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R. E. STONG

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## SEMI-CHARACTERISTICS AND FREE GROUP ACTIONS

R. E. Stong

### 1. Introduction

Recently, Ronnie Lee [5] has introduced a semi-characteristic homomorphism

$$\chi_{\frac{1}{2}} : \mathfrak{N}_{2n+1}(G) \rightarrow \tilde{R}_{GL, \text{ev}}(G)$$

from the unoriented bordism group of free  $G$  actions,  $G$  a finite group, into a Grothendieck group of representations of  $G$  over a finite field  $K$  of characteristic 2. One of the questions he raises is to compute this invariant in terms of Stiefel-Whitney numbers, and that question will be answered here.

Perhaps more interesting is the fact that  $\chi_{\frac{1}{2}}$  can be computed quite simply. Specifically, there is a class  $i_*(K) \in \tilde{R}_{GL, \text{ev}}(G)$  obtained by extension from the Sylow 2 subgroup of  $G$ , so that for any free  $G$  action  $(M, \phi)$ ,

$$\chi_{\frac{1}{2}}(M; K) = s\chi(M) \cdot i_*(K)$$

where  $s\chi(M)$  is the Kervaire semi-characteristic [4]

$$s\chi(M) = \sum_{i=0}^n (-1)^i \dim H^i(M; Z_2)$$

in  $Z_2$ ,  $\dim M = 2n + 1$ . Except when  $G$  has odd order, so that  $i_*(K) = 0$ , Lee's invariant then reduces to the usual semicharacteristic.

A direct proof that  $s\chi(M)$  is a cobordism invariant of  $(M, \phi)$ , for  $G$  of even order, will be given. This involves showing that for a free involution  $T : M^{2n+1} \rightarrow M^{2n+1}$   $s\chi(M)$  is just the Euler characteristic of the submanifold  $N^{2n} \subset M^{2n+1}/T$  which defines the double cover of  $M/T$  by  $M$ .

An analogous result holds for arbitrary sphere bundles, and this will be used to show that for even dimensional manifolds with involution which is free on the boundary,

$$s\chi(\partial V) = \chi(F) + \chi(F \cap F)$$

where  $T$  is an involution on  $V$  with  $F$  the fixed set of  $T$ , and  $F \cap F$  the self intersection of  $F$  in  $V$ .

As a corollary, one obtains a more geometric proof of a result of Conner and Floyd [2]: If  $T : M^{2n} \rightarrow M^{2n}$  is an involution on a manifold of odd Euler characteristic, then some component of the fixed set has dimension at least  $n$ .

Finally, the semicharacteristics for oriented manifolds introduced by Lee will be examined. Unfortunately, the algebraic problems are much harder, and the results are far from complete. For groups with abelian Sylow 2 subgroup, the invariants always vanish (Proposition 5.4) for  $4k+3$  dimensional manifolds. For abelian groups and manifolds of dimension  $4k+1$ , the invariants are determined in Propositions 5.5 and 5.6.

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## 2. Lee's invariant

In order to understand Lee's invariant, one needs primarily to define the Grothendieck group  $\tilde{R}_{GL, ev}(G)$ . Let  $K$  be a finite field of characteristic 2, and  $G$  a finite group.  $R_K(G)$  denotes the Grothendieck group of finite dimensional  $G$  representations over  $K$ .

If  $V$  is a  $G$ -representation over  $K$ , a quadratic form  $(V, \phi)$  is a symmetric bilinear pairing  $\phi : V \times V \rightarrow K$  such that

$$\phi(gx, gy) = \phi(x, y).$$

The form is *even* if for all  $t \in G$ ,  $t \neq e$  and  $t$  of order 2,

$$\phi(x, tx) = 0$$

for every  $x \in V$ . The form is non-singular if the homomorphism  $ad \phi : V \rightarrow V^*$  given by  $(ad \phi)(x)(y) = \phi(x, y)$  is an isomorphism.

$R_{GL, ev}(G)$  is the quotient group of  $R_K(G)$  obtained by dividing out the subgroup generated by the classes of those  $V$  which admit a non-singular even quadratic form.

If  $H \subset G$ , one has a transfer homomorphism

$$i^* : R_{GL, ev}(G) \rightarrow R_{GL, ev}(H)$$

obtained by considering a  $G$  representation as an  $H$ -representation, and an extension homomorphism

$$i_* : R_{GL, ev}(H) \rightarrow R_{GL, ev}(G)$$

obtained by sending  $W$  to  $KG \otimes_{KH} W$ .

Then  $\tilde{R}_{GL, ev}(G)$  is defined to be the cokernel of

$$i_* : R_{GL, ev}(\{e\}) \rightarrow R_{GL, ev}(G).$$

Thus  $\tilde{R}_{GL, ev}(G)$  is obtained from  $R_K(G)$  by dividing out the subgroup generated by the non-singular even forms and the free  $KG$  modules.

The homomorphism

$$\chi_{\frac{1}{2}} : \mathfrak{N}_{2n+1}(G) \rightarrow \tilde{R}_{GL, ev}(G)$$

assigns to  $(M^{2n+1}, \phi)$  the class  $\sum_{i=0}^n (-1)^i [H^i(M; K)]$ , where  $G$  acts on  $H^i(M; K)$  via  $\phi$ .

Now for  $H \subset G$ ,  $i^*$  and  $i_*$  induce homomorphisms

$$i^* : \tilde{R}_{GL, ev}(G) \rightarrow \tilde{R}_{GL, ev}(H)$$

and

$$i_* : \tilde{R}_{GL, ev}(H) \rightarrow \tilde{R}_{GL, ev}(G).$$

Letting

$$i^* : \mathfrak{N}_*(G) \rightarrow \mathfrak{N}_*(H)$$

by sending  $(M, \phi)$  to  $(M, \phi/H \times M)$  and

$$i_* : \mathfrak{N}_*(H) \rightarrow \mathfrak{N}_*(G)$$

by sending  $(N, \psi)$  to the class of  $G \times N/(gh^{-1}, hx) \sim (g, x)$  with action  $g'(g, x) = (g'g, x)$ , one has a commutative diagram (Lemma 4.10 of [5])

$$\begin{array}{ccccc} \mathfrak{N}_{2n+1}(H) & \xrightarrow{i_*} & \mathfrak{N}_{2n+1}(G) & \xrightarrow{i^*} & \mathfrak{N}_{2n+1}(H) \\ \downarrow \chi_{\frac{1}{2}} & & \downarrow \chi_{\frac{1}{2}} & & \downarrow \chi_{\frac{1}{2}} \\ \tilde{R}_{GL, ev}(H) & \xrightarrow{i_*} & \tilde{R}_{GL, ev}(G) & \xrightarrow{i^*} & \tilde{R}_{GL, ev}(H). \end{array}$$

The other fact needed here is that if  $S \subset G$  is the Sylow 2-subgroup of  $G$ , then the composite

$$i_* \circ i^* : \mathfrak{N}_*(G) \rightarrow \mathfrak{N}_*(S) \rightarrow \mathfrak{N}_*(G)$$

is the identity. (Note: This is Lemma 4.11 (3) of [5]; beware that parts (1) and (2) of the Lemma do not hold for arbitrary  $G$ ). To see this one notes that if  $f : M \rightarrow BG$  represents  $\alpha \in \mathfrak{N}_*(G)$  then  $i_* \circ i^*(\alpha)$  is represented by  $f \circ \pi : \tilde{M} \rightarrow BG$  where  $\tilde{M}$  is the bundle induced by

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & BS \\ \downarrow \pi & & \downarrow \pi' \\ M & \xrightarrow{f} & BG \end{array}$$

Then for  $x \in H^*(BG; Z_2)$ ,

$$\begin{aligned} \langle w_\omega(\tilde{M})(f \circ \pi)^*(x), [\tilde{M}] \rangle &= \langle \pi^*(w_\omega(M)f^*(x)), [\tilde{M}] \rangle \\ &= [G : S] \langle w_\omega(M)f^*(x), [M] \rangle \end{aligned}$$

and  $[G : S] = \text{index of } S \text{ in } G = 1 \pmod{2}$ ).

LEMMA 2.1: *If  $S$  is a 2 group, then  $\tilde{R}_{GL, \text{ev}}(S)$  is isomorphic to  $Z_2$  if  $S \neq \{e\}$  and is the zero group if  $S = \{e\}$ .*

PROOF: If  $S = \{e\}$ ,  $i_* : R_{GL, \text{ev}}(\{e\}) \rightarrow R_{GL, \text{ev}}(S)$  is the identity, so the cokernel,  $\tilde{R}_{GL, \text{ev}}(S)$ , is the zero group.

Thus suppose  $S \neq \{e\}$ . If  $V$  is any representation space for  $S$ ,  $S$  acts on the underlying set of  $V$  which has an even number of elements, and each orbit has  $2^j$  elements for some  $j$ . Since  $S$  fixes  $\{0\}$ ,  $S$  must also fix a nonzero vector  $x$ . Thus  $V$  contains a trivial representation,  $Kx$ . Then  $[V] = [K] + [V/Kx]$ , and inductively  $R_K(S) \cong Z$  assigning to  $V$  its dimension over  $K$ .

On  $K \oplus K$  with trivial  $S$  action one has the hyperbolic form  $\phi((x_1, x_2), (y_1, y_2)) = x_1y_2 + x_2y_1$ , which is even. On the other hand,  $KS \oplus_K W$  has dimension divisible by  $2^s = \text{order of } S$ , and any even form has even dimension, so  $\tilde{R}_{GL, \text{ev}}(S) \cong Z_2$ .

To see that any even form has even dimension, it suffices to restrict  $(V, \phi)$  to some subgroup of order 2 in  $S$ . If  $t$  is the element of order 2, the form  $\psi : V \times V \rightarrow K$  defined by  $\psi(x, y) = \phi(x, ty) = \phi(tx, y)$  is then non-singular and  $\psi(x, x) = 0$ . One may then choose a symplectic base for  $(V, \psi)$ . \*

PROPOSITION 2.2: *The homomorphism*

$$\chi_{\frac{1}{2}} : \mathfrak{R}_{2n+1}(G) \rightarrow \tilde{R}_{GL, \text{ev}}(G)$$

sends  $(M, \phi)$  to  $s\chi(M) \cdot i_*(K)$  where

$$s\chi(M) = \sum_{i=0}^n (-1)^i \dim H^i(M; Z_2)$$

and  $i_*(K)$  is the class obtained by applying

$$i_* : \tilde{R}_{GL, \text{ev}}(S) \rightarrow \tilde{R}_{GL, \text{ev}}(G),$$

$S$  the Sylow 2-subgroup of  $G$  to the 1-dimensional trivial  $S$  representation.

PROOF: This is essentially the proof given in Theorem 4.13 of [5]. First,  $H^i(M; K) \cong H^i(M; Z_2) \otimes_{Z_2} K$ , so

$$\begin{aligned}
 \chi_{\frac{1}{2}}(M; K) &= \chi_{\frac{1}{2}}(i_* i^* M; K) \\
 &= i_* \chi_{\frac{1}{2}}(i^* M; K) \\
 &= i_* \left( \sum_{i=0}^n (-1)^i [H^i(M; K)] \right) \\
 &= i_* \left( \sum_{i=0}^n (-1)^i \dim_K H^i(M; K) \cdot [K] \right) \\
 &= i_* (s\chi(M) \cdot [K]) \\
 &= s\chi(M) \cdot i_*([K]). \quad *
 \end{aligned}$$

*Note:* If  $G$  has odd order,  $S = \{e\}$ , and  $i_*(K) = 0$ . If  $G$  has even order,  $i^* i_*(K)$  is represented by  $KG \otimes_{KS} K$  which has dimension  $[G : S] = \text{odd}$ . Thus  $i^* i_*(K) \neq 0$  and so  $i_*(K) \neq 0$ . Thus, the Kervaire semi-characteristic is an invariant of free  $G$  bordism, if  $G$  has even order. It is definitely not an invariant when  $G$  has odd order.

It should be remarked that Lee’s invariant is stronger than just the Kervaire semi-characteristic. His arguments make heavy use of the fact that  $i_*(K)$  is not in general the class of the trivial  $G$  representation. The formula  $\chi_{\frac{1}{2}}(M, \phi) = s\chi \cdot (M) i_*(K)$  contains more geometric information than the value of the semicharacteristic alone.

### 3. Kervaire’s semicharacteristic

The basic result needed to analyze the Kervaire semicharacteristic will be:

**PROPOSITION 3.1:** *Let  $M$  be a closed manifold of dimension  $2n+r$  and  $\xi$  an  $r$ -plane bundle over  $M$ . Then the Kervaire semicharacteristic of the sphere bundle of  $\xi$ ,  $s\chi(S(\xi))$ , is the sum of the Euler characteristics of  $M$  and  $N$ , where  $N \subset M$  is the submanifold dual to  $\xi$ ; i.e.  $s\chi(S(\xi)) = \chi(M) + \chi(N)$ .*

**PROOF:** The Gysin sequence of the bundle  $\xi$  gives an exact sequence

$$\begin{aligned}
 0 \leftarrow A \leftarrow H^{n+r-1}(S(\xi)) \leftarrow H^{n+r-1}(M) \leftarrow H^{n-1}(M) \leftarrow H^{n+r-2}(S(\xi)) \leftarrow \\
 \cdots \leftarrow H^r(S(\xi)) \leftarrow H^r(M) \leftarrow H^0(M) \leftarrow H^{r-1}(S(\xi)) \leftarrow H^{r-1}(M) \leftarrow 0 \leftarrow \\
 \leftarrow H^{r-2}(S(\xi)) \leftarrow H^{r-2}(M) \leftarrow \cdots \leftarrow 0 \leftarrow H^0(S(\xi)) \leftarrow H^0(M) \leftarrow 0.
 \end{aligned}$$

where

$$A = \ker \{ \cup w_r(\xi) : H^n(M) \rightarrow H^{n+r}(M) \}.$$

The usual rule for Euler characteristics in an exact sequence gives

$$\begin{aligned}
 s\chi(S(\xi)) &= \sum_0^{n+r-1} (-1)^i \dim H^i(S(\xi)) \\
 &= \sum_0^{n+r-1} (-1)^i \dim H^i(M) + (-1)^{n+r-1} \dim A \\
 &\qquad\qquad\qquad + (-1)^{r-1} \sum_0^{n-1} (-1)^i \dim H^i(M) \\
 &= \chi(M) - \dim H^n(M) + \dim A \pmod{2} \\
 &= \chi(M) + \dim \text{im} \{ \cup w_r(\xi) : H^n(M) \rightarrow H^{n+r}(M) \}
 \end{aligned}$$

Now consider the symmetric quadratic form

$$\phi : H^n(M) \times H^n(M) \rightarrow Z_2$$

defined by  $\phi(x, y) = \langle w_r(\xi) \cup x \cup y, [M] \rangle = \langle f^*(x) \cup f^*(y), [N] \rangle$ . where  $f : N \rightarrow M$  is the inclusion. Clearly, the rank of  $\phi$  is equal to the dimension of the image of  $\{ \cup w_r(\xi) : H^n(M) \rightarrow H^{n+r}(M) \}$ . On the other hand, there exist classes  $v \in H^n(M)$  so that  $\phi(x, x) = \phi(x, v)$  for all  $x \in H^n(M)$ , and for any such  $v$ ,  $\text{rank}(\phi) = \phi(v, v)$  in  $Z_2$ . Now the Stiefel-Whitney class of  $N$  is given by  $f^*(w(M)/w(\xi))$ , and so there is a class  $v' \in H^n(M)$  with  $f^*(v') = v_n(N)$  being the  $n$ -th Wu class of  $N$ . Thus, for any  $x \in H^n(M)$ ,

$$\begin{aligned}
 \phi(x, x) &= \langle f^*(x) \cup f^*(x), [N] \rangle = \langle v_n(N) \cup f^*(x), [N] \rangle \\
 &= \langle f^*(x) \cup f^*(v'), [N] \rangle = \phi(x, v')
 \end{aligned}$$

and

$$\begin{aligned}
 \text{rank}(\phi) &= \langle f^*(v') \cup f^*(v'), [N] \rangle = \langle v_n(N) \cup v_n(N), [N] \rangle \\
 &= \langle w_{2n}(N), [N] \rangle = \chi(N).
 \end{aligned}$$

Hence,  $s\chi(S(\xi)) = \chi(M) + \chi(N)$ . \*

*Note:* One would like to prove this using only the cohomology structure, but it seems to depend heavily on the fact that the Wu class  $v_n(N)$  belongs to the image of  $f^*$ .

**COROLLARY 3.2:** *If  $M^{2n+1}$  is a closed manifold and  $T : M \rightarrow M$  is a free involution, then  $s\chi(M) = \chi(N)$  where  $N^{2n} \subset M^{2n+1}/T$  is the submanifold which defines the double cover of  $M/T$  by  $M$ .*

(See [1], Prop (3.4), and [3], Cor. 2.7).

**PROOF:**  $M = S(\lambda)$  where  $\lambda \rightarrow M/T$  is the line bundle associated to the double cover of  $M/T$  by  $M$ , and  $N$  is the submanifold dual to  $\lambda$ . Since  $M/T$  has odd dimension,  $\chi(M/T) = 0$ . \*

**COROLLARY 3.3:** *If  $G$  is a finite group of even order, then assigning to  $(M^{2n+1}, \phi)$  the semi-characteristic  $s\chi(M)$  defines a homomorphism*

$$s\chi : \mathfrak{N}_{2n+1}(G) \rightarrow Z_2.$$

**PROOF:** Letting  $Z_2 \subset G$  be any subgroup of order 2,  $s\chi$  is given by the composite of

$$i^* : \mathfrak{N}_{2n+1}(G) \rightarrow \mathfrak{N}_{2n+1}(Z_2)$$

and the Smith homomorphism ([1] §26)

$$\Delta : \mathfrak{N}_{2n+1}(Z_2) \rightarrow \mathfrak{N}_{2n}(Z_2)$$

and the usual isomorphism

$$\mathfrak{N}_{2n}(Z_2) \cong \mathfrak{N}_{2n}(BZ_2)$$

and the augmentation

$$\varepsilon : \mathfrak{N}_{2n}(BZ_2) \rightarrow \mathfrak{N}_{2n}$$

and the Euler characteristic

$$\chi : \mathfrak{N}_{2n} \rightarrow Z_2. \quad *$$

One may now write down a characteristic number description of the semi-characteristic, as was asked for by Lee. Being given  $(M^{2n+1}, \phi)$ , let  $h : M/G \rightarrow BG$  classify the principal bundle  $M \rightarrow M/G$ . Let  $Z_2 \subset G$  be any subgroup of order 2,  $c \in H^1(BZ_2, Z_2)$  the nonzero class, and  $i_* : H^*(BZ_2, Z_2) \rightarrow H^*(BG; Z_2)$  the extension homomorphism. Then

$$s\chi(M) = \left\langle \sum_{j=0}^{2n+1} w_{2n+1-j}(M/G) h^* i_*(c^j); [M/G] \right\rangle$$

i.e.  $s\chi$  is associated with the characteristic class

$$\sum_{j=0}^{2n+1} w_{2n+1-j} i_*(c^j).$$

To see this, one notes that the diagram

$$\begin{array}{ccc} M/Z_2 & \xrightarrow{\tau \times \tilde{h}} & BO \times BZ_2 \\ \downarrow \pi' & & \downarrow 1 \times \pi \\ M/G & \xrightarrow{\tau \times h} & BO \times BG \end{array}$$

commutes. Thus

$$\begin{aligned}
 & \langle \sum_0^{2n+1} w_{2n+1-j}(M/G)h^*i_*(c^j); [M/G] \rangle \\
 &= \langle \sum_0^{2n+1} w_{2n+1-j} \otimes i_*(c^j), (\tau \times h)_*([M/G]) \rangle \\
 &= \langle (1 \times \pi)_* \left( \sum_0^{2n+1} w_{2n+1-j} \otimes c^j \right), (\tau \times h)_*([M/G]) \rangle \\
 &= \langle \sum_0^{2n+1} w_{2n+1-j} \otimes c^j, (\tau \times \tilde{h})_*([M/Z_2]) \rangle \\
 &= \langle \sum_0^{2n+1} w_{2n+1-j}(M/Z_2)\tilde{h}^*(c^j), [M/Z_2] \rangle
 \end{aligned}$$

where

$$(1 \times \pi)_* : H^*(BO \times BZ_2; Z_2) \rightarrow H^*(BO \times BG; Z_2)$$

is the cohomology ‘transfer’ of a finite cover. Now

$$\langle w_{2n+1}(M/Z_2), [M/Z_2] \rangle = \chi(M/Z_2),$$

and

$$\begin{aligned}
 & \langle \sum_1^{2n+1} w_{2n+1-j}(M/Z_2)\tilde{h}^*(c^j), [M/Z_2] \rangle \\
 &= \langle h^*(c) \cdot \sum_1^{2n+1} w_{2n+1-j}(M/Z_2)h^*(c^{j-1}), [M/Z_2] \rangle \\
 &= \langle f^* \left( \sum_1^{2n+1} w_{2n+1-j}(M/Z_2)h^*(c^{j-1}) \right), [N] \rangle \\
 &= \langle w_{2n}(N), [N] \rangle \\
 &= \chi(N).
 \end{aligned}$$

Since  $\chi(M/Z_2) + \chi(N) = s\chi(M)$ , the result follows.

The characteristic number formulation seems to depend heavily on the choice of the subgroup  $Z_2$ ; in fact it does not.

LEMMA 3.4: *If  $M^{2n+1}$  admits a free action of  $Z_2 \times Z_2$ , then  $s\chi(M) = 0$ .*

PROOF: Take  $T_1, T_2$  as generators of  $Z_2 \times Z_2$ . Then  $s\chi(M) = \chi(N_1)$  where  $N_1 \subset M/T_1$  is dual to the double cover. However in  $M/Z_2 \times Z_2$ , one may take  $N_2$  dual to the double cover by  $M/T_2$  and if

$$\pi : M/T_1 \rightarrow M/Z_2 \times Z_2,$$

$\pi^{-1}(N_2)$  may be taken to be  $N_1$ ; thus  $N_1$  may be taken to have a free involution induced by  $T_2$ , so  $N_1$  bounds and  $\chi(N_1) = 0$ . \*

Thus if the semi-characteristic is non-trivial on free  $G$  bordism, then  $G$  can contain no subgroup  $Z_2 \times Z_2$ , in particular, the Sylow 2 subgroup  $S$  of  $G$  can contain no such subgroup. Thus, every abelian subgroup of  $S$  is cyclic which implies that  $S$  is either cyclic or generalized quaternion. If  $S$  is cyclic or generalized quaternion, it contains a unique element of order 2, and since any two Sylow 2 subgroups are conjugate, any two elements of order 2 in  $G$  are conjugate.

Restated, either the semi-characteristic is trivial for  $G$  or up to conjugacy, there is a unique element of order 2.

If  $G$  contains a subgroup  $Z_2 \times Z_2$ , and  $H$  is a subgroup of order 2 lying in the Sylow subgroup  $S$ , then  $S$  contains a central subgroup  $K$  of order 2. If  $H = K$ , and  $L$  is any other subgroup of order 2 in  $S$ ,  $H \times L \subset S$ , while if  $H \neq K$ ,  $H \times K \subset S$ . Thus  $H$  lies in a subgroup isomorphic to  $Z_2 \times Z_2$ . Now  $i^* : H^*(B(Z_2 \times Z_2); Z_2) \rightarrow H^*(BZ_2, Z_2)$  is epic so  $i_*$  is zero ( $i_* i^* = 0$ ), but  $i_* : H^*(BZ_2, Z_2) \rightarrow H^*(BG; Z_2)$  factors through  $B(Z_2 \times Z_2)$ , hence is zero.

If  $G$  contains no subgroup  $Z_2 \times Z_2$ , then the classes  $i_*(c^j)$  and  $i_*(\bar{c}^j)$  for two different subgroups  $Z_2$  differ by the action of an inner automorphism on  $G$ , but inner automorphisms are trivial on cohomology, so  $i_*(c^j) = i_*(\bar{c}^j)$ .

#### 4. Self-intersections

The cobordism invariance of the semi-characteristic for free involutions on odd dimensional manifolds gives rise to a cobordism invariant of even dimensional manifolds with involution which is free on the boundary. Denoting this cobordism group by  $\mathfrak{N}_*^{Z_2}(\text{Free } \partial)$ , the composite

$$\mathfrak{N}_{2n}^{Z_2}(\text{Free } \partial) \xrightarrow{\hat{\sigma}} \mathfrak{N}_{2n-1}(Z_2) \xrightarrow{s\chi} Z_2$$

is the homomorphism of interest.

The cobordism group  $\mathfrak{N}_{2n}^{Z_2}(\text{Free } \partial)$  has been analyzed thoroughly by Conner and Floyd [2] (28.1). It may be identified via the fixed point homomorphism with  $\bigoplus_{j=0}^{2n} \mathfrak{N}_{2n-j}(BO_j)$ , by assigning to  $(V^{2n}, T)$  the cobordism classes  $F^{2n-j} \xrightarrow{\nu} BO_j$  of the maps classifying the normal bundle to the codimension  $j$  part of the fixed set of  $T$ .

From Corollary 3.3,  $s\chi(\partial V)$  is given as the sum of the semi-characteristics of the sphere bundles of the normal bundles of the  $F^{2n-j}$ , and by Proposition 3.1, these semi-characteristics are the sum of the Euler characteristics of  $F^{2n-j}$  and the submanifold dual to  $\nu$ . The submanifold dual to  $\nu$  may also be described as the self-intersection of  $F^{2n-j}$  in the disc of  $\nu$ .

Being given  $(V^{2n}, T)$  with fixed set  $F$ , one may consider the self-intersection  $F \cap F$  of  $F$  in  $V$ , i.e. the submanifold of  $F$  obtained by deforming  $F$  to be transverse regular to itself within  $V$ , and taking the intersection. The cobordism class of  $F \cap F$  is a cobordism invariant of  $(V, T)$ . (To see this, make the fixed set of a cobordism from  $(V, T)$  to  $(V', T')$  transverse to itself). In fact, the self-intersection of  $F^{2n-j}$  with itself is the submanifold dual to  $v$ . Thus one has:

**PROPOSITION 4.1:** *If  $(V^{2n}, T)$  is a manifold with involution which is free on  $\partial V$ , then*

$$s\chi(\partial V) = \chi(F) + \chi(F \cap F),$$

where  $F$  is the fixed set of  $T$  and  $F \cap F$  is the self-intersection of  $F$  in  $V$ .

In particular, if  $V$  is closed,  $s\chi(\partial V) = 0$ , and  $\chi(F) \equiv \chi(F \cap F) \pmod 2$ . Combining this with  $\chi(V) \equiv \chi(F) \pmod 2$ , one has  $\chi(F \cap F) \equiv \chi(V)$ . (See Conner and Floyd [2] (27.2), or note that if  $T$  is simplicial on  $V$ , the simplices of  $V$  consist of pairs  $\sigma, T\sigma \neq \sigma$  and simplices of  $F$ ). Thus one has:

**PROPOSITION 4.2:** ([2], (27.4)). *If  $T : M^{2n} \rightarrow M^{2n}$  is an involution on a closed manifold of odd Euler characteristic, then some component of the fixed set of  $T$  has dimension at least  $n$ .*

**PROOF:** If the fixed set has dimension less than  $n$ , then the normal bundle of the fixed component  $F^i$  has dimension greater than  $i$ , so has a section. Thus,  $F \cap F$  can be taken empty, and  $\chi(F \cap F) = 0$ . Then  $\chi(M) \equiv \chi(F \cap F)$  and  $M$  has even Euler characteristic.     \*

### 5. Lee's oriented invariants

Lee also introduced semicharacteristic invariants

$$\chi_{\frac{1}{2}} : \Omega_{2n+1}(G, \omega) \rightarrow \tilde{R}_{GL, Sp}(G, \omega) \quad n \text{ even}$$

and

$$\chi_{\frac{1}{2}} : \Omega_{2n+1}(G, \omega) \rightarrow \tilde{R}_{GL, 0}(G, \omega) \quad n \text{ odd}$$

for free  $G$  actions on oriented manifolds, using cohomology with  $K$  coefficients, where  $K$  is a field of characteristic not 2. He characterizes these invariants as 'remarkably useless' and yet they are far from trivial.

Being given a finite group  $G$  and homomorphism  $\omega : G \rightarrow Z_2 = \{+1, -1\}$ ,  $\Omega_*(G, \omega)$  denotes the cobordism group of free  $G$  actions on oriented manifolds for which each  $g \in G$  preserves or reverses orientation as  $\omega(g)$  is respectively  $+1$  or  $-1$ . When  $\omega$  is trivial, this is the usual

oriented  $G$  bordism group  $\Omega_*(BG)$ ; when  $\omega$  is non-trivial, the kernel of  $\omega$  is a normal subgroup  $H \subset G$  of index 2 giving a double cover  $BH \xrightarrow{\pi} BG$ , and the group  $\Omega_*(G, \omega)$  is the oriented bordism group  $\tilde{\Omega}_{*+1}(M_\pi, BH)$  where  $M_\pi$  is the mapping cone of  $\pi$ . (Note: given  $V \xrightarrow{f} M_\pi, \partial V \xrightarrow{f} BH$ ,  $f$  may be made transverse to  $BG$  giving an unoriented manifold  $N$  with principal  $G$  bundle  $P$  so that  $P/H$  is the orientation cover of  $N$ ; thus  $[V, f]$  gives the action of  $G$  on  $P$ ).

One has a restriction homomorphism  $i^* : \Omega_*(G, \omega) \rightarrow \Omega_*(S, \omega/S)$  for a subgroup  $S \subset G$  by restricting the action to  $S$ , and an extension homomorphism  $i_* : \Omega_*(S, \omega/S) \rightarrow \Omega_*(G, \omega)$  assigning to  $(M, S)$  the action on  $G \times M/(g, m) \sim (gs^{-1}, sm)$  given by  $g'(g, m) = (g'g, m)$ , where  $G$  is oriented by  $\omega$  so that  $g \in G$  is a positively oriented point if  $\omega(g) = +1$ , and is negatively oriented if  $\omega(g) = -1$ . (Note: The  $S$  action  $s_*(g, m) = (gs^{-1}, sm)$  is then orientation preserving making  $G \times M/\sim$  oriented).

PROPOSITION 5.1: *The semicharacteristic*

$$\chi_{\frac{1}{2}} : \Omega_*(G, \omega) \rightarrow \tilde{R}_{GL,x}(G, \omega)$$

depends only on the Sylow 2-subgroup of  $G$ ; specifically

$$\chi_{\frac{1}{2}}(M; K) = i_* \chi_{\frac{1}{2}}(i^* M; K)$$

where  $i_*, i^*$  are extension and restriction from a Sylow 2-subgroup  $S$  of  $G$ .

PROOF: One has a commutative diagram

$$\begin{array}{ccccc} \Omega_*(S, \omega/S) & \xrightarrow{i_*} & \Omega_*(G, \omega) & \xrightarrow{i^*} & \Omega_*(S, \omega/S) \\ \downarrow \chi_{\frac{1}{2}} & & \downarrow \chi_{\frac{1}{2}} & & \downarrow \chi_{\frac{1}{2}} \\ \tilde{R}_{GL,x}(S, \omega/S) & \xrightarrow{i_*} & \tilde{R}_{GL,x}(G, \omega) & \xrightarrow{i^*} & \tilde{R}_{GL,x}(S, \omega/S) \end{array}$$

and so one wants  $M \equiv i_* i^* M \pmod{\text{kernel } \{\chi_{\frac{1}{2}}(\ ; K)\}}$ . Now Lee notes that  $\chi_{\frac{1}{2}}$  has image in the subgroup of  $\tilde{R}_{GL,x}(G, \omega)$  consisting of elements of order 2, so kernel  $\{\chi_{\frac{1}{2}}(\ ; K)\} \supset 2\Omega_*(G, \omega)$ .

One now has a commutative diagram

$$\begin{array}{ccccc} \Omega_*(S, \omega/S) & \xrightarrow{i_*} & \Omega_*(G, \omega) & \xrightarrow{i^*} & \Omega_*(S, \omega/S) \\ \downarrow 2 & & \downarrow 2 & & \downarrow 2 \\ \Omega_*(S, \omega/S) & \xrightarrow{i_*} & \Omega_*(G, \omega) & \xrightarrow{i^*} & \Omega_*(S, \omega/S) \\ \downarrow \rho & & \downarrow \rho & & \downarrow \rho \\ \mathfrak{N}_*(S) & \xrightarrow{i_*} & \mathfrak{N}_*(G) & \xrightarrow{i^*} & \mathfrak{N}_*(S) \end{array}$$

where  $\rho$  is reduction, and the columns are exact (when  $\omega$  is trivial, this is the exact Rohlin sequence ([2] (16.2))  $\Omega_*(BG) \xrightarrow{\rho} \Omega_*(BG) \xrightarrow{f} \mathfrak{R}_*(BG)$ , while if  $\omega$  is non-trivial, it is the Rohlin sequence for  $(M_\pi, BH)$  combined with the Thom isomorphism  $\tilde{\mathfrak{R}}_{*+1}(M_\pi, BH) \cong \mathfrak{R}_*(BG)$ ).

Since  $i_*i^* = 1$  on  $\mathfrak{R}_*(G)$ ,  $i_*i^* = 1 \pmod{2\Omega_*(G, \omega)}$  on  $\Omega_*(G, \omega)$ . \*

*Note:* There are no non-trivial semicharacteristic invariants for a group of odd order, for  $\tilde{R}_{GL,x}([1], \omega/1)$  is the zero group.

The major advantage of this result is that one need only consider ordinary representations; i.e. representations of a 2-group on a field of characteristic different from 2, and may largely ignore the odd part of  $G$  which might have led to modular representations.

**PROPOSITION 5.2:** *If  $G$  is a finite group with non-trivial cyclic Sylow 2-subgroup  $S$ , and  $1 : G \rightarrow Z_2$  is the trivial homomorphism, then*

$$\chi_{\frac{1}{2}} : \Omega_{2n+1}(G, 1) \rightarrow \tilde{R}_{GL,0}(G, 1) \quad n \text{ odd}$$

*is the zero homomorphism, and*

$$\chi_{\frac{1}{2}} : \Omega_{2n+1}(G, 1) \rightarrow \tilde{R}_{GL,Sp}(G, 1) \quad n \text{ even}$$

*is given by*

$$\chi_{\frac{1}{2}}(M; K) = s\chi(M) \cdot i_*(K)$$

*where  $i_*$  is the extension from  $S$ .*

*Note:* It will be shown that  $i_*(K) \neq 0$ .

**PROOF:** The proof will be somewhat involved, needing first the case  $G = Z_2$ .

Let  $K$  be a field of characteristic not equal to 2. The irreducible  $K$  representations of  $Z_2$  are  $K_+$ ,  $K_-$ , the one dimensional representations with  $tx = x$  and  $tx = -x$  respectively, where  $t$  is the non-trivial element of  $Z_2$  and  $x \in K$ .  $R_K(Z_2)$  is then isomorphic to  $Z \oplus Z$ , where the isomorphism assigns the dimensions of image  $(\frac{1}{2}(1+t))$  and image  $(\frac{1}{2}(1-t))$ ; i.e. the number of copies of  $K_+$  and  $K_-$ .

Each of  $K_+$  and  $K_-$  has the nonsingular symmetric form  $\phi : K \times K \rightarrow K$  given by  $\phi(x, y) = xy$ , and so  $R_{GL,0}(Z_2, 1) = 0$ .

A skew form which is nonsingular on  $V$  makes  $\text{im}(\frac{1}{2}(1+t))$  and  $\text{im}(\frac{1}{2}(1-t))$  orthogonal and induces nonsingular skew forms on each, so each is even dimensional, with  $2K_+$  and  $2K_-$  having the hyperbolic forms. Thus  $R_{GL,Sp}(Z_2, 1) \cong Z_2 \oplus Z_2$ . Extending  $K$  from the trivial group to  $Z_2$  gives  $K_+ \oplus K_-$ , so  $\tilde{R}_{GL,Sp}(Z_2, 1) \cong Z_2$  and the isomorphism sends  $V$  to  $\dim V \cdot [K]$ , where  $K = K_+$  is the trivial representation.

Thus for  $G = Z_2$ ,  $\chi_{\frac{1}{2}}$  is zero on  $\Omega_{2n+1}(Z_2, 1)$  if  $n$  is odd, and on  $\Omega_{2n+1}(Z_2, 1)$ , with  $n$  even,

$$\begin{aligned} \chi_{\frac{1}{2}}(M; K) &= \sum_0^n (-1)^i [H^i(M; K)] \\ &= \left\{ \sum_0^i (-1)^i \dim_K H^i(M; K) \right\} \cdot [K] \end{aligned}$$

By the work of Lusztig, Milnor, and Peterson [6] an oriented manifold of dimension  $4r + 1$  which bounds as an unoriented manifold has the property that its semicharacteristic is independent of the field with which it is computed. Thus, the equation becomes  $\chi_{\frac{1}{2}}(M; K) = s\chi(M) \cdot [K]$ .

Now let  $G = Z_{2^s}$ ,  $s \geq 1$ . Let  $\gamma$  denote the standard complex line bundle over  $CP(\infty) = BS^1$ . Then the sphere bundle of  $\gamma^{2^s} = \gamma \otimes_C \cdots \otimes_C \gamma$  ( $2^s$  times) may be identified with  $BZ_{2^s}$  and the cofibration

$$S(\gamma^{2^s}) \rightarrow D(\gamma^{2^s}) \rightarrow T(\gamma^{2^s})$$

gives an exact sequence

$$\begin{array}{ccccc} \Omega_*(S(\gamma^{2^s})) & \rightarrow & \Omega_*(D(\gamma^{2^s})) & \rightarrow & \tilde{\Omega}_*(T(\gamma^{2^s})) \\ & & \uparrow & & \downarrow \\ & & \Omega_*(CP(\infty)) & & \tilde{\Omega}_*(BS^1) \end{array}$$

Projection is a homotopy equivalence, and identifies  $\Omega_*(D(\gamma^{2^s}))$  with  $\Omega_*(CP(\infty))$ , while the Thom isomorphism identifies  $\tilde{\Omega}_*(T(\gamma^{2^s}))$  with  $\Omega_{*-2}(CP(\infty))$ . Thus, one has an exact sequence

$$\begin{array}{ccccc} \Omega_*(BZ_{2^s}) & \xrightarrow{\pi_*} & \Omega_*(BS^1) & \xrightarrow{\alpha} & \Omega_*(BS^1) \\ & & \uparrow & & \downarrow \\ & & \Omega_*(CP(\infty)) & & \tilde{\Omega}_*(BS^1) \end{array}$$

$\beta$

Now  $\Omega_*(BZ_{2^s}) \cong \Omega_* \oplus \tilde{\Omega}_*(BZ_{2^s})$ , where the  $\Omega_*$  summand is obtained from the inclusion of a point and  $\tilde{\Omega}_*(BZ_{2^s})$  consists of 2-torsion. The  $\Omega_*$  summand maps isomorphically to the similar  $\Omega_*$  summand of  $\Omega_*(BS^1)$ .

In the special case  $s = 1$ ,  $\pi_* : \Omega_*(BZ_2) \rightarrow \Omega_*(BS^1)$  maps onto the torsion subgroup (Note: The torsion in  $\Omega_*(BS^1)$  maps monomorphically into unoriented bordism of  $BS^1$ , but  $\pi^* : H^*(BS^1; Z_2) \rightarrow H^*(BZ_2; Z_2)$  is monic, so  $\pi_*$  is epic in unoriented bordism, and  $\alpha$  is zero. Thus if  $x$  is a torsion class  $\rho\alpha x = \alpha\rho x = 0$ , but  $\alpha x$  is torsion so  $\rho\alpha x = 0$  implies  $\alpha x = 0$ ). One then has, for any  $s$ ,

$$\Omega_*(BZ_2) \xrightarrow{\pi'_*} \Omega_*(BZ_{2^s}) \xrightarrow{\pi_*} \Omega_*(BS^1)$$

and the image of  $\pi_*$  is contained in the image of  $\pi_* \circ \pi'_*$ . Thus

$$\beta + \pi'_* : \Omega_*(BS^1) \oplus \Omega_*(BZ_2) \rightarrow \Omega_*(BZ_{2^s})$$

is epic; i.e. every free  $Z_{2^s}$  action is bordant to a sum of restrictions of free  $S^1$  actions and extensions of free  $Z_2$  actions.

*Note:* For further discussion of the cofibration, one may see [7]. The fact that  $\beta + \pi'_*$  is epic was worked out in a joint discussion with Russell J. Rowlett, for a theorem on which he was working.

Now consider an element in  $\Omega_{2n+1}(Z_{2^s}, 1)$  with  $n$  odd, and write it as  $(M, \phi) + (N, \psi)$  where  $(M, \phi)$  is the restriction of an  $S^1$  action, and  $(N, \psi)$  is the extension of a  $Z_2$  action  $(N', \psi')$ . Then  $\chi_{\frac{1}{2}}(N; K) = i_* \chi_{\frac{1}{2}}(N', K)$ , but  $\chi_{\frac{1}{2}}(N', K) = 0$ . Also  $\chi_{\frac{1}{2}}(M, K) = \{\sum_0^n (-1)^i \dim H^i(M, K)\} \cdot [K]$  for  $Z_{2^s}$  acts trivially on  $H^*(M; K)$ , being the restriction of an  $S^1$  action. Since the trivial representation admits the nonsingular symmetric form  $\phi : K \times K \rightarrow K : (x, y) \rightarrow xy, [K] = 0$ . Thus

$$\chi_{\frac{1}{2}} : \Omega_{2n+1}(Z_{2^s}, 1) \rightarrow \tilde{R}_{GL, 0}(Z_{2^s}, 1)$$

is the zero homomorphism, ( $n$  odd).

Letting  $n$  be even, an element in  $\Omega_{2n+1}(Z_{2^s}, 1), s > 1$ , may be written as  $(M, \phi) + (N, \psi)$  as above. Then

$$\chi_{\frac{1}{2}}(N, K) = i_* \chi_{\frac{1}{2}}(N', K) = i_*(s\chi(N') \cdot [K]) = s\chi(N')i_*[K].$$

In particular, if  $N'$  is the sphere  $S^{2n+1}$  with antipodal action,

$$i_*[K] = \chi_{\frac{1}{2}}(i_*(S^{2n+1}); K) = \chi_{\frac{1}{2}}(i_*i^*(S^{2n+1}, \theta); K)$$

where  $\theta$  is the standard free  $Z_{2^s}$  action, but  $i_*i^*$  is trivial on unoriented bordism, so  $i_*i^*(S^{2n+1}, \theta)$  is divisible by 2. Thus  $i_*[K] = 0$  and  $\chi_{\frac{1}{2}}(N, K) = 0$ . Note that  $s\chi(N) = 2^{s-1}s\chi(N') = 0$ . Since  $Z_{2^s}$  acts trivially on  $H^*(M; K)$ , one has  $\chi_{\frac{1}{2}}(M; K) = s\chi(M) \cdot [K]$ , and combining

$$\chi_{\frac{1}{2}}(M \cup N; K) = s\chi(M \cup N) \cdot [K].$$

Thus the proposition is true for  $G = Z_{2^s}$ , and applying Proposition 5.1 gives the result for all  $G$  with cyclic Sylow 2-subgroup.

To see that  $i_*[K] \neq 0$ , consider the restriction to  $Z_2 \subset G. KG \otimes_{KS} K$  has dimension  $[G : S] = \text{odd}$  over  $K$ , so restricts to the nonzero class in  $\tilde{R}_{GL, Sp}(Z_2, 1).$  \*

Now turning to homomorphisms  $\omega : G \rightarrow Z_2$  which are non-trivial, one has

PROPOSITION 5.3: *If  $\omega : G \rightarrow Z_2$  is non-trivial, then the composite*

$$\Omega_{2n+1}(G, \omega) \xrightarrow{\rho} \mathfrak{N}_{2n+1}(G) \xrightarrow{\chi_{\frac{1}{2}}} \tilde{R}_{GL, \text{ev}}(G)$$

*is the zero homomorphism.*

PROOF:  $\chi_{\frac{1}{2}}(\rho M; K) = s\chi(M)i_*[K]$ , and so one wants  $s\chi(M) = 0$ . Since

$\omega$  is non-trivial, there is an  $x$  with  $\omega(x) = -1$ , and  $\omega(x^{2^{j+1}}) = -1$  so by taking a suitable odd power of  $x$ , one may find  $x$  with  $\omega(x) = -1$  and  $x^{2^s} = 1$ ; i.e. it is sufficient to consider  $G$  cyclic of order  $2^s$ .

If  $s = 1$ ,  $M \xrightarrow{\pi} M/Z_2$  is the orientation cover, and

$$\begin{aligned} s\chi(M) &= \langle w_{2n}c + w_{2n-1}c^2 + \cdots + c^{2n+1}, [M/Z_2] \rangle \\ &= \langle cv'v', [M/Z_2] \rangle = \langle w_1v'v', [M/Z_2] \rangle \\ &= \langle S_q^{-1}((v')^2), [M/Z_2] \rangle = 0, \end{aligned}$$

or alternately, the submanifold  $N \subset M/Z_2$  dual to  $w_1$  is a torsion element of  $\Omega_*$ , but  $\chi(N) = \text{Index}(N) \pmod 2$  and the index vanishes on torsion classes.

If  $s > 1$ , one has a diagram

$$\begin{array}{ccc} M & & \\ \downarrow & & \\ M/Z_2 & \longrightarrow & BZ_2 \\ \downarrow & & \downarrow \\ M/Z_{2^{s-1}} & \longrightarrow & BZ_{2^{s-1}} \\ \downarrow & & \downarrow \\ M' = M/Z_{2^s} & \longrightarrow & BZ_{2^s} \\ & & \downarrow \\ & & BS^1 = CP(\infty) \end{array}$$

and

$$\begin{aligned} s\chi(M) &= \langle w_{2n}c + w_{2n-1}c^2 + \cdots + c^{2n+1}, [M/Z_2] \rangle \\ &= \langle w_{2n}i_*(c) + w_{2n-1}i_*(c^2) + \cdots + i_*(c^{2n+1}), [M'] \rangle. \end{aligned}$$

Now  $H^*(BZ_{2^s}; Z_2)$  is generated by a 1-dimensional class  $d$  and a 2-dimensional class  $\alpha$  (a Bockstein of  $d$ ) with  $d^2 = 0$ . The class  $\alpha$  comes from  $CP(\infty)$  and restricts to  $c^2$  in  $BZ_2$ . One then has  $i_*(c^{2^j}) = 0$  and  $i_*(c^{2^{j+1}}) = d\alpha^j$ . The condition that  $\omega$  is non-trivial is that  $M/Z_{2^{s-1}}$  is the orientation cover of  $M'$ , so  $d$  restricts to  $w_1$ . Thus

$$s\chi(M) = \langle w_{2n}w_1 + w_{2n-2}w_1\alpha + \cdots + w_1\alpha^n, [M'] \rangle.$$

Letting  $N \subset M'$  be the codimension 2 submanifold dual to the complex line bundle coming from  $CP(\infty)$ ,

$$w(N) = w(M)/1 + \alpha$$

so

$$w_1(N) = w_1, w_{2n-2}(N) = w_{2n-2} + w_{2n-4}\alpha + \dots + \alpha^{n-1}$$

and

$$s\chi(M) = \langle w_{2n}w_1, [M'] \rangle + \langle w_{2n-2}w_1, [N] \rangle.$$

For a manifold  $V$  of dimension  $2j+1$ ,  $w_{2j} = v_j^2$  so

$$\langle w_{2j}w_1, [V] \rangle = \langle w_1v_j^2, [V] \rangle = \langle S_q^{-1}(v_j^2), [V] \rangle = 0,$$

and so  $s\chi(M) = 0$ . \*

Now consider an *abelian* group  $G$  with  $\omega : G \rightarrow Z_2$  a homomorphism, and let  $K$  be a field having characteristic zero or relatively prime to the order of  $G$ .

If  $V$  is an irreducible  $K$  representation of  $G$ , then  $V$  is a module over the commutative ring  $KG$  and has the property that if  $x \neq 0$  is an element of  $V$ , then  $(KG)x = V$ . For any nonzero element  $x$  in  $V$ ,  $I_x = \{\lambda \in KG | \lambda x = 0\}$  is a (two sided) ideal in  $KG$ , and  $KG/I_x$  is a field (Note: If  $\mu \notin I_x$ ,  $\mu x \neq 0$  and  $(KG)\mu x = V$  so there is a  $\lambda \in KG$  with  $\lambda\mu x = x$ ). Further,  $I_x$  is independent of  $x$ . One may then identify  $V$  with a finite extension  $\tilde{K} = KG/I$  of the field  $K$ .

Letting  $1 \in \tilde{K}$  be the multiplicative unit, let  $H \subset G$  be the isotropy group  $\{g \in G | g1 = 1\}$ , so that the orbit  $G \cdot 1$  is identifiable with  $G/H$  and consists of  $[G : H] = [G/H : 1]$  elements of  $\tilde{K}$ . If  $g \cdot 1 = \lambda_g \in \tilde{K}$ , action by  $g$  on  $V$  is given by multiplication by  $\lambda_g \in \tilde{K}$ . In particular, if  $e$  is the exponent of  $G/H$ , i.e.  $z^e = 1$  for all  $z \in G/H$ , then  $G \cdot 1$  consists of  $e$ -th roots of unity in  $\tilde{K}$ , but there are at most  $e$   $e$ -th roots of unity. Thus the exponent and order of  $G/H$  are the same, and  $G/H$  is cyclic.

Then  $\tilde{K}$  is a splitting field for  $x^e - 1$  over  $K$ , i.e.  $x^e - 1$  factors as  $\prod(x - \rho)$  where  $\rho \in G \cdot 1$  and  $\tilde{K}$  is generated over  $K$  by  $G$  and hence by the elements in  $G \cdot 1$ . Further, the polynomial  $x^e - 1$  is separable over  $K$  for the roots  $\rho \in G \cdot 1$  are distinct. Thus  $\tilde{K}$  is a finite dimensional Galois extension of  $K$  and hence is a separable extension. In particular,  $\tilde{K}$  has a non-singular symmetric bilinear form given by  $\phi(x, y) = \text{trace}_{\tilde{K}/K}(xy)$ , the trace of the  $K$ -linear map given by multiplication by  $xy$ .

Now define an automorphism  $\sigma : KG \rightarrow KG$  by

$$\sigma(\sum \alpha_g g) = \sum \omega(g)\alpha_g g^{-1}$$

(an anti-automorphism if  $G$  is nonabelian), so that the  $KG$  module structure on the  $\omega$ -dual of  $V$  is given by  $(\lambda f)(x) = f(\sigma(\lambda)x)$  for  $f \in \text{Hom}(V, K)$ .

CLAIM: If  $V$  is isomorphic to its  $\omega$ -dual  $V^*$ , then  $\sigma(I) = I$ , where  $I = \{\lambda \in KG \mid \lambda x = 0 \forall x \in V\}$ . To see this, let  $\psi : V \rightarrow V^*$  be an isomorphism of  $KG$  modules. Then for  $v, v' \in V, \lambda \in KG$ ,

$$\psi(\lambda v)(v') = \{\lambda \psi(v)\}(v') = \psi(v)(\sigma(\lambda)v')$$

so if  $\lambda \in I, \psi(v)(\sigma(\lambda)v') = 0$  for all  $v$  and so  $\sigma(\lambda)v' = 0$  and  $\sigma(\lambda) \in I$ , while if  $\sigma(\lambda) \in I, \psi(\lambda v)(v') = 0$  for all  $v'$  and so  $\psi(\lambda v) = 0$  or  $\lambda v = 0$  and so  $\lambda \in I$ .

Thus, if  $V \cong V^*, \sigma$  induces an automorphism  $\sigma \cdot \tilde{K} \rightarrow \tilde{K}$ .

CLAIM: The form  $\theta(x, y) = \text{trace}_{\tilde{K}/K}(x \cdot \sigma(y))$  on  $\tilde{K}$  is a symmetric non-singular  $\omega$ -form on  $\tilde{K}$ . To see this,

$$\begin{aligned} \theta(y, x) &= \text{trace}_{\tilde{K}/K}(y \cdot \sigma(x)) = \text{trace}_{\tilde{K}/K}(\sigma(x \cdot \sigma(y))) \\ &= \text{trace}_{\tilde{K}/K}(x\sigma(y)) = \theta(x, y) \end{aligned}$$

and

$$\begin{aligned} \theta(gx, gy) &= \text{trace}_{\tilde{K}/K}(gx\sigma(y)\omega(g)g^{-1}) = \omega(g) \text{trace}_{\tilde{K}/K}(x\sigma(y)) \\ &= \omega(g)\theta(x, y) \end{aligned}$$

while  $\{x \mid \theta(x, y) = 0 \text{ for all } y\}$  is a  $G$  invariant subspace of  $V$  and is proper since  $\text{trace}_{\tilde{K}/K}(xy)$  is nonsingular, so is the zero subspace.

From this one has:

PROPOSITION 5.4: *If the Sylow 2 subgroup of  $G$  is abelian, then*

$$\chi_{\frac{1}{2}} : \Omega_{2n+1}(G, \omega) \rightarrow \tilde{R}_{GL,0}(G, \omega)$$

*is the zero homomorphism.*

PROOF: It suffices to verify this on the Sylow 2 subgroup,  $S$ . Then  $R_K(S)$  is the free abelian group with base the irreducible representations, which one may list as  $\{[V] \mid V \cong V^*\} = T_0$  and  $\{[V] \mid V \not\cong V^*\} = T_1$ . Divide  $T_1$  into two disjoint classes  $T_+$  and  $T_-$  so that if  $[V] \in T_+$  then  $[V^*] \in T_-$ . By the above discussion,  $[V] = 0$  in  $R_{GL,0}(S, \omega|S)$  if  $[V] \in T_0$ , and thus  $R_{GL,0}(S, \omega|S)$  is the free abelian group with base the classes  $[V]$  with  $[V] \in T_+$  (and  $[V^*] = -[V]$ ). Since  $(KG)^* = KG, KG$  is zero in  $R_{GL,0}(S, \omega|S)$ , and so  $\tilde{R}_{GL,0}(S, \omega|S) = R_{GL,0}(S, \omega|S)$  is torsion free. Since  $\chi_{\frac{1}{2}}(\Omega_{2n+1}(S, \omega|S))$  consists of 2 torsion, it is the zero group. \*

Note: To see that  $(KG)^* = KG$ , one need only consider the form  $\theta(\sum \alpha_g g, \sum \beta_g g) = \sum \omega(g)\alpha_g \beta_{g^{-1}}$ , which is an orthogonal form.

Now returning to an irreducible representation  $V$  of  $G$  with  $V \cong V^*$ , suppose there is an element  $\zeta \in \tilde{K}$  with  $\sigma(\zeta) = -\zeta$ . Then

$$\tau(x, y) = \text{trace}_{\tilde{K}/K}(\zeta x \sigma(y))$$

is a nonsingular skew  $\omega$ -form on  $V$ . To see this,

$$\begin{aligned} \tau(y, x) &= \text{trace}_{\tilde{K}/K}(\zeta y \sigma(x)) = \text{trace}_{\tilde{K}/K}(\sigma(\zeta y \sigma(x))) = \text{trace}_{\tilde{K}/K}(\sigma(\zeta) x \sigma(y)) \\ &= -\text{trace}_{\tilde{K}/K}(\zeta x \sigma(y)) = -\tau(x, y) \end{aligned}$$

and

$$\tau(gx, gy) = \text{trace}_{\tilde{K}/K}(\zeta gx \sigma(y) \omega(g) g^{-1}) = \omega(g) \tau(x, y),$$

while  $\{x | \tau(x, y) = 0 \forall y\}$  is a proper  $G$  invariant subspace of  $V$  and so is zero.

Now  $\sigma : \tilde{K} \rightarrow \tilde{K}$  is an involution, so decomposes  $\tilde{K}$  into  $\pm 1$  eigenspaces. Thus if  $\sigma(\zeta) = -\zeta$  has no solution, then  $\sigma(\lambda) = \lambda$  for all  $\lambda$ . Applying this to  $g \in G$ ,  $gx = \omega(g)g^{-1}x$  for all  $x \in V$  or  $g^2x = \omega(g)x$ , i.e.  $g^2$  acts on  $V$  as multiplication by  $\omega(g)$ .

There are now several cases to consider.

First, suppose  $\omega : G \rightarrow Z_2 = \{1, -1\}$  is the trivial homomorphism. Then supposing  $V \cong V^*$  and that there is no element  $\zeta \in \tilde{K}$  with  $\sigma(\zeta) = -\zeta$ ,  $g^2$  acts trivially on  $V$  for all  $G$ . Thus  $H = \{g | g^2 = 1\}$  is a subgroup of index 2 in  $G$  or  $G$  itself and there is a homomorphism  $\phi : G \rightarrow Z_2$  with kernel  $H$  so that the representation  $V$  is the representation  $K_\phi$  of  $G$  on  $K$  given by  $gx = \phi(g) \cdot x$ .

In order to analyze  $\tilde{R}_{GL, Sp}(G, 1)$ , divide the irreducible  $K$  representations into four classes,  $T_+$  and  $T_-$  consisting of two disjoint collections of  $V$  with  $V \not\cong V^*$ , so that if  $V \in T_+$ ,  $V^* \in T_-$ ,  $T_0$  the collection of those  $V \cong V^*$  for which there is a  $\zeta \in \tilde{K}$  with  $\sigma(\zeta) = -\zeta$ , and  $\Phi$ , the collection of  $K_\phi$  with  $\phi \in \text{Hom}(G; Z_2)$ . Then  $R_K(G, 1)$  is free abelian with base  $[V]$ , with  $V$  in  $\Phi \cup T_0 \cup T_+ \cup T_-$ . Any representation  $W$  with a symplectic form decomposes into sums of irreducible summands corresponding to the different irreducibles and must pair  $nV$  against  $nV^*$ ,  $V$  being irreducible. In particular, if  $V \in T_+$ , the number of copies of  $V$  and  $V^*$  in  $W$  is the same, and of course  $V \oplus V^*$  has a hyperbolic form, and the number of copies of  $K_\phi$  in  $W$  is even, for a nonsingular skew form on a  $K$  vector space must have even rank, while  $K_\phi \oplus K_\phi$  has a hyperbolic form. Thus  $R_{GL, Sp}(G, 1)$  is the direct sum of a free abelian group on  $[V]$ ,  $V \in T_+$  (with  $[V^*] = -[V]$ ) and a  $Z_2$  vector space with base the  $[V]$ ,  $V \in \Phi$ .

Now turning to  $KG$ ,  $(KG)^* \cong KG$  so the number of occurrences of  $V$  and  $V^*$  in  $KG$  is the same. Further,  $K_\phi$  is one-dimensional so absolutely irreducible and hence occurs exactly once in  $KG$ . Thus

$$[KG] = \sum [K_\phi] \in R_{GL, Sp}(G, 1)$$

and  $\tilde{R}_{GL, Sp}(G, 1)$  is the direct sum of a free abelian group on the classes  $[V]$  for  $V \in T_+$  and a  $Z_2$  vector space on the classes  $[K_\phi]$  for  $\Phi \in \text{Hom}(G; Z_2)$  a nontrivial homomorphism. The class of  $[K_1] = [K]$ , the trivial representation is  $\sum_{\phi \neq 1} [K_\phi]$ .

Being given a manifold  $M^{2n+1}$  with free  $G$  action, the coefficient of  $[K_\phi] \in \tilde{R}_{GL, Sp}(G, 1)$  is the sum of the dimensions of the subspaces of the  $H^i(M, K)$  on which  $G$  acts trivially (the number of copies of  $K_1$ ) and as multiplication via  $\phi$  (the number of copies of  $K_\phi$ ), which is the dimension of the subspace on which the kernel of  $\phi$  acts trivially. However, the projection  $\pi : M \rightarrow M/\ker \phi$  onto the orbit space of the action of the kernel of  $\phi$  induces an isomorphism of  $H^i(M/\ker \phi; K)$  onto the elements of  $H^i(M; K)$  invariant under  $\ker \phi$ . Thus one has:

**PROPOSITION 5.5:** *If  $G$  is abelian and  $K$  is a field of characteristic zero or prime to the order of  $G$ , then the 2-torsion subgroup of  $\tilde{R}_{GL, Sp}(G, 1)$  is a  $Z_2$  vector space with a base  $\{[K_\phi]\}$  where  $\phi$  is a nontrivial homomorphism of  $G$  to  $Z_2$ . The homomorphism*

$$\chi_{\frac{1}{2}} : \Omega_{2n+1}(G, 1) \rightarrow \tilde{R}_{GL, Sp}(G, 1)$$

sends the class of  $M^{2n+1}$  into

$$\sum_{\phi} s\chi(M/\ker \phi) \cdot [K_\phi].$$

Notes:

(1) This applies via 5.1 to any  $G$  with abelian Sylow 2 subgroup. However, the  $s\chi(M/\ker \phi)$  may satisfy dependence relations for the action of the normalizer of  $S$  may carry  $\phi$  into some other homomorphism. When  $G$  is abelian,  $i_*[K_{\phi/s}] = [K_\phi]$ , and the result looks nicer.

(2) This shows that Lee's impressions were incorrect; one can obtain nontrivial invariants from these semicharacteristics. Taking  $G$  to be  $Z_2 \times Z_2$ , the unoriented invariants were trivial, but these are not. In particular, if  $M$  is a manifold with involution  $t$  and  $\tilde{M}$  is its extension to  $Z_2 \times Z_2$ , then  $s\chi(\tilde{M}/\ker \phi) = s\chi(M)$  if  $\phi(t) \neq 1$ , while

$$s\chi(\tilde{M}/\ker \phi) = s\chi(2(M/Z_2)) = 0$$

if  $\phi(t) = 1$ .

(3) This result should be compared with 5.2 for  $G = Z_{2^s}$ , for the two results give  $s\chi(M) \cdot [K]$  and  $s\chi(M/Z_{2^{s-1}})[K_\phi]$  where

$$\phi : Z_{2^s} \rightarrow Z_{2^s}/Z_{2^{s-1}} \cong Z_2$$

is the unique non-trivial homomorphism. Since  $[K] = [K_\phi]$ , this simply asserts equality of the semicharacteristics. One may obtain this equality using either approach.

From a cobordism point of view  $M$  may be written as a sum of terms  $N^{2j} \times (S^{2k+1}, \theta)$  with  $N$  oriented and  $2j + 2k = 2n$ ,  $n$  odd and  $\tilde{M}$  where  $\tilde{M}$  is an extension from  $Z_{2^{s-1}}$  (in fact from  $Z_2$ ). Now the semicharacteristic

of  $\tilde{M}$  is trivial, and  $\tilde{M}/Z_{2^{s-1}}$  is two copies of the same manifold so has trivial semicharacteristic. Now  $s\chi(N \times S^{2k+1}) = \chi(N) \cdot s\chi(S^{2k+1})$  vanishes if  $j$  is odd (for an oriented manifold has  $\chi(N) \equiv \text{Index}(N)$  which vanishes if  $j$  is odd) and similarly  $s\chi(N \times (S^{2k+1}/Z_{2^{s-1}}))$  vanishes. Thus it suffices to show  $s\chi(S^{2k+1}/Z_{2^{s-1}}) = 1$  if  $k$  is odd, but this is trivial.

One may also give a purely representation theoretic proof of the result, computing  $s\chi(M)$  and  $s\chi(M/Z_{2^{s-1}})$  over any field  $K$  of characteristic not 2. From Lee's result ([5], Lemma 2.4),  $\chi_{\frac{1}{2}}(M; K) \cong \chi_{\frac{1}{2}}(M, K)^*$  in  $\tilde{R}_K(Z_2)$  and  $(KZ_{2^s})^* = KZ_{2^s}$ , so writing  $\chi_{\frac{1}{2}}(M; K)$  in  $R_K(Z_{2^s})$  as

$$nK_1 + mK_\phi + p_v V + \sum (q_{v'} V' + r_{v'} V'^*)$$

with  $V \in T_0, V' \in T_+, q_{v'} = r_{v'} \pmod 2$ , giving  $s\chi(M) = n + m + \sum p_v \dim V$ . On the other hand  $s\chi(M/Z_{2^{s-1}}) = n + m$  and so it suffices to show that  $\dim V$  is even for all  $V \in T_0$ ; i.e. that every self dual irreducible representation of  $Z_{2^s}$  other than  $K$  and  $K_\phi$  is even dimensional. (Note: If  $s = 1$ ,  $K$  and  $K_\phi$  are the only irreducibles, so there is nothing to prove. Thus one may suppose  $s > 1$ .)

First, if  $x^{2^{s-1}} = -1$  is solvable in  $K$ , then every irreducible representation has the form  $K_\beta$  and is given by  $K$  with the generator of  $Z_{2^s}$  acting as multiplication by  $\beta$  where  $\beta^{2^s} = 1$ . Since  $(K_\beta)^* = K_{\beta^{-1}}$ ,  $K_\beta$  is self dual only if  $\beta = \beta^{-1}$  or  $\beta^2 = 1$ . Thus only  $K_1$  and  $K_\phi$  are self dual.

Thus, one may suppose  $x^{2^{r-1}} = -1$  is solvable in  $K$  but  $x^{2^r} = -1$  is not, where  $1 \leq r < s$ . The irreducible representations of  $K$  are then of the form  $K_\beta, \beta^{2^r} = 1$ , or have a base  $x, tx, t^2x, \dots, t^{2^p-1}x$  with  $t^{2^p}x = \theta x$  where  $\theta^{2^{r-1}} = -1, \theta \in K$ , and  $p+r \leq s, p \geq 1$ . The dual of the latter may be similarly described but corresponds to  $\theta^{-1}$ , so is self dual only if  $\theta = \theta^{-1}$  or  $\theta^2 = 1$  and  $r = 1$ . Similarly,  $(K_\beta)^* = K_{\beta^{-1}}$  and  $K_\beta$  is self dual only if  $\beta^2 = 1$ . Thus  $r = 1$  or the only self duals are  $K_1$  and  $K_\phi$ .

Assuming  $r = 1$ , the irreducibles are  $K_1, K_\phi$  or of the form with a base  $x, tx, \dots, t^{2^p-1}x$  with  $t^{2^p}x = -x$  and with  $1 \leq p < s$ . In this case, all are self dual, but only  $K_1$  and  $K_\phi$  have odd dimension. \*

The referee observes that  $S\chi$  is invariant under field extension, and by [6], is independent of the characteristic for manifolds of dimension  $4k+1$ . Thus, one may compute over the reals. Considering the representation of  $Z_{2^s}$  on  $H^i(M; R)$  and splitting into irreducible representations,  $H^i(M/Z_{2^{s-1}}; R)$  is clearly isomorphic to the sum of the representation spaces where the generator acts as multiplication by  $\pm 1$ . The remaining components are all two dimensional.

Now returning to the general situation, consider the case with  $\omega : G \rightarrow Z_2$  nontrivial, with  $V \cong V^*$  and  $\tilde{K}$  containing no element  $\zeta$  with  $\sigma(\zeta) = -\zeta$ , so that  $g^2x = \omega(g)x$  for all  $g$  in  $G$ . In particular,  $g^4x = x$  and for some  $g, g^2x = -x$ . Letting  $H = \{g|g1 = 1\}$ , it follows that  $G/H$

is cyclic of order 4, and that  $V$  is given by a representation of  $G/H = Z_4$  for which the subgroup  $Z_2$  acts as multiplication by  $-1$ .

The first obvious case is when there is no homomorphism  $\theta : G \rightarrow Z_4$  for which  $\theta(g^2) = \omega(g) \in Z_2$ . Noting that the epimorphism  $\pi : Z_4 \rightarrow Z_2$  is given by  $\pi(x) = x^2$  (considering  $Z_2 \subset Z_4$  as the squares), this is the case in which  $\omega : G \rightarrow Z_2$  cannot be written in the form  $\pi \circ \phi$  with  $\phi : G \rightarrow Z_4$ . Then every self dual representation is symplectic and letting the set of irreducible representations of  $G$  be decomposed into  $T_0, T_+$  and  $T_-$ ,  $\tilde{R}_{GL,Sp}(G, \omega)$  is free abelian on the classes  $[V]$  with  $V$  in  $T_+$ , and so  $\chi_{\frac{3}{2}}$  is zero.

If there is an element  $t \in G$  of order 2 with  $\omega(t) \neq 1$ , there can be no homomorphism  $\phi : G \rightarrow Z_4$  with  $\pi \circ \phi = \omega$ . The converse is also true; if there is no element  $t \in G$  of order 2 with  $\omega(t) \neq 1$ , then there is a homomorphism  $\phi : G \rightarrow Z_4$  with  $\pi \circ \phi = \omega$ . (To see this, write

$$G = Z_{2^s} \oplus \cdots \oplus Z_{2^{s_n}} \oplus Z_{r_1} \oplus \cdots \oplus Z_{r_j}$$

where  $r_i$  are odd. If  $t_i$  generates the summand  $Z_{2^{s_i}}$ , there is a  $t_i$  of minimal order for which  $\omega(t_i) \neq 1$ . If  $\omega(t_j) \neq 1$  for some other  $t_j$ ,  $t_j$  may be replaced by  $t_j t_i$  giving a new generator for a summand on which  $\omega$  is trivial. After iterating,  $\omega$  factors through projection on the  $t_i$  summand.)

Suppose there is a homomorphism  $\phi : G \rightarrow Z_4$  with  $\pi \circ \phi = \omega$ . The irreducible representations of  $Z_4$  may be described as follows:

*Case I:* If the equation  $x^2 = -1$  is solvable in  $K$  then every irreducible representation of  $Z_4$  is of the form  $K_\beta$  with the generator of  $Z_4$  acting on  $K$  as multiplication by  $\beta$ , where  $\beta^4 = 1$ . Those  $\beta$  with  $\beta^2 = -1$  give representations with  $Z_2$  acting as  $-1$ .  $K_\beta$  is its own  $\pi$ -dual. Choosing one specific  $\beta \in K$  with  $\beta^2 = -1$  as generator of  $Z_4$ , the nonsymplectic self dual irreducible representations of  $G$  are then in one-to-one correspondence with  $\{\phi : G \rightarrow Z_4 | \pi \circ \phi = \omega\} = \Phi$  with  $G$  acting on  $K$  by  $gx = \phi(g) \cdot x$ . This will be denoted  $K\langle\phi\rangle$ . Now  $R_K(G)$  is free abelian with a base given by the  $K\langle\phi\rangle$ ,  $\phi \in \Phi$ , those  $V \cong V^*$  not in  $\Phi$ , called  $T_0$ , and  $T_+$ ,  $T_-$  which decompose those  $V \not\cong V^*$ .  $R_{GL,Sp}(G, \omega)$  is the direct sum of the free abelian group on  $T_+$  and the  $Z_2$  vector space on  $\Phi$  (a skew form on  $W$  makes  $W$  self dual so  $V$  and  $V^*$  occur with the same multiplicity: if  $nK\langle\phi\rangle$  occurs in  $W$   $nK\langle\phi\rangle$  has a skew form so  $n$  is even). Each  $K\langle\phi\rangle$  occurs once in  $KG$ , since  $K\langle\phi\rangle$  is absolutely irreducible, and so  $[KG] = \sum [K\langle\phi\rangle]$ .

*Note:* Writing  $Z_4$  additively,  $\phi$  and  $\theta$  taking  $G$  into  $Z_4$  with  $\pi \circ \phi = \pi \circ \theta = \omega$  differ by a homomorphism of  $G$  into  $Z_2$  i.e.  $\theta = \phi + \lambda$ . Thus fixing one  $\phi_0 : G \rightarrow Z_4$ ,  $\phi \rightarrow \phi - \phi_0$  defines a one-to-one correspondence between  $\Phi$  and  $\text{Hom}(G; Z_2)$ . Thus  $\tilde{R}_{GL,Sp}(G, \omega)$  is the direct sum of the

free abelian group on  $T_+$  and the  $Z_2$  vector space with base the  $K\langle\phi_0 + \lambda\rangle$  where  $\lambda \in \text{Hom}(G; Z_2)$  is nontrivial, and  $[K\langle\phi_0\rangle] = \sum_{\lambda} [K\langle\phi_0 + \lambda\rangle]$ . Notice that  $\phi_0 + \lambda + \omega$  is the negative of  $\phi_0 + \lambda$ .

Being given a manifold  $M^{2n+1}$ ,  $n$  even, with a free  $G$  action and  $\phi : G \rightarrow Z_4$  with  $\pi \circ \phi = \omega$ ,  $H^*(M/\ker \phi; K)$  may be identified with the elements of  $H^*(M; K)$  invariant under  $\ker \phi$ , i.e. with the summands  $K_1, K_{\omega}, K\langle\phi\rangle$  and  $K\langle\phi + \omega\rangle$ , while  $H^*(M/\ker \omega; K)$  is identifiable with the summands  $K_1$  and  $K_{\omega}$ . Thus letting  $n\langle\phi\rangle$  be the number of summands of  $K\langle\phi\rangle$  in

$$\sum_0^n (-1)^i H^i(M; K), \quad n\langle\phi\rangle + n\langle\phi + \omega\rangle = s\chi(M/\ker \phi) - s\chi(M/\ker \omega).$$

Now  $M/\ker \phi$  and  $M/\ker \omega$  admit free orientation reversing  $Z_4$  and  $Z_2$  actions, so by 5.3  $n\langle\phi\rangle \equiv n\langle\phi + \omega\rangle$  in  $Z_2$ . Letting  $\phi_0$  be fixed as above, the coefficient of  $[K\langle\phi_0 + \omega\rangle]$  in  $\chi_{\frac{1}{2}}(M; K)$  is  $n\langle\phi_0\rangle + n\langle\phi_0 + \omega\rangle = 0$ , while for  $\lambda \neq 1, \omega$ , the coefficients of  $[K\langle\phi_0 + \lambda\rangle]$  and  $[K\langle\phi_0 + \lambda + \omega\rangle]$  are equal and are given by

$$\begin{aligned} & \frac{1}{2} \left\{ \sum_0^n (-1)^i \dim H^i(M/\ker \phi_0; K) - \sum_0^n (-1)^i \dim H^i(M/\ker \omega; K) \right. \\ & \left. + \sum_0^n (-1)^i \dim H^i(M/\ker (\phi_0 + \lambda); K) - \sum_0^n (-1)^i \dim (M/\ker \omega; K) \right\}. \end{aligned}$$

Letting

$$s\chi_K(M) = \sum_0^n (-1)^i \dim H^i(M; K)$$

in  $Z$ , this gives

$$\chi_{\frac{1}{2}}(M; K) \sum \frac{1}{2} (s\chi_K(M/\ker \phi_0) + s\chi_K(M/\ker (\phi_0 + \lambda))) \times \{[K\langle\phi_0 + \lambda\rangle] + [K\langle\phi_0 + \lambda + \omega\rangle]\}$$

where the sum is over representatives  $\lambda$  for the pairs  $\lambda, \lambda + \omega$ , where  $\lambda \neq 1, \omega$ .

*Case II:* If the equation  $x^2 = -1$  is not solvable in  $K$ , then every irreducible representation of  $Z_4$  is one of the forms  $K_1, K_{-1}$  or  $V$  where  $V$  is the 2 dimensional  $K$  representation given by  $t(x, y) = (-y, x)$  (Note: If  $c(x, y) = (x, -y)$ ,  $tc = -ct$ , so this is equivalent to the representation with the generator of  $Z_4$  acting as  $-t$ ). Thus, for each pair of homomorphisms  $\phi$  and  $\phi + \omega$  sending  $G$  to  $Z_4$  and lifting  $\omega$  there is an irreducible 2 dimensional representation,  $V\langle\phi, \phi + \omega\rangle$ . Decomposing the non-self duals into  $T_+$  and  $T_-$  and letting  $\Phi = \{\phi : G \rightarrow Z_4 | \pi \circ \phi = \omega\}$ ,

$R_{GL,Sp}(G, \omega)$  is the direct sum of the free abelian group on  $T_+$  and a  $Z_2$  vector space with base the  $V\langle\phi, \phi + \omega\rangle$  for the pairs  $\{\phi, \phi + \omega\}$  of elements of  $\Phi$ . (Note: If  $nV\langle\phi, \phi + \omega\rangle$  admits a symplectic form, then extending  $K$  to a splitting field  $K'$  for  $x^2 + 1$ ,  $nK'_\phi + nK'_{\phi + \omega}$  has a symplectic form, so  $n$  is even.) Now  $KG$  has each  $V\langle\phi, \phi + \omega\rangle$  appearing exactly once (extending to  $K'$ ,  $K'_\phi$  and  $K'_{\phi + \omega}$  appear exactly once in  $K'G$ ) so  $\tilde{R}_{GL,Sp}(G, \omega)$  is the direct sum of a free abelian group on  $T_+$  and a  $Z_2$  vector space with base the  $[V\langle\phi_0 + \lambda, \phi_0 + \lambda + \omega\rangle]$ ,  $\lambda \neq 1, \omega$ , and with

$$[V\langle\phi_0, \phi_0 + \omega\rangle] = \sum_{\lambda} [V\langle\phi_0 + \lambda, \phi_0 + \lambda + \omega\rangle].$$

Since the number of copies of  $V\langle\phi, \phi + \omega\rangle$  in  $\sum (-1)^i H^i(M; K)$  is  $\frac{1}{2}(s\chi_K(M/\ker \phi) - s\chi_K(M/\ker \omega))$ , one has

$$\chi_{\frac{1}{2}}(M, K) = \sum \left\{ \frac{1}{2}(s\chi_K(M/\ker \phi_0) + s\chi_K(M/\ker(\phi_0 + \lambda))) \right\} [V\langle\phi_0 + \lambda, \phi_0 + \lambda + \omega\rangle].$$

This completes the list of cases, with a full understanding of each of the  $\tilde{R}_{GL,Sp}(G, \omega)$ , but with several cases. One may obtain a clean result:

**PROPOSITION 5.6:** *If  $G$  is abelian and  $K$  is a field of characteristic zero or prime to the order of  $G$  and  $\omega : G \rightarrow Z_2$  is a nontrivial homomorphism then  $\chi_{\frac{1}{2}}(M^{2n+1}, K) \in \tilde{R}_{GL,Sp}(G, \omega)$  is determined by the numbers*

$$\frac{1}{2}\{s\chi_K(M/\ker \phi) + s\chi_K(M/\ker \phi')\} \in Z_2$$

where

$$s\chi_K(M^{2n+1}) = \sum_0^n (-1)^i \dim H^i(M; K) \in Z$$

and where  $\phi, \phi' : G \rightarrow Z_4$  are liftings of  $\omega$ .

**COROLLARY 5.7:** *If the Sylow 2 subgroup of  $G$  is either  $Z_2 \times \dots \times Z_2$  or cyclic, and if  $\omega : G \rightarrow Z_2$  is nontrivial, then*

$$\chi_{\frac{1}{2}} : \Omega_{2n+1}(G, \omega) \rightarrow \tilde{R}_{GL,Sp}(G, \omega)$$

is zero.

Notes:

(1)  $\chi_{\frac{1}{2}}$  can be nontrivial. Let  $G = Z_4 \times Z_2$  generated by  $t, s$  with  $t^4 = s^2 = 1, ts = st$ . Let  $\omega(t) = -1, \omega(s) = 1$ . If  $M_0^{2n+1}$  is a manifold with free involution  $s'$ , consider  $Z_4 \times M_0$  with  $t(x, y) = (tx, y)$  and  $s(x, y) = (x, s'y)$  and the obvious  $\omega$  orientation; i.e. the extension from  $Z_2$  to  $G$  of  $M_0$ . There are two classes of liftings of  $\omega, \phi_0$  with kernel

$\{s\}$  and  $\phi_1$  with kernel  $\{st^2\}$ . One has  $M/\ker \phi_0 \cong Z_4 \times (M_0/Z_2)$  and  $M/\ker \phi_1 \cong 2$  copies of  $M$ , so  $\frac{1}{2}\{s\chi_K(M/\ker \phi_0) + s\chi_K(M/\ker \phi_1)\}$  is  $2s\chi(M_0/Z_2) + s\chi(M_0) \equiv s\chi(M_0)$ .

(2) It would be nice to know if the expression

$$\frac{1}{2}\{s\chi_K(M/\ker \phi) + s\chi_K(M/\ker \phi')\}$$

is independent of  $K$ . This is in fact true. First consider  $\omega : G \rightarrow Z_2$  and two liftings  $\phi, \phi' : G \rightarrow Z_4$ . Let  $H = \ker \phi \cap \ker \phi'$ , and then  $G/H$  acts on  $M/H$  and is a free action of  $Z_4 \times Z_2$  of the sort in Note 1 above. Thus one need only check this on  $Z_4 \times Z_2$  actions.

First, one needs to compute  $\Omega_*(Z_4 \times Z_2, \omega)$ . If  $\rho : BZ_2 \rightarrow BZ_4$ ,  $\Omega_*(Z_4 \times Z_2, \omega) \cong \Omega_{*+1}(D(\rho) \times BZ_2, S(\rho) \times BZ_2)$  where  $D, S$  denote disc and sphere of the line bundle of  $\rho$ . The homomorphism given by inclusion of  $(D(\rho) \times pt, S(\rho) \times pt)$  may be identified with the extension from  $\Omega_*(Z_4, \pi)$ , and the complementary summand is identifiable with

$$\begin{aligned} \tilde{\Omega}_{*+1}(M(\rho) \wedge BZ_2) &= \lim \pi_{*+r+1}(M(\rho) \wedge BZ_2 \wedge MSO(r)) \\ &= \lim \pi_{*-r+1}(M(\rho) \wedge MO(r+1)) \\ &= \tilde{\mathfrak{N}}_*(M(\rho)) \\ &\cong \mathfrak{N}_{*-1}(BZ_4) \end{aligned}$$

where the homomorphism to  $\tilde{\mathfrak{N}}_*(M(\rho))$  is obtained by dualizing the line bundle given by the map into  $BZ_2$  and the last is the Thom isomorphism.

Now  $\mathfrak{N}_*(BZ_4)$  is generated as  $\mathfrak{N}_*$  module by the spheres  $(S^{2n+1}, i)$  and by the extensions from  $Z_2$  of  $(S^{2n}, a)$  which will be denoted  $2S^{2n}$ ,  $t(x, 0) = (x, 1)$ ,  $t(x, 1) = (-x, 0)$  giving the action. Now let  $M$  be a closed manifold, not necessarily orientable and consider  $S(\det \tau \oplus 1) \times S^{2n+1}$  or  $S(\det \tau \oplus 1) \times 2S^{2n}$ , where  $\det \tau$  is the determinant of the tangent bundle of  $M$ . Let  $s$  act as the antipodal map in the fibers of  $S(\det \tau \oplus 1)$  and let  $t$  act diagonally, by multiplication by  $-1$  in the fibers of  $\det \tau$ ,  $1$  in those of the trivial bundle and with the given action on  $S^{2n+1}$  or  $2S^{2n}$ . The double cover of the action of  $Z_2 = \{s\}$  has base  $RP(\det \tau \oplus 1) \times X$  and dualizing this line bundle gives  $RP(\det \tau) \times X$ ; i.e.  $M \times X$  and in  $\mathfrak{N}_{*-1}(BZ_4)$  this gives the class  $M \times (S^{2n+1}, i)$  or  $M \times (2S^{2n}, t)$ . Thus these classes in  $\Omega_*(Z_4 \times Z_2, \omega)$  are generators modulo extensions from  $(Z_4, \pi)$ .

For  $S(\det \tau \oplus 1) \times S^{2n+1} = N$ , the cohomology of  $N/\ker \phi_0$  and  $N/\ker \phi_1$  are identifiable with the elements in  $H^*(N; K)$  invariant under  $s$  and  $st^2$ , but  $t^2$  is trivial on cohomology, so these quotients have the same  $K$  cohomology. Thus

$$\frac{1}{2}\{s\chi_K(N/\ker \phi_0) + s\chi_K(N/\ker \phi_1)\} = s\chi_K(N/\ker \phi_0)$$

which is even; i.e.  $\chi_{\frac{1}{2}}(N, K)$  is zero.

For  $S(\det \tau \oplus 1) \times 2S^{2n} = N$ ,  $s$  and  $st^2$  act preserving the components of  $N$ . Thus  $N/\ker \phi_0$  consists of 2 copies of  $RP(\det \tau \oplus 1) \times S^{2n}$  and  $N/\ker \phi_1$  consists of 2 copies of  $S((\det \tau \oplus 1) \otimes \gamma)$  over  $M \times RP(2n)$ , where  $\gamma$  is the nontrivial line bundle over  $RP(2n)$ . Thus

$$\frac{1}{2}\{s\chi_K(N/\ker \phi_0) + s\chi_K(N/\ker \phi_1)\}$$

is

$$s\chi_K(RP(\det \tau \oplus 1) \times S^{2n}) + s\chi_K(S((\det \tau \oplus 1) \otimes \gamma)).$$

These bound  $RP(\det \tau \oplus 1) \times D^{2n+1}$  and  $D((\det \tau \oplus 1) \otimes \gamma)$  unorientedly and so the semicharacteristics are independent of  $K$ .

For an extension, let  $M_0$  have a free  $Z_4$  action and let  $M = M_0 \times Z_2$  with  $t(x, y) = (tx, y)$ ,  $s(x, y) = (x, -y)$  which gives the extension. Then  $M/\ker \phi_0$  and  $M/\ker \phi_1$  may each be identified with  $M_0$  for  $s$  and  $st^2$  interchange components. Thus

$$\frac{1}{2}\{s\chi_K(M/\ker \phi_0) + s\chi_K(M/\ker \phi_1)\} = s\chi_K(M_0)$$

which is even since  $M_0$  has an orientation reversing  $Z_4$  action.

Since the invariants  $\frac{1}{2}\{s\chi_K(M/\ker \phi_0) + s\chi_K(M/\ker \phi_1)\}$  are cobordism invariants and agree on a base of  $\Omega_*(Z_4 \times Z_2, \omega)$  they agree. Thus the value is independent of  $K$ .

*Beware:* The independence of  $K$  assumed throughout that the characteristic of  $K$  is *not* 2. The expression

$$\frac{1}{2}\{s\chi_{Z_2}(M/\ker \phi_0) + s\chi_{Z_2}(M/\ker \phi_1)\}$$

is not a cobordism invariant, as one may verify by considering  $S(\det \tau \oplus 1) \times S^1 = M$  for the bundle over  $S^6 \times S^7 \times RP(2)$ ; the invariant is 1, but the manifold bounds – bounding  $S(\det \tau \oplus 1) \times S^1$  for the bundle over  $D^7 \times S^7 \times RP(2)$ .

To compute the invariant,  $M/\{s\} = RP(\det \tau \oplus 1) \times S^1$  has mod 2 cohomology a free module over that of  $S^6 \times S^7 \times RP(2) \times S^1$  on a 1-dimensional class. Thus,  $\dim H^i(M/\{s\}; Z_2)$  is given by 1, 3, 4, 3, 1, 0, 1, 4, 7 in dimensions 0 through 8 and  $s\chi_{Z_2}(M/\{s\}) = 4$ . For  $M/\{st^2\}$ , one has  $S^6 \times S^7 \times S((\det \tau \oplus 1) \otimes \gamma)$  where the sphere bundle is over  $RP(2) \times RP(1)$ . In the spectral sequence for the sphere bundle the fiber class transgresses to  $\alpha \cdot \sigma$  (the product of the generators, so  $\dim H^i(S((\det \tau \oplus 1) \otimes \gamma); Z_2)$  is 1, 2, 2, 2, 1 in dimensions 0 through 4, and  $\dim H^i(M/\{st^2\}; Z_2)$  is 1, 2, 2, 2, 1, 0, 1, 3, 4 so  $s\chi_{Z_2}(M/\{st^2\}) = 2$ .

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University of Virginia  
Charlottesville, Virginia 22903