.\]

We note \( m_{i-1} m_{i+1}/m_i = p_1^{k_1+1} p_2^{k_2-1} p_3^{k_3+1} \cdots p_r^{k_r+1} \). If (11) holds, then \( k_3 + l_3 = \cdots = k_r + l_r = 0 \), and, hence, \( k_3 = \cdots = k_r = 0 \) and \( l_3 = \cdots = l_r = 0 \). In this case both \( m_{i-1} \in N \) and \( m_{i+1} \in N \). If (12) holds, then
\[(13) \quad k_1 + l_1 - a < 0 \quad \text{or} \quad k_2 + l_2 - b < 0.\]

Suppose \( k_1 \leq a \) and \( k_2 \leq b \). By (1), \( p_1^{a-k_1} p_2^{b-k_2} \) is preceded in \( M \) by \( p_3^{k_3} \cdots p_r^{k_r} \). Since
\[
p_1^{a-k_1} p_2^{b-k_2} = \frac{m_i}{(m_{i-1}, m_i)} > m_{i_0},
\]
this is a contradiction with the condition of the lemma. Hence, \( k_1 > a \) or
Similarly, \( l_1 > a \) or \( l_2 > b \). Without loss of generality we may assume \( k_2 > b \). Then, by (13), \( k_1 < a \) and \( l_1 < a \). Thus \( l_2 > b \). So we obtain

\[ k_2 > b, \quad l_2 > b. \]

We define a sequence of positive integers \( \{a_j\}_{j=0}^{\infty} \) by

\[ m_{a_j} = p_1^j p_2^j \quad \text{for } j = 0, 1, 2, \cdots. \]

We have, by (1) and (14),

\[ m_{a_{j-1}} = p_1^{k_1 - b + j} p_2^{k_2} \cdots p_r^{k_r} \quad \text{and} \quad m_{a_{j+1}} = p_1^{l_1 - b + j} p_2^{l_2} \cdots p_r^{l_r}, \]

for \( j = 0, 1, \cdots, b \). Consider the pairs of quotients

\[ \left( \frac{m_{a_{j-1}}}{m_{a_j}}, \frac{m_{a_{j+1}}}{m_{a_j}} \right) \quad \text{for } j = 0, 1, 2, \cdots. \]

We know

\[ \frac{m_{a_{j-1}}}{m_{a_j}} = p_1^{k_1 - a} p_2^{k_2 - b} \cdots p_r^{k_r} \quad \text{and} \quad \frac{m_{a_{j+1}}}{m_{a_j}} = p_1^{l_1 - a} p_2^{l_2 - b} \cdots p_r^{l_r} \]

for \( j = 0, \cdots, b \). Let \( J_0 \) be the smallest value of \( j \) for which one of the quotients in (15) assumes another value. This \( J_0 \) exists, since, by Lemma 1, \( m_{i+1}/m_i \) tends to 1 as \( i \to \infty \). We assume that the first quotient changes firstly. Thus

\[ 1 > \frac{m_{a_{j-1}}}{m_{a_j}} > \frac{m_{a_{j-1}-1}}{m_{a_{j-1}}} = \cdots = \frac{m_{a_{0}-1}}{m_{a_{0}}} = \frac{m_{i-1}}{m_{i}} \]

and

\[ \frac{m_{a_{j-1}+1}}{m_{a_{j-1}}} = \cdots = \frac{m_{a_{0}+1}}{m_{a_{0}}} = \frac{m_{i+1}}{m_{i}}. \]

Put \( m_{a_{j-1}} = p_1^{k_1} \cdots p_r^{k_r} \) and \( m_{a_{j+1}} = p_1^{l_1} \cdots p_r^{l_r} \). The following argument shows \( \kappa_2 = 0 \). If \( \kappa_2 > 0 \), then, by (1), \( m_{a_{j-1}}/p_2 \) is the predecessor of \( m_{a_{j}}/p_2 = m_{a_{j-1}} \), and, hence, \( m_{a_{j-1}}/m_{a_{j}} = m_{a_{j-1}-1}/m_{a_{j-1}} \) in contradiction with (16). Since we know from the argument preceding formula (14) that both \( \kappa_1 \leq a \) and \( \kappa_2 \leq J \) is impossible, we have
Consider

\[ m = \frac{m_{a_j-1} + 1}{m_{a_j}} \cdot \frac{m_{a_j-1}}{m_{a_j}} = \frac{m_{i+1}}{m_i} \cdot \frac{m_{a_j-1}}{m_{a_j}} \]

\[ = p_1^{l_1 + \kappa_1} p_2^{l_2 + \kappa_2} \cdots p_r^{l_r + \kappa_r} \cdot m_{i+1} \cdot m_{a_j-1} \cdot m_{a_j-1}. \]

We have, by (17)

\[ m = \frac{m_{a_j-1} + 1}{m_{a_j-1}} \cdot \frac{m_{a_j-1} - 1}{m_{a_j}} = m_{a_j-1} \cdot m_{a_j-1} \cdot m_{a_j-1} \]

\[ = m_{a_j-1} \cdot m_{a_j-1} \cdot m_{a_j-1}. \]

From (18) and (14) we see that \( m \in M \). Moreover,

\[ m_{a_j-1} + 1 > m > \frac{m_{a_j-1} + 1}{m_{a_j}} > m_{a_j-1}. \]

Hence,

\[ m = m_{a_j-1}. \]

This implies \( l_1 + \kappa_1 = \cdots = l_r + \kappa_r = 0 \). Thus \( l_1 = \cdots = l_r = 0 \) and \( m_{i+1} \in \mathbb{N} \). Furthermore, in view of (17), (19), the definition of \( m \) and (16),

\[ \frac{m_{i+1}}{m_i} = \frac{m_{a_j-1} + 1}{m_{a_j-1}} = \frac{m_{a_j-1} + 1}{m} \]

\[ = \frac{m_{a_j-1}}{m_{a_j-1}} \cdot \frac{m_{a_j-1}}{m_{a_j-1}} \cdot m_{i+1} \cdot m_{a_j-1} \cdot m_{a_j-1}. \]

Similarly, the assumption that the second quotient in (15) changes firstly leads to \( m_{i+1} \in \mathbb{N} \) and \( m_{i-1}/m_i > m_{i}/m_{i+1} \). This completes the proof of the lemma.

**Lemma 7**: Let \( r > 2 \). Let \( M = \{m_1, m_2, \cdots\} \) be the sequence composed of the primes \( p_1, \cdots, p_r \). Let \( p \) and \( q \) be two arbitrary primes from \( p_1, \cdots, p_r \) with \( p > q \) and let \( N = \{n_1, n_2, \cdots\} \) be the sequence composed of \( p \) and \( q \). Suppose there exists an \( i_0 \) such that for every \( i \geq i_0 \)

\[ m_i \in \mathbb{N} \Rightarrow (m_{i-1}, pq) > 1 \quad \text{and} \quad (m_{i+1}, pq) > 1. \]
Then there exists a monotonically increasing, unbounded sequence \(T_1, T_2, T_3, \ldots\) such that no interval \([T_H, qT_H]\) contains an element of \(M \setminus N\).

**Proof:** Let \([a_0, a_1, a_2, \ldots]\) be the continued fraction of \(\xi = \log q / \log p\). Put \(A_h/B_h = [a_0, \ldots, a_h]\) for \(h = 0, 1, 2, \ldots\). It follows from the Gel'fond-Schneider theorem [4, Satz 14], that \(\xi\) is transcendental. Hence, the sequence \(a_0, a_1, a_2, \ldots\) is not periodical [2, Satz 3.1]. There therefore exist infinitely many values \(h\) with \(a_h > 1\).

Let \(H\) be such that \(a_H > 1\). It is no loss of generality to assume \(A_H/B_H < \xi\). We consider the subsequence \(N_1\) of \(N\) beginning with

\[T_H = p^{A_H} q^{B_H - 1} - 1\]

and ending with \(qT_H = p^{A_H} q^{B_H - 1}\).

(If \(A_H/B_H > \xi\), we may choose \(T_H = p^{A_H - 1} q^{B_H}\) and consider the interval \([T_H, pT_H]\).)

Let \(n_i = p^c q^d\) be in \(N_1\), \(n_i \neq qT_H\). Since \(qT_H < q^{B_H + B_H - 1} < p^{A_H + A_H - 1}\), we have

\[c < A_H + A_H - 1\text{ and } d < B_H + B_H - 1.\]

We distinguish two cases.

(i) \(c \geq A_H\). We assert that \(n_{i + 1} = p^{c - A_H} q^{d + B_H}\). Since \(A_H/B_H < \xi\), we have

\[n_i < p^{c - A_H} q^{d + B_H} \in N_1.\]

Suppose

\[n_{i + 1} = p^c q^d < p^{c - A_H} q^{d + B_H}.\]

This implies

\[A_H - B_H \xi < (c - s) - (t - d) \xi < 0.\]

By Lemma 3(b), \(|t - d| \geq B_H + 1\). Hence, \(d \geq B_H + 1\) or \(t \geq B_H + 1\) in contradiction with (20).

(ii) \(c < A_H\). Since \(d \leq B_H - 1\) implies \(p^c q^d < p^{A_H} q^{B_H - 1} - 1 = T_H\), we have \(d \geq B_H - 1\). We assert that \(n_{i + 1} = p^{c + A_H - 1} q^{d - B_H - 1}\). Since \(A_H - 1/B_H - 1 > \xi\), we have

\[n_i < p^{c + A_H - 1} q^{d - B_H - 1} \in N_1.\]
Suppose

\[ n_{i+1} = p^{q}q' < p^{A_H -1}q^{d-B_H} \]

Then

\[ 0 < (s-c) - (d-t)\xi < A_{H-1} - B_{H-1} \xi. \]

By Lemma 3(b), \(|d-t| \geq B_H\). If \(d-t < 0\), then \(t > d + B_H \geq B_H + B_{H-1}\) and \(p^{q}q' > q^{B_H + B_{H-1}} > qT_H\), which is false. Hence, \(d-t \geq 0\). This implies \(s-c > 0\) and \(d \geq B_H + t\). By Lemma 5(b) \((s-c)/(d-t)\) is a right convergent to \(\xi\). Since \(A_H/B_H\) is a left convergent to \(\xi\), we obtain \(d \geq d-t \geq B_H + B_{H-1}\), which is impossible in view of (20). Summarizing we see that among the quotients \(n_{i+1}/n_i\) for \(n_i \in N_1\) only \(p^{A_H}q^{B_H}\) and \(p^{A_H-1}q^{-B_H-1}\) occur. Note

\[ p^{A_H-1}q^{-B_H-1} > p^{-A_H}q^{B_H} > 1. \]

We now assert that

\[ n_{i+1}/n_i = p^{A_H-1}q^{-B_H-1} \Rightarrow \frac{n_i}{n_{i-1}} = \frac{n_{i+1}}{n_i} \text{ or } \frac{n_{i+1}}{n_i} = \frac{n_{i+2}}{n_{i+1}}. \]

Since \(n_i = T_H\) implies \(n_{i+1} = p^{-A_H}q^{B_H}n_i\), we have \(n_i > T_H\). Hence, \(n_{i-1} \in N_1\). Suppose

\[ \frac{n_i}{n_{i-1}} = \frac{n_{i+2}}{n_{i+1}} = p^{-A_H}q^{B_H}. \]

Then

\[ n_{i+2} = p^{-2A_H + A_H -1}q^{2B_H - B_H - 1}n_{i-1}. \]

By \(a_H \geq 2\), it follows that \(n_{i+2} \geq q^{B_H + B_{H-1} + B_{H-2}}\). This is a contradiction.

We now turn our attention to the subsequence \(M_1\) of \(M\) starting with \(T_H\) and ending with \(qT_H\). Let \(m_i \in N_1, m_i \neq qT_H\). Put \(m_i = n_j\). Hence, \(n_{j+1} \in N_1\). Note that \(n_{j-1} \leq m_{i-1} < m_i < m_{i+1} \leq n_{j+1}\). The condition of Lemma 7 enables us to apply Lemma 6. Hence, \(n_{j-1} = n_{j+1}\) if \(m_i^2 \leq m_{i-1}m_{i+1}\) and \(m_{i+1} = n_{j+1}\) if \(m_i^2 \geq m_{i-1}m_{i+1}\). It follows that

\[ m_{i-1} = n_{j-1} \text{ if } n_j^2 \leq n_{j-1}m_{i+1} \leq n_{j-1}n_{j+1} \]
and

\[(24) \quad m_{i+1} = n_{j+1} \quad \text{if} \quad n_j^2 \geq m_{i-1} n_{j+1} \geq n_{j-1} n_{j+1}.\]

We can now prove that all elements \(m_i\) with \(T_H \leq m_i \leq q T_H\) belong to \(N_1\). Suppose \(T_H = m_1\) and all integers \(m_1, m_{i+1}, \ldots, m_i\) belong to \(N_1\), while \(m_i < q T_H\). We shall prove that \(m_{i+1} \in N_1\). Put \(m_i = n_j\). We distinguish two cases.

(i) \(n_{j-1} n_{j+1} \leq n_j^2\). It follows from (24) that \(m_{i+1} = n_{j+1} \in N_1\).

(ii) \(n_{j-1} n_{j+1} \geq n_j^2\). It follows from formula (21) and the lines before that

\[\frac{n_j}{n_{j-1}} = p^{-A_H} q^{B_H}, \quad \frac{n_{j+1}}{n_j} = p^{A_{H-1}} q^{-B_{H-1}}.\]

Since \(n_j = T_H\) implies \(n_{j+1} = p^{-A_H} q^{B_H} n_j\), we have \(n_j > T_H\) and, hence, \(n_{j-1} \in N_1\). By (22) we have \(n_{j+2}/n_{j+1} = p^{A_{H-1}} q^{-B_{H-1}}\). Let \(n_{j+1} = m_i\).

Since \(n_{j+2}/n_{j+1} = n_{j+1}/n_j\), we obtain from (23) that \(n_j = m_{i+1}\). Hence, \(m_{i+1} = m_i\) and \(i^* - 1 = i\). It follows that \(m_{i+1} = m_i = n_{j+1} \in N\).

Since we have constructed an infinite number of \(T_H\)'s such that all integers \(m_i \in [T_H, q T_H]\) belong to \(N\), the lemma has been proved.

We are now going to prove the main result.

**Proof of Theorem 3**: It is no restriction to assume \(p = p_1, q = p_2, p > q\). Suppose that there are only a finite number of values \(i\) for which the statement of the theorem holds. Then the condition of Lemma 7 is fulfilled for some \(i_0\). It follows that there exists an unbounded sequence \(T_1, T_2, T_3, \ldots\) such that each element \(m_i \in [T_H, q T_H]\) belongs to the sequence \(N\) composed of \(p\) and \(q\). Let \(N = \{n_1, n_2, n_3, \ldots\}\). We know from Lemma 1 that \(n_{i+1}/n_i \to 1\) as \(i \to \infty\). Consider the sequence \(p_3 n_1, p_3 n_2, p_3 n_3, \ldots\). These elements belong to \(M\) \(|\ N\). However, \(p_3 n_{i+1}/p_3 n_i \to 1\) as \(i \to \infty\). This is a contradiction.
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